ERASMUS UNIVERSITY ROTTERDAM

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Bachelor Thesis [Program IBEB]

Multiple Principals Problems with Incentive Contract in a Moral Hazard Setting in Principal-Agent Model.

Name: Ziyi Lin

Student Number: 536661

Supervisor: Dr. Josse Delfgaauw

Second assessor: Dr. Vladimir Karamychev

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Contents

1. Abstract

In some situations, an agent performs different tasks for different principals. In a moral hazard setting, this creates an interaction between the agent's moral hazard problem and the externalities that the principals impose on each other. This paper assumes a principal-agent setting with multiple principals with incentive pay to an agent who is protected by limited liability. Principals only care about their own benefits because of the information asymmetry. It shows that limited liabilities decrease the effort from the agent. However, since there is asymmetric information between principals meaning that the externalities are imposed. It can be problematic because principals only consider their own actions. The results shows that principals will bid for the higher bonuses to make the agent exert the corresponding efforts of their tasks. However, from the agent side, increasing the effort from one task will decrease the effort from other tasks. At the same time, because the agent will always prioritize the task with higher bonuses, it can subsequentially hurt other principals which contributes inefficiency. Alternatively, principals can also try to collaborate with each other to make the agent achieve the first best effort.

2. Introduction

Collaboration happens in every aspect of life. Wherever between the individual inside the organization or between the different organizations or different stakeholders. However, it should be highly noted that to achieve a well-organized and smooth collaboration, different goals and different objectives from different stakeholders should also be considered. Without these conditions, it might decrease the efficiency of these collaborations or subsequentially this collaboration would fail, since the different goals of these stakeholders are not aligned.

Over the years, many people claim that to design specific projects or policies, it is important to bring in citizen participation and collaboration. Kuzior, A., & Kuzior, P. (2020) argue that to make the smart city project work, it is essential that the different stakeholders, namely the government, knowledge institutions, citizens, and businesses collaborate. In addition, Lodge, M., & Wegrich, K. (2015) shows that by using crowdsourcing, the government can achieve specific goals more efficiently. However, these claims not only painted an over-idealistic graph about the collaboration with different stakeholders whether inside or outside the organization but also ignored the fundamental behaviors of the corresponding stakeholders because the strategic games or actions are taking place in this situation which can lead to an inaccurate conclusion. In this situation, the identification of who are the principals and who is the agent is important and the situation of multiple principals needs to be considered first. In the example of Kuzior, A., & Kuzior, P. (2020), if there is a situation where the government, knowledge institution, and the business are all the principals and the citizen is the agent and the citizens will perform different tasks in this process, what will be the choice for the agent and each principal? In the case of crowdsourcing, the crowdsourcing worker will not only perform a single task and in some situations, but these workers also work for different principals in crowdsourcing. Both situations can create some efficiency troubles when the agent has different attitude to each task, or the incentives are not correctly given by the principals.

Therefore, this paper will show the analysis of a situation where multiple principals provide incentive pay to an agent in a moral hazard setting. These principals care about various aspects of the agent's work and, hence, their incentives are not aligned. Through a complementarity in the agent's cost of effort, stronger incentives provided by one principal reduce the value of output for the other principal. In addition, assuming the agent is protected by limited liability, it will be beneficial to study whether and when the lack of alignment between principals reduces efficiency.

Knowing the effect of the agent with multiple principals has scientific relevance because it is indispensable to check the behaviors of the agent and the outcome of these types of behavior. This phenomenon is also claimed by Zhang, Y., Gu, Y., Liu, L., Pan, M., Dawy, Z., & Han, Z. (2015) among the crowdsourcing workers. It is also essential to check the condition that moral hazard happened in a more general way. This also means that it is socially relevant to understand this effect since, within some projects or organizations, this kind of structure will create problems when the agent has preferences for specific tasks and can exploit the contract from the given incentives.

Therefore, it is vital to check the effect of multiple principal situations to decrease the inefficiency in the project or in the organization. In the following analysis, the result will show the situations which induce the inefficiencies.

In this paper, section three will discuss the related literatures, including agency theories, multitasking and multiple principals and the next section the model and its assumption are presented. In section five, it will show analysis and results from both principal and agent sides. In the last section this paper will discuss some concluding remarks and limitations.

3. Related Literatures

The agency theory will be introduced which covers the relationship between the principals and the agents. Starting from this, the principal gives the contract to the agent and the agent will be rewarded according to their efforts. In this case, it can create some situations which can contribute to the moral hazard when using incentive pay. At the same time, it is also possible that each agent is performing more than one task at the same time which means that they need to divide their efforts across the tasks, or an agent has multiple principals. The following paragraphs will be expanding these points.

Agency theory

One of the most essential elements in agency theory is moral hazard. This can be defined that one individual has the incentive to take risks for exploiting the contract and getting higher private gains while other parties need to pay for these expenses. Therefore, since the agent serves on behalf of the principal, it can be inferred that there will be some hidden actions because the interests of those two parties are not perfectly aligned. This will cause the moral hazard issue, where Stiglitz (1983) uses the insurance market as an example for illustration. Inside the principal-agent theory, the performance measurements, incentives, and multitasking elements will strongly affect the contract and the outcome. When the principals can provide the incentive pay to the agent, principals will use the performance measurement to assess the agent actions since there will be asymmetric information between the parties and principals need to know the output regarding these actions from the agent.

Baker (1992) assumes that the objective of the principle is not contractable but tied to the action of the agent. The principal will enforce a contractable performance measurement which also serves as the action of the agent and the state of the world. This performance measurement is included in the linear contract which principals give to the agent. In addition, the agent is asymmetrically informed of the information structure. This incentive contract is for the single agent and single task. With this setting, the author has found out that the first best option for a risk-neutral agent will be setting the optimal bonus equal to one, resulting in the expected marginal product of the effort equaling to expected marginal value for the efforts, given that the performance measurement is normalized. When the effort is unobservable, the piece rate is determined by the correlation between the performance measurement and the marginal value of the effort. If the performance measurement and the marginal value of the effort are perfectly correlated and both have the same variance, the piece rate will be one. If correlation between the performance measurement and the marginal value of the effort is lower, then optimal piece rate will be lower since the agent’s effort level does not match the desired effort level of the principal. To conclude, in the incentive contract, only if the marginal product is aligned with the objectives of the principal, the contract is efficient, and it will yield a reliable performance measurement. Otherwise, the asymmetric information will determine the effort level of the agent. In addition, there will be a tradeoff between risk and distortion in the incentive contract based on different elements (Baker, 2002). This tradeoff indicates that higher risks result in lower distortion and higher distortion gives lower risks.

Datar et al. (2001) assumes that principals can only determine agents' compensation by using the signals from agents from different tasks. These signals also served as performance measurements. Agents have negative exponential utility structure, and their actions will not affect the variance and covariance of the performance structures. The principal gives out the linear compensation contract. By using these assumptions, the authors have found out that when the agent is risk neutral and the performance measurements are noiseless, then it is not important to consider the risk. At the same time, if there is no perfect congruity, meaning that the performance measurements are not perfectly aligned with agent actions, then the principal prefers to achieve the maximum congruity by considering the weight of the performance measurements from different tasks. The performance measurements have a relationship with other variables in the contract, as well as the balance between the agents' compensation and the firm's goals when the contract is optimal and the principal needs to think about these variables in the contract. In addition, there are some tradeoffs between congruity and sensitivity as well. The principal can reduce the congruity when the sensitivity of the agent actions is higher. However, if the sensitivity is too much from a task, the weight on that task-specific performance measurement will decrease since the agent does not exert useful efforts.

When designing a contract with asymmetric information, it can lead to moral hazard. This phenomenon also appears in the principal-agent relationship in different fields and industries. For instance, Vera-Hernandez (2003) concentrates on the moral hazard problems in medical insurance contract by using the experimental data. She found out that the people have low copayment rate, meaning that they share less cost with others in their medical insurance contract, they are likely to seek more medical care compared to the people who have higher copayment rate and the average cost on treatment decreases when the copayment rate increases. Meanwhile, the health status has the relationship with the treatment cost. The better health status indicates the lower health penalty and lower treatment cost. In this situation, it will create moral hazard problem since the insurance provider cannot contract the health status of the people because this variable is unobservable. Therefore, the people will have the treatment when the health penalty is low, while the treatment cost is large which will create welfare loss.

Ollier and Thomas (2013) states that when the output is binary (either success and fail) and the type of the agent is clear, the efforts and type are complements for the agent. For the principal, they pay the agent the fixed wage and a bonus when the output is "success". The principal also considers the density and distribution of the agent types. Since agents can potentially misreport their type and for the principal. This will lead to moral hazard problem, and it is hard to differentiate good or bad type agents for the principal. In this case, the bonus offered by the principal can be seen as a type screening instrument and the principal will provide a decreasing information rent contract to minimizing the hazardous situation.

Cohen (1987) describes the moral hazard issue on polluting the environment by using the example of oil spills. He illustrates that in this specific situation, the government acts as principal and maximizes social welfare instead of the profit and agents are the firms who maximize their own profit. In this situation, moral hazard problem arises since the firms has the incentive to pollute and the government needs to pay for the damage. Therefore, the principal must monitor agents and setting some penalties for agents, when agents are detected making pollution, which will push agents to perform the social optimal effort level properly. However, if the monitoring effort from the principal and the penalties are improperly set, the outcome will still be inefficient and not optimal. Therefore, the principal needs to carefully consider their monitoring effort and the penalties on agents and agents' incentives to getting the social optimal.

Ghatak and Pandey (2000) focus on the moral hazard in the agricultural contract with limited liabilities. When the landlord, acts as principal, cannot observe the action on the agent such as the quantity and quality of using fertilizer and using unnecessary amount of water, moral hazard issue arises. Therefore, the landlord needs to draw a contract to encourage the agent exert optimal efforts while decreasing the harmful actions from agents the principal cannot punish the agent for unwanted behavior due to the limited liabilities. In addition, the landlord can monitor the effort and risk of agents to minimize the effect of moral hazard. In the context of managerial and entrepreneurial businesses,

Nygaard and Myrtveit, (2000) assume the company is the principal who owns the trademark, and the dealer is the agent. The principal will design different kind of contract to the agent in different cases. They found that the key factor for the principal to design a contract is the monitoring cost. If the monitoring cost for the principal is low in certain types of contracts, they are more likely to use these types to minimize the damage of hidden actions from the agent. Baker, G., Gibbons, R., & Murphy, K. J. (1994) argue that principals can also use subjective performance measurements in the contract. In this case, incentives are implicit, and it will give different optimal compared to the objective performance measurements.

Multitasking

It is also vital to consider the multitasking situation about the influence of different tasks. Holmstrom and Milgrom (1991) argue that in the multitasking principal – agent relationship, incentives from different tasks will influence the effort level of the agent and these incentives will be allocated according to the exerted effort from the agent. Therefore, the key issue in the multitasking setting is that when the principal incentivizes an agent to exert more effort to one task, the agent can reduce the effort on some other tasks. This indicates that it is essential for the principal to induce a balance of efforts in determining incentive pay. For the principal side, choosing the proper instruments to control and monitor the agent's performance is also important. In addition, the way of incentivizing these performances is important as well. Feltham and Xie (1994) state that there is a positive relationship between the incentives and risk premiums and the compensation to the agents will affect their own effort, which are the noises in this type of contract. They suggest that referencing the market price at the contract termination data will be beneficial as one of the performance measurements but act as the supplement with other performance measurements since the price can be influenced by some uncontrollable events. It needs to be said that these measurements will be more valuable when they are aligned to the benefit only if the effort cost is identical across the tasks and they are independent (Schnedler, 2008). The author also reveals that the different weighs on different tasks in the performance measurements are important and there is a trade of between the allocation on effort and the insurance.

Hong, Hossain, List, and Tanaka (2018) have conducted in a natural field experiment of multitasking theory in Chinese factories. They have found out that the workers who previously worked in the fixed wages, their effort were moving into the tasks which are incentivized which lower the effort for the non-incentivized part. For the workers previously have already in the piece rate scheme, there is a slight change in output when they give out extra incentive for performance. If the tasks cannot be observed and rewarded, the worker will decrease the effort exertion on these tasks. On the contrary, when the effort for the unrewarded task is observable, the worker might not decrease their effort level. These characteristics reveal that when the tasks have different dimensions, it is hard to get the implications and for the principal it is hard to do performance pay as well. The Hawthorne type effect on the workers is also observed which can also influence the effort from the agent. In another word, when the workers notice that they are being observed for their work and effort, then they will change their own behaviors and subsequentially increase productivity.

Multiple Principals

In some situations, there will be more than one principal that exists, which means that it can have common agency problems. Lawrence, P. R., Kolodny, H. F., and Davis, S. M. (1977) illustrate that when there are multiple principals, the agent can play their principals if the interests are not aligned. Bernheim and Whinston (1986) state that there are two types of common agency, namely the delegated one and the intrinsic one. In both situations, each principal only cares about their interests and influences the decision of the agent which can yield inefficiency because the different principal will compete for the offering when they do not cooperate. Therefore, the main problems with the common agency are that principals' interests are not aligned and they do not consider others' behaviors since there are not only asymmetric information between principals and agent, but these principals as well. This will create moral hazard problem and contribute inefficiencies since all principals are going to pursuit their own profit and agents can also have some hidden actions which only benefit themselves as well. Martimort, D. (1992) states that when an agent is contracted by multiple principals and performing different tasks for these principals, the result will depend on the characteristics of these tasks given by principals. He also explains this by giving an example of wholesaler-retailer relationship when two different wholesalers(principals) contract the same agent(retailer), the equilibrium from these contracts are determined by the nature of two tasks, namely the substitute task and the complement task. In addition, Martimort, D., & Stole, L. (2002) argue that it is difficult for the principals to reveal their truth type in multiple principals setting. In this case, agents can play the communication games with the different principals and their effort choice from one specific task will also depends on offer from other principals in other tasks instead of only the corresponding task. Meanwhile, one principal can ask the agent to lie to other principals to get more private benefits and then raises inefficiency.

Since the agency problem with the effort will create moral hazard and decrease the efficiency, it is also vital to check when the agent has different principals, the behavior between agent and principals, and between two different principals, and how they play the game with each other. Therefore, in the next part, the model considering this multiple principals' phenomenon will be discussed.

4. The Model

Consider the multitasking model noted by Gibbons, R. (2010), which is based on Datar, S., Kulp, S. C., & Lambert, R. A. (2001) and Feltham, G. A., & Xie, J. (1994), where both principals and the agent are risk neutral. The agent will perform two tasks for the principals. Therefore, the following assumptions are made

Production: $y_i = f_i e_i$ $(f > 0)$, while f_i is the production of effort

Performance measurements: $p_i \in \{0,1\}$

With probability $q_i \geq 0$, $p_i = 1$ for any e_i and $1 - q_i, p_i = 1$ with probability $g_i e_i$

Here, q_i measures the extent of moral hazard problem, if q_i is larger, it will be more costly for principals to make the agent exert more effort and g_i measures the marginal effort of the effort measured by this performance measurement.

Wage of the agent: $w_i = s_i + b_i p_i$ while b_i is the bonus

Assuming the agent is protected by limited liability, which means that the agent will not be given a negative wage payment. In this case, the wage $w_i \geq 0$ and it is optimal to set the fixed wage $s_i = 0$ and the bonus $b_i \geq 0$

Each task has its target principal. This means that different tasks are managed by different principals. In this situation, the production and performance measurement will be evaluated separately for different tasks. The agent values two tasks differently which can be indicated by their own cost of effort for each task θ_1, θ_2 and both are bigger than zero. At the same time, doing one task also influences the cost of effort on the other task. In this case, the interaction term of two tasks is added. The expected utility function of the agent is given by the following:

$$EU_A = w_1 + w_2 - \theta_1 e_1^2 - \theta_2 e_2^2 - \gamma e_1 e_2$$

Rewrite this gives

$$EU_A = (1 - q_1) b_1 p_1 + (1 - q_2) b_2 p_2 + q_1 b_1 + q_2 b_2 - \theta_1 e_1^2 - \theta_2 e_2^2 - \gamma e_1 e_2$$

The principals cannot observe the effort generated by the agent. Instead, they will use the performance measurement p_i to indicate the exerted effort of the agent. The bonuses will be given based on the performance in accordance with the effort. However, the principals can only screen the outcome of the task. When there is no imput for the task, the principal will get the payoff of zero for the corresponding task. This means that the principal only pays the wages and bonuses to the agent when the task has the imput. Furthermore, each principal can only pay a bonus for their own task. They can neither give a negative bonus nor pay for the failure on the other task. Each principal only cares about the state of success towards their own task. Hence, the expected payoff for the principal can be written in the following:

$$E\pi_i = f_i e_i - b_i p_i$$

Where $i \in \{1,2\}$, meaning that there are two principals in this setting.

The following timeline is set for this game:

1. Each principal sends their own contract to the agent
2. The agent either accepts or rejects the contract, if the agent rejects the contract, the payoff of the agent is zero.
3. The agent chooses the effort of each task
4. The states of the tasks revealed
5. Payoff realized.

5. Analysis[1]

5.1. First Best Option

Since the information is complete, the principal will directly contract the effort itself, therefore for each principal, the following maximization will be used

$$\max_{e_1,e_2} \sum_i y_i + EU_A$$

This gives the optimal effort of $e_1^{FB} = \frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}$ and $e_2^{FB} = \frac{2f_2\theta_1-\gamma f_1}{4\theta_1\theta_2-\gamma^2}$ which means that the optimal effort for each task is dependent on the productivity of two tasks, the marginal effort of productivity on another task and externalities γ between the tasks (See Proof 1A in Appendix). Therefore, it can be said in this situation, $e_1 \geq e_2$, then the condition $\gamma \leq \frac{2(f_1\theta_2-f_2\theta_1)}{f_2-f_1}$ needs to be satisfied and if $\gamma = \frac{2(f_1\theta_2-f_2\theta_1)}{f_2-f_1}$, the agent will exert the same effort. When there are no externalities, and $e_1 \geq e_2$, this will become $\frac{f_1}{\theta_1} \geq \frac{f_2}{\theta_2}$, which means that it depends on the ratio of the output parameters and marginal effort of productivity of each task. Both effort level e_1 and e_2 should be bigger than zero. Therefore, this translates to $e_1, e_2 \geq 0$ which gives $\frac{f_1}{f_2} \geq \frac{\gamma}{2\theta_2}$ for task one and $\frac{f_2}{f_1} \geq \frac{\gamma}{2\theta_1}$ for task two. If $f_1 = f_2$ and $\theta_1 = \theta_2$, then for both tasks $\gamma \leq 2\theta_1$.

In this case, since it is complete information, the principal will never introduce the performance measurement. This means $q_i = 1$, the performance $p_i = 1$. In this situation, the agent will receive the wage according to their effort level on each task. When the agent performs the first best effort level, the amount of the wage paid to the agent will be equal to the agent's cost of effort,

which will be $b_1 = \theta_1\left(\frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}\right)^2 + \frac{\gamma\left(\frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}\right)^2\left(\frac{2f_2\theta_1-\gamma f_1}{4\theta_1\theta_2-\gamma^2}\right)}{\left(\frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}\right)+\left(\frac{2f_2\theta_1-\gamma f_1}{4\theta_1\theta_2-\gamma^2}\right)}$ and $b_2 = \theta_2\left(\frac{2f_2\theta_1-\gamma f_1}{4\theta_1\theta_2-\gamma^2}\right)^2 + \frac{\gamma\left(\frac{2f_2\theta_1-\gamma f_1}{4\theta_1\theta_2-\gamma^2}\right)^2\left(\frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}\right)}{\left(\frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}\right)+\left(\frac{2f_2\theta_1-\gamma f_1}{4\theta_1\theta_2-\gamma^2}\right)}$ for the bonus level (See Proof 1B in Appendix) since both principal divided the part of the wage in the term $\gamma e_1 e_2$ by the fraction of their own tasks. In another word, principal one pays the fraction of $\frac{\gamma e_1^2 e_2}{e_1+e_2}$ and principal two pays the fraction of $\frac{\gamma e_1 e_2^2}{e_1+e_2}$ from the cost of effort with externalities.

[1] All the detailed proofs in this section can be found in the Appendix

Only if the agent performs the first best effort level, then the agent will receive these bonuses. If $b_1 \geq b_2$, then it will be

$$\frac{\theta_1 e_1^2(e_1+e_2)+\gamma e_1^2 e_2}{\theta_2 e_2^2(e_1+e_2)+\gamma e_1 e_2^2} \geq 1$$

Rewrite this and substitute e_1, e_2 gives

$$\frac{(2f_1\theta_2-\gamma f_2)^2}{(2f_2\theta_1-\gamma f_1)^2} \geq \frac{\theta_2(2f_2\theta_1-\gamma f_1+2f_1\theta_2-\gamma f_2)+\gamma(2f_1\theta_2-\gamma f_2)}{\theta_1(2f_2\theta_1-\gamma f_1+2f_1\theta_2-\gamma f_2)+\gamma(2f_2\theta_1-\gamma f_1)}$$

If $f_1 = f_2$ and $\theta_1 = \theta_2$, this ratio is equal to one and $b_1 = b_2 = \left(\theta_1 + \frac{\gamma}{2}\right)\left[\frac{f_1(2\theta_1-\gamma)}{4\theta_1^2-\gamma^2}\right]^2$.

5.2. Second Best Option: Agent's effort

In this situation, it can be argued that by using the performance measurement from the model part, the agent will optimize the following function

$$EU_A = (1-q_1)b_1p_1 + (1-q_2)b_2p_2 + q_1b_1 + q_2b_2 - \theta_1e_1^2 - \theta_2e_2^2 - \gamma e_1e_2$$

Which gives the optimal for each effort $e_1^* = \frac{(1-q_1)2\theta_2b_1g_1-\gamma b_2g_2(1-q_2)}{4\theta_1\theta_2-\gamma^2}$ and $e_2^* = \frac{(1-q_2)2\theta_1b_2g_2-\gamma b_1g_1(1-q_1)}{4\theta_1\theta_2-\gamma^2}$. Here, it should be noted that for task one $(1-q_1)2\theta_2b_1g_1 \geq \gamma b_2g_2(1-q_2)$ needs to be satisfied in order to get $e_1 \geq 0$ and for task two, $(1-q_2)2\theta_1b_2g_2 \geq \gamma b_1g_1(1-q_1)$ will be the condition to get $e_2 \geq 0$ (See Proof 2 in Appendix). This means that the effort for each task of the agent is depending on the bonus level of two different tasks, the performance maturement, the probability and the externalities which also means that it is also depending on the effort level from another task. These can also be written in the form of bonus ratio $\frac{b_1}{b_2} \geq \frac{\gamma g_2(1-q_2)}{2\theta_2g_1(1-q_1)}$ and $\frac{b_2}{b_1} \geq \frac{\gamma g_1(1-q_1)}{2\theta_1g_2(1-q_2)}$

If $e_1 \geq e_2$, then it will be $\frac{(1-q_1)2\theta_2b_1g_1-\gamma b_2g_2(1-q_2)}{(1-q_2)2\theta_1b_2g_2-\gamma b_1g_1(1-q_1)} \geq 1$. This gives an insight of when asking an agent to exert more effort for a certain task the corresponding principal must pay a higher bonus for that. If $q_1 = q_2 = 1$, then the effort level for both tasks is zero because principals do not enact performance measurements so that the agent has incentives to exert no effort for both tasks.

5.3. Situation for Principals

For the principals, since they will consider how the agent responds to the bonuses which have mentioned before, and each principal will maximize the following

$$\max_{b_i} f_ie_i - b_ip_i$$

Which gives

$$b_1^* = \frac{(1-q_1)2\theta_2 g_1 f_1 - (4\theta_1\theta_2 - \gamma^2)q_1 + \gamma(1-q_1)(1-q_2)b_2 g_1 g_2}{4\theta_2 g_1^2(1-q_1)^2}$$

And

$$b_2^* = \frac{(1-q_2)2\theta_1 g_2 f_2 - (4\theta_1\theta_2 - \gamma^2)q_2 + \gamma(1-q_1)(1-q_2)b_1 g_1 g_2}{4\theta_1 g_2^2(1-q_2)^2}$$

Substitute the bonus level of principal two into the equation of optimal bonus level for principal one which yields

$$b_1^* = \frac{g_2(1-q_2)}{g_1(1-q_1)}\{\frac{(1-q_1)(1-q_2)g_1 g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)g_1 g_2^2(1-q_1)(1-q_2)^2} - \frac{(4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1 q_2]}{(16\theta_1\theta_2 - \gamma^2)g_1 g_2^2(1-q_1)(1-q_2)^2}\}$$

Same for principal two

$$b_2^* = \frac{g_1(1-q_1)}{g_2(1-q_2)}\{\frac{(1-q_1)(1-q_2)\theta_2 g_1 g_2(8\theta_1 f_2 + 2\gamma f_1)}{(16\theta_1\theta_2 - \gamma^2)g_1^2 g_2(1-q_1)^2(1-q_2)} - \frac{(4\theta_1\theta_2 - \gamma^2)[q_2 4\theta_2 g_1(1-q_1) + \gamma(1-q_2)g_2 q_1]}{(16\theta_1\theta_2 - \gamma^2)g_1^2 g_2(1-q_1)^2(1-q_2)}\}$$

Since the agent is protected by the limited liability, therefore, both bonus levels need to be bigger than zero, which means these conditions need to be satisfied (See Proof 3 in Appendix).

$$(1-q_1)(1-q_2)g_1 g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2) \geq (4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1 q_2]$$

$$(1-q_1)(1-q_2)\theta_2 g_1 g_2(8\theta_1 f_2 + 2\gamma f_1) \geq (4\theta_1\theta_2 - \gamma^2)[q_2 4\theta_2 g_1(1-q_1) + \gamma(1-q_2)g_2 q_1]$$

In this situation, principals will bid for the bonus to make the agent exert this effort. Therefore, if $b_1 \geq b_2$, then it gives

$$\frac{(1-q_1)(1-q_2)g_1 g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2) - (4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1 q_2]}{(1-q_1)(1-q_2)\theta_2 g_1 g_2(8\theta_1 f_2 + 2\gamma f_1) - (4\theta_1\theta_2 - \gamma^2)[q_2 4\theta_2 g_1(1-q_1) + \gamma(1-q_2)g_2 q_1]} \geq \frac{g_2(1-q_2)}{g_1(1-q_1)}$$

In this case, this condition

$$\frac{b_1}{b_2} \geq \frac{\gamma g_2(1-q_2)}{2\theta_2 g_1(1-q_1)}$$

also needs to be satisfied since there should be more effort than zero in the principal's perspective which gives

$$\frac{\gamma g_2(1-q_2)}{2\theta_2 g_1(1-q_1)} \leq 1$$

Similarly, it also can have,

$$\frac{\gamma g_1(1-q_1)}{2\theta_1 g_2(1-q_2)} \leq 1$$

(See Proof 4 in Appendix)

And if the case is symmetric, meaning that $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$, it will give the result of $b_1^* = b_2^* = \frac{\theta_1^2 f_1(8+2\gamma)}{(16\theta_1^2-\gamma^2)g_1(1-q_1)} - \frac{(4\theta_1^2-\gamma^2)q_1[4\theta_1+\gamma]}{(16\theta_1^2-\gamma^2)g_1^2(1-q_1)^2}$, meaning that the bonus level of both tasks is equal.

Therefore, this result indicates that only if $\gamma > 0$, which means externalities exist, the optimal bonus level for each principal also depends on the bonus level of another principal. Therefore, this indicates that each principal tends to set the bonus higher than other principals to make the agent concentrate on the tasks to whom the principal charges, since the externalities and the effort cost for the other task is presenting here. This can potentially create problems for the principals since they want to maximize their own profit, however giving a huge amount of the bonus to the agent will be harmful for that since this decreases their expected profits. For the agent, since this agent knows that two principals will bid for the bonus, it can be inferred that the agent will also take advantage of this and ask a higher bonus for the task to exert the effort which yield inefficiency.

5.4. Agent Actions

Substitute the optimal bonus into the effort level which gives the following (See Proof 5 in Appendix)

$$e_1^* = \frac{1}{4\theta_1\theta_2 - \gamma^2}\left[\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)} - \frac{(4\theta_1\theta_2 - \gamma^2)q_1(8\theta_1\theta_2 - \gamma^2)}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)} + \frac{3\gamma\theta_2(4\theta_1\theta_2 - \gamma^2)q_2}{(16\theta_1\theta_2 - \gamma^2)g_2(1-q_2)}\right]$$

$$e_2^* = \frac{1}{4\theta_1\theta_2 - \gamma^2}\left\{\frac{[2\theta_1(8\theta_1\theta_2 f_2 - \gamma^2 f_2 - 2\theta_2\gamma f_1)]}{(16\theta_1\theta_2 - \gamma^2)} - \frac{(4\theta_1\theta_2 - \gamma^2)q_2(8\theta_1\theta_2 - \gamma^2)}{(16\theta_1\theta_2 - \gamma^2)g_2(1-q_2)} + \frac{3\gamma\theta_2(4\theta_1\theta_2 - \gamma^2)q_2}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)}\right\}$$

Notice that there are two effects on the effort level, namely the effect of limited liabilities and the effect from the externalities between two principals. The effect of limited liabilities indicates that the efficient effort level is larger than the second optimal effort level since the agent knows that principals cannot introduce negative bonuses on them. However, since the principals are bidding on a bonus and then giving high bonuses to the agent. The agent can increase the effort on the corresponding task. This makes the net effect on effort unclear. In this situation, it is interesting to compare the second best and first best effort levels and then determine which effect will be larger because when q_1, q_2 become larger, the effect on limited liabilities becomes larger. In the meantime, increasing γ indicates greater effect of externalities. Therefore, denote the first best option as e_1^{FB} and second-best option as e_1^*, the following inequality will be checked to compare these two effort levels.

$$e_1^* \leq e_1^{FB}$$

Then substituting the corresponding effort level, it gives

$$\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{2\theta_2(8\theta_1\theta_2 f_1-\gamma^2 f_1-2\theta_1\gamma f_2)}{(16\theta_1\theta_2-\gamma^2)}-\frac{(4\theta_1\theta_2-\gamma^2)q_1(8\theta_1\theta_2-\gamma^2)}{(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)}+\frac{3\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-\gamma^2)g_2(1-q_2)}\right] \leq \frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2} \quad (1)$$

Then it gives this inequality

$$2\theta_2(8\theta_1\theta_2 f_1-\gamma^2 f_1-2\theta_1\gamma f_2)]g_1(1-q_1)g_2(1-q_2)-(4\theta_1\theta_2-\gamma^2)[(8\theta_1\theta_2-\gamma^2)q_1 g_2(1-q_2)-3\gamma\theta_2 q_2 g_1(1-q_1)] \leq (2f_1\theta_2-\gamma f_2)(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)g_2(1-q_2) \quad (2)$$

Then

$$\frac{2\theta_2(8\theta_1\theta_2 f_1-\gamma^2 f_1-2\theta_1\gamma f_2)-(2f_1\theta_2-\gamma f_2)(16\theta_1\theta_2-\gamma^2)}{(8\theta_1\theta_2-\gamma^2)q_1 g_2(1-q_2)-3\gamma\theta_2 q_2 g_1(1-q_1)} \leq \frac{(4\theta_1\theta_2-\gamma^2)}{g_1(1-q_1)g_2(1-q_2)}$$

When $q_1, q_2 \in (0,1)$ and $\gamma \in (0,+\infty)$ and if this inequality holds, then the effect on limited liabilities is larger than the externalities and when both sides are equal, the first best effort level can be achieved. Otherwise, the externalities have larger effects on limited liabilities.

From the inequality of the effort level, if $q_1 \to 1$, then the effort level $e_1 \to 0$ and when $q_1 = 1$ then $e_1 = 0$. This can be explained that in this situation, the performance measurements are not enacted by the principal for any e_1, which infers that the agent has an incentive to exert no effort and the principal offer zero bonus for that. When setting $\gamma = 0$, the inequality becomes

$$\frac{1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)} \leq \frac{1}{2\theta_1}$$

This indicates that the condition $e_1^* < e^{FB}$ always holds which also means the inequality can be rewritten as $\frac{1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)} < \frac{1}{2\theta_1}$. It also infers that the limited liabilities lead to lower effort. If $q_1 = q_2 = 0$, then the effort level becomes $e_1 = \frac{1}{4\theta_1\theta_2-\gamma^2}\frac{2\theta_2(8\theta_1\theta_2 f_1-\gamma^2 f_1-2\theta_1\gamma f_2)}{(16\theta_1\theta_2-\gamma^2)}$. Compare this with the first best effort level gives

$$\frac{2\theta_2(8\theta_1\theta_2 f_1-\gamma^2 f_1-2\theta_1\gamma f_2)}{(16\theta_1\theta_2-\gamma^2)} \leq 2f_1\theta_2-\gamma f_2$$

Reorganize the inequality

$$2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2) \le (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)$$

Simplify this expression gives

$$\gamma^3 - 12\theta_1\theta_2\gamma f_2 + 16\theta_1\theta_2^2 f_1 \ge 0$$

However, here it is still hard to determine the effect of the externalities from this inequality directly. In another words, it is hard to see the effect directly from the inequality of the externalities on the first best effort and the second-best option since the second-best option can be larger or smaller than the first best effort level. In this situation, recall the inequality from expression (2)

$$\begin{aligned} 2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)]g_1(1-q_1)g_2(1-q_2) - (4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) \\ - 3\gamma\theta_2 q_2 g_1(1-q_1)] \le (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2) \end{aligned}$$

And then set the following functions

$$\begin{aligned} F(q_1, q_2, \gamma) = 2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)]g_1(1-q_1)g_2(1-q_2) \\ - (4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)] \end{aligned}$$

And

$$G(q_1, q_2, \gamma) = (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)$$

Therefore, the function $F(q_1, q_2, \gamma)$ is derived from the second-best effort level and the function $G(q_1, q_2, \gamma)$ is derived from the first best option. In this case, it is intuitive to see the marginal effect of the externalities by setting both $q_1, q_2 = 0$ and then taking the derivative in terms of γ for both two functions $F(q_1, q_2, \gamma)$ and $G(q_1, q_2, \gamma)$. The marginal effect of limited liabilities on first best and second-best effort can also be determined by using the same way. It is also beneficial to see the change from the functions dynamically when increasing q_1, q_2, γ and formally access the effect of limited liabilities and externalities on both first best and second-best option. Therefore, take the following steps (See Proof 6A in Appendix):

$$\begin{cases} F'_{q_1}(q_1, 0, 0) = -16\theta_1\theta_2^2 g_1 g_2 f_1 - 32\theta_1^2\theta_2^2 g_2 \ (Set\ q_2, \gamma = 0) \\ F'_{q2}(0, q_2, 0) = -16\theta_1\theta_2^2 g_1 g_2 f_1 \ (Set\ q_1, \gamma = 0) \\ F'_{\gamma}(0, 0, \gamma) = [-4\theta_2 f_1\gamma - 2\theta_1\theta_2 f_2]g_1 g_2 \ \ (Set\ q_1, q_2 = 0) \end{cases}$$

And

$$\begin{cases} G'_{q_1}(q_1, 0, 0) = -32 g_1 g_2 \theta_1\theta_2^2 f_1 \ (Set\ q_2, \gamma = 0) \\ G'_{q2}(0, q_2, 0) = -32 g_1 g_2 \theta_1\theta_2^2 f_1 \ (Set\ q_1, \gamma = 0) \\ G'_{\gamma}(0, 0, \gamma) = g_1 g_2(-16\theta_1\theta_2 f_2 + 3f_2\gamma^2 - 4\gamma f_1\theta_2) \ (Set\ q_1, q_2 = 0) \end{cases}$$

According to these results, it can be revealed that the marginal effect of q_1, q_2 is negative. Hence, when q_1, q_2 increase, the effort level will decrease which means it has a stronger effect on limited liabilities and when $q_1 = q_2 = 1$, the effort level will be zero. The marginal effect of externalities is affected by the production, performance measurement and marginal cost of effort from both tasks and it is smaller than zero, since $f_1, f_2, \theta_1, \theta_2, \gamma, g_1, g_2$ are all bigger than zero. For the function F, it has the decreasing marginal effect on externalities and $F_\gamma'(0,0,\gamma)$ is a decreasing linear function. For function G, recall that in Section 5.1., the first best effort level should be larger than zero which means that the expressions $\frac{f_1}{f_2} \geq \frac{\gamma}{2\theta_2}$ and $\frac{f_2}{f_1} \geq \frac{\gamma}{2\theta_1}$ needs to be satisfied. Therefore, Function G is also a decreasing function in $\gamma \in (0, \frac{2\theta_2 f_1}{f_2})$ since $G_\gamma'\left(0,0,\frac{2\theta_2 f_1}{f_2}\right)$ is smaller than zero.

It is also essential to introduce the symmetric case, which means $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$. In this situation, compare the second best and the first best option gives the following by rewriting the expression (1). (See Proof 6B in Appendix):

$$\frac{1}{4\theta_1^2 - \gamma^2}\left\{\frac{[2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1)]}{(16\theta_1^2 - \gamma^2)} - \left[\frac{(4\theta_1^2 - \gamma^2)[(8\theta_1^2 - \gamma^2)q_1 g_1(1-q_1) - 3\gamma\theta_1 q_1 g_1(1-q_1)]}{(16\theta_1^2 - \gamma^2)g_1(1-q_1)g_1(1-q_1)}\right]\right\} \leq \frac{2f_1\theta_1 - \gamma f_1}{4\theta_1^2 - \gamma^2}$$

This gives the following inequality:

$$\frac{[2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1)]}{(16\theta_1^2 - \gamma^2)} - \frac{q_1[(4\theta_1^2 - \gamma^2)(8\theta_1^2 - \gamma^2) - 3\gamma\theta_1]}{(16\theta_1^2 - \gamma^2)g_1(1-q_1)} \leq 2f_1\theta_1 - \gamma f_1 \qquad (3)$$

Overall, if this inequality holds, it implies that the effect on limited liabilities is larger than the externalities between principals and when this ratio is equal to one, the first best effort level can be achieved. Otherwise, the effect on limited liabilities is smaller than the externalities. Similarly, when $q_1 = 1$, then the agent will not exert any efforts and the effect on limited liabilities will also be stronger when q_1 increases. Next the effect on externalities will be checked. In this case, set $q_1 = 0$ and the inequality of expression (3) becomes

$$2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1) \leq (2f_1\theta_1 - \gamma f_1)(16\theta_1^2 - \gamma^2)$$

Simplify this expression gives

$$\gamma^3 - 12\theta_1^2\gamma + 16\theta_1^3 \geq 0$$

And set the function

$$H(\gamma) = \gamma^3 - 12\theta_1^2\gamma + 16\theta_1^3$$

Take the derivative in terms of γ gives

$$H'(\gamma) = 3\gamma^2 - 12\theta_1^2$$

Recall that in the denominator $4\theta_1^2 - \gamma^2$and $16\theta_1^2 - \gamma^2$ in the effort function should not be zero. This means $\gamma \neq 2\theta_1$ and $\gamma \neq 4\theta_1$. The function $H(\gamma)$ implies the difference of the effort level between second-best option and the first best option. Recall that the following condition mentioned before $\frac{\gamma g_2(1-q_2)}{2\theta_2 g_1(1-q_1)} \leq 1$ also must be satisfied since the effort level needs to be larger than zero (See the part Situation for Principals). Since it is the symmetric case, this becomes $\frac{\gamma}{2\theta_1} \leq 1$ which implies

$$\gamma \leq 2\theta_1$$

This indicates that the domain for both functions $H(\gamma)$, $H'(\gamma)$ are $\gamma \in (0, 2\theta_1)$. This also means that $H'(\gamma) < 0 \; \forall \, \gamma \in (0, 2\theta_1)$ which can be interpreted that the marginal effect of externalities is always negative for the effort level and when $\gamma = 2\theta_1$, $H'(\gamma) = 0$. Therefore, the inequality $\gamma^3 - 12\theta_1^2\gamma + 16\theta_1^3 \geq 0$ always hold when $\gamma \in (0, 2\theta_1)$. This means that in the symmetric case, the agent cannot achieve the first best effort.

These obtained results can have the following potential explanations. The variable γ indicates the externalities, not only between the principals, but also between two tasks. Therefore, when the externalities stay the same, if one of the principals increases the bonus, the agent will exert more effort on that task and decreases the effort from another task. However, In the cost function, increasing externalities γ will increase the cost of effort and reduces the effort of the agent since externalities also exist between two tasks. Since the second-best option is always smaller than the first best effort level, this infers the effect of externalities between two tasks influence the effort level more than the externalities between the principals. Therefore, even if principals can try to give higher bonuses in order to make the agent exert more effort on the corresponding task, the agent still cannot achieve the first best effort level.

In short, these characteristics indicate that whether the agent can achieve the first best option effort level is dependent on the marginal effort of productivity of both tasks, the productivity, the performance measurements, and the externalities. When it is the symmetric case, it is dependent on the probability of the principal not enforcing the performance measurement, marginal effort of productivity, the marginal cost of effort, and the externalities. Limited liabilities affect negatively on effort level, and it makes the second-best option smaller than the first best effort level. The externalities between the principals might potentially increase the efforts of the agent since both principals do not consider that increasing bonuses reduces the effort on the other principal's task. However, the marginal externalities are also negative. This infers that the externalities between tasks have a larger effect on the externalities on effort level than the externalities between principals. Since there are externalities behind this process, considering the increase of the effort from the agent in the corresponding task, the principals can potentially harm their profit when they are paying a huge amount of bonus to the agent.

Alternatively, if $\gamma = 0$ for all the cases, then it means that there are no externalities between two principals and they do not bid for the bonuses, then the effort levels become the following (See Proof 6C in Appendix):

First best effort level

$$e_1^{FB} = \frac{f_1}{2\theta_1}$$

$$e_2^{FB} = \frac{f_2}{2\theta_2}$$

The second-best option effort

$$e_1 = \frac{(1-q_1)b_1 g_1}{2\theta_1}$$

$$e_2 = \frac{(1-q_2)b_2 g_2}{2\theta_2}$$

And then

$$e_1 = \frac{f_1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)}$$

$$e_2 = \frac{f_2}{4\theta_2} - \frac{q_2}{2g_2(1-q_2)}$$

Bonus level from both principals

$$b_1 = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$$

$$b_2 = \frac{(1-q_2)g_2 f_2 - 2\theta_2 q_2}{2g_2^2(1-q_2)^2}$$

In this situation, the agent will exert more effort if the effort cost on that corresponding task is lower than another task. For the first best option, if the productivity on one task is higher, the agent will have a higher effort level on that task. This also holds for the second-best effort level. The bonuses given by each principal are also important in the second-best option and higher bonuses yield a higher amount of effort. However, in this case, since no externalities exist, the effort level on another task will not change because both principals do not have bidding power for the bonuses. For both principals, their bonus level is also dependent on the productivity of their tasks and the cost of effort, if the productivity is higher and the effort cost is lower for the task, they will give higher bonuses. The bonus setting from one principal will not influence another principal's bonus setting. If the case is symmetric, meaning that $\theta_1 = \theta_2$, $g_1 = g_2$, $q_1 = q_2$, $f_1 = f_2$, then it indicates that the agent has the same effort level on both tasks, and both principals set the same bonus level. In another word, $e_1 = e_2 = \frac{f_1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)}$ and $b_1 = b_2 = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$. Limited liabilities still affect effort levels in this case and the agent still cannot achieve the first best option on both tasks.

5.5. A Possibility of Collaboration

There is still a possibility that both principals can cooperate with each other. In this situation both principals internalize problems and function as one principal which means that both principals' interests, no matter actively or passively, are now aligned. It also means that the principal now maximizes bonuses b_1, b_2 on the following function

$$E\pi = f_1 e_1 + f_2 e_2 - b_1 p_1 - b_2 p_2$$

Therefore, if $\gamma = 0$, this gives the first best effort (See Proof 7A in Appendix):

$$e_1^{FB} = \frac{f_1}{2\theta_1}$$

$$e_2^{FB} = \frac{f_2}{2\theta_2}$$

Second best effort

$$e_1^* = \frac{(1-q_1)b_1 g_1}{2\theta_1}$$

$$e_2^* = \frac{(1-q_2)b_2 g_2}{2\theta_2}$$

And then

$$e_1^* = \frac{f_1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)}$$

$$e_2^* = \frac{f_2}{4\theta_2} - \frac{q_2}{2g_2(1-q_2)}$$

And the bonus level for both tasks are

$$b_1 = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$$

$$b_2 = \frac{(1-q_2)g_2 f_2 - 2\theta_2 q_2}{2g_2^2(1-q_2)^2}$$

And if $\theta_1 = \theta_2$, $g_1 = g_2$, $q_1 = q_2$, $f_1 = f_2$, then $e_1 = e_2 = \frac{f_1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)}$ and $b_1 = b_2 = \frac{(1-q_1)g_1f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$

This gives the same result as previous section where there are no externalities between the principals without collaboration. In this situation, it is indifferent for both principal to either chooses to collaborate or not collaborate. If both principals choose to collaborate, then this problem will reduce to a standard multi-tasking principal-agent problem.

If $\gamma > 0$, this infers that both principals need to negotiate first for setting the bonus level of each task, since they cannot pay for the other task for which they are not in charging after setting the bonus level. After the negotiation of bonus levels, they will act like as if there were only one principal. Assuming the transaction cost is negligible, then this gives the bonus level of both tasks after the negotiation (See Proof 7B in Appendix)

$$b_1^C = \frac{g_2(1-q_2)}{g_1(1-q_1)}\left\{\frac{(1-q_1)(1-q_2)g_1g_2 4\theta_1(2\theta_2 f_1 - \gamma f_2 + 4\gamma\theta_1 f_2 - 2\gamma^2 f_1)}{(16\theta_1\theta_2 - 4\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2} - \frac{(4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + 2\gamma(1-q_1)g_1q_2]}{(16\theta_1\theta_2 - 4\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}\right\}$$

And

$$b_2^C = \frac{g_1(1-q_1)}{g_2(1-q_2)}\left\{\frac{(1-q_1)(1-q_2)4\theta_2 g_1g_2(2\theta_1 f_2 - \gamma f_1 + 4\gamma\theta_2 f_1 - 2\gamma^2 f_2)}{(16\theta_1\theta_2 - 4\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)} - \frac{(4\theta_1\theta_2 - \gamma^2)[q_2 4\theta_2 g_1(1-q_1) + 2\gamma(1-q_2)g_2q_1]}{(16\theta_1\theta_2 - 4\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}\right\}$$

If $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$, this gives $b_1^C = b_2^C = \frac{4\theta_1(2\theta_1 f_1 - \gamma f_1 + 4\gamma\theta_1 f_1 - 2\gamma^2 f_1)}{(16\theta_1^2 - 4\gamma^2)g_1(1-q_1)} - \frac{(4\theta_1^2 - \gamma^2)(q_1 4\theta_1 + 2\gamma q_1)}{(16\theta_1^2 - 4\gamma^2)g_1^2(1-q_1)^2}$

Compare this bonus level with non-collaborative bonus level gives

$$b_1^* \leq b_1^C$$

Which can be written as

$$\frac{b_1^*}{b_1^C} \leq 1$$

This gives the following inequality and when $\gamma = 0$, then both sides are equal to one. If the inequality holds, the collaborative bonus level is bigger than the non-collaborative bonus level.

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2) - (4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)g_1g_2 4\theta_1(2\theta_2 f_1 - \gamma f_2 + 4\gamma\theta_1 f_2 - 2\gamma^2 f_1) - (4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + 2\gamma(1-q_1)g_1q_2]} \leq \frac{16\theta_1\theta_2 - \gamma^2}{16\theta_1\theta_2 - 4\gamma^2}$$

If the case is symmetric, then inequality becomes

$$\frac{\theta_1^2 f_1(8+2\gamma)g_1(1-q_1)-(4\theta_1^2-\gamma^2)q_1[4\theta_1+\gamma]}{4\theta_1(2\theta_1 f_1-\gamma f_1+4\gamma\theta_1 f_1-2\gamma^2 f_1)g_1(1-q_1)-(4\theta_1^2-\gamma^2)(q_1 4\theta_1+2\gamma q_1)}\leq\frac{16\theta_1^2-\gamma^2}{16\theta_1^2-4\gamma^2}$$

Following the given bonus in the collaboration setting, the agent will exert the following effort (See Proof 7C-I in Appendix)

$$e_1^C=\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{8\gamma^3\theta_2 f_2-(16\theta_1\theta_2+16\theta_2^2-4\theta_2)f_1\gamma^2+20\theta_1\theta_2 f_2\gamma+16\theta_1\theta_2^2 f_1}{16\theta_1\theta_2-4\gamma^2}-\frac{(4\theta_1\theta_2-\gamma^2)q_1(8\theta_1\theta_2-2\gamma^2)}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}\right]$$

$$e_2^C=\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{8\gamma^3\theta_1 f_1-(16\theta_1\theta_2+16\theta_1^2-4\theta_1)f_2\gamma^2+20\theta_1\theta_2 f_1\gamma+16\theta_1^2\theta_2 f_2}{16\theta_1\theta_2-4\gamma^2}-\frac{(4\theta_1\theta_2-\gamma^2)q_2(8\theta_1\theta_2-2\gamma^2)}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}\right]$$

From the previous sections, the marginal effect of limited liabilities is always negative. Meaning that if $\gamma=0$, then it is not possible to achieve the first best effort and if $q_1,q_2=1$, the effort level will be zero. Hence, it is still interesting to compare the collaborative effort level with the first best effort level and find out the effect of externalities in case of collaboration. Therefore, evaluate $e_1^C\leq e^{FB}$ and set $q_1=0$ gives (See Proof 7C-II in Appendix)

$$\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{8\gamma^3\theta_2 f_2-(16\theta_1\theta_2+16\theta_2^2-4\theta_2)f_1\gamma^2+20\theta_1\theta_2 f_2\gamma+16\theta_1\theta_2^2 f_1}{16\theta_1\theta_2-4\gamma^2}\right]\leq\frac{2f_1\theta_2-\gamma f_2}{4\theta_1\theta_2-\gamma^2}\tag{4}$$

Rewrite the inequality then

$$8\gamma^3\theta_2 f_2-(16\theta_1\theta_2+16\theta_2^2-4\theta_2)f_1\gamma^2+20\theta_1\theta_2 f_2\gamma+16\theta_1\theta_2^2 f_1\leq(2f_1\theta_2-\gamma f_2)(16\theta_1\theta_2-4\gamma^2)$$

Which gives the following inequality

$$(8\theta_2 f_2-4f_2)\gamma^3-(16\theta_1\theta_2+16\theta_2^2-12\theta_2)f_1\gamma^2+36\theta_1\theta_2 f_2\gamma-16\theta_1\theta_2^2 f_1\leq 0$$

When this inequality holds, it indicates that the agent effort level is smaller than the first best level in case of collaboration and then set the function

$$M(\gamma)=(2\theta_2 f_2-f_2)\gamma^3-(4\theta_1\theta_2+4\theta_2^2-3\theta_2)f_1\gamma^2+9\theta_1\theta_2 f_2\gamma-4\theta_1\theta_2^2 f_1$$

In this situation, the equation $M(\gamma)=0$ can potentially have a solution, meaning that in the situation of collaboration, the agent can reach the first best effort. However, in the equation it is still hard to determine whether it has a root or not, therefore, by using the expression (4), consider the symmetric case which

$q_1=q_2$, $g_1=g_2$, $\theta_1=\theta_2$ and $f_1=f_2$ and the effort level becomes (See Proof 7C-III in Appendix)

$$e_1^C = \frac{1}{4\theta_1^2 - \gamma^2}\left[\frac{8\gamma^3\theta_1 f_2 - (16\theta_1^2 + 16\theta_1^2 - 4\theta_1)f_1\gamma^2 + 20\theta_1^2 f_2\gamma + 16\theta_1^3 f_1}{16\theta_1^2 - 4\gamma^2} - \frac{(4\theta_1^2 - \gamma^2)q_1(8\theta_1^2 - 2\gamma^2)}{(16\theta_1\theta_2 - 4\gamma^2)g_1(1 - q_1)}\right]$$

And the function $M(\gamma)$ becomes

$$N(\gamma) = [(2\theta_1 - 1)\gamma^3 - (8\theta_1^2 - 3\theta_1)\gamma^2 + 9\theta_1^2\gamma - 4\theta_1^3]f_1$$

Where the condition $\gamma \in (0, 2\theta_1)$ needs to be satisfied.

Therefore, if $N(\gamma) = 0$ has a root on $(0, 2\theta_1)$ then the condition $N(0)N(2\theta_1) < 0$ must holds. This gives

$$(-4\theta_1^3 f_1)(-16\theta_1^4 f_1 + 18\theta_1^3 f_1) < 0$$

Note that this inequality can achieve in certain condition. In this case, when the equation $N(\gamma) = 0$ has a root on $(0, 2\theta_1)$, it indicates that $\theta_1 \in (0, \frac{8}{9})$ and the following statement is true

$$\forall\, \theta_1 \in \left(0, \frac{8}{9}\right);\ \exists\, \gamma \in (0, 2\theta_1), N(\gamma) = 0$$

In another word, when marginal cost of effort $\theta_1 \in (0, \frac{8}{9})$, the agent can achieve first best option under the situation of both principals collaborate with each other, which is an interesting result in the symmetric case. One of the explanations behind this result is that since both principals negotiate and internalize their problem first even though there are still externalities, they act like one principal and balance incentives on both tasks, given that the transaction cost is negligible. This makes the marginal cost of effort of the agent for each task crucial for achieving the first best effort level, but this marginal cost of effort cannot be too high otherwise the agent will not achieve the first best effort level. Another explanation behind this result is that in collaboration, both principals have little incentive to boost the bonus to increase the effort from the agent since they are satisfied with their bonus level after the negotiation since they have internalized problems. This also means that after both principals have mitigated the externalities and internalized problems, the agent can choose their corresponding effort and potentially achieve the first best effort when both principals give reasonable bonuses. However, in the non-symmetric case, it is still tough to determine whether the effort level of the agent will reach the first best level because it also needs to consider productivities from both tasks.

6. Concluding Takeaways

In short, with the situation of multiple principals. The agent knows that since the two principals' interests are not perfectly aligned which means that the principals start bidding with each other and try to give the agent a higher bonus as possible in their corresponding task to make the agent exert the effort since they only care about their own task. The agent can also play the principals from the effort which means that the agent will decrease the effort if the task has less incentives. This yields inefficiency since the agent will lower the effort when the corresponding principal offers a lower bonus which is harmful. The marginal cost of the productivity for the effort cost of the agent towards the task also plays a role in this situation of determining their desired bonuses. Alternatively, when no externalities exist, the effort level on one task will not affect the effort level on another task. At the same time, both principals cannot bid with each other for the bonus either. They can also try to collaborate with each other to make the agent achieve the first best effort level by negotiating or bargaining between them when the transaction cost is neglectable. Coase, R. H. (2013) has discussed similar situations regarding negotiation or bargaining as well.

This multiple principals' situation can be applied in the crowdsourcing situation which has been mentioned before, when the workers are not only contracted by the single principal and these principals to whom they are contracted do not have perfectly aligned interests, creating moral hazard. It can also be applied to some projects involving different tasks in the process, where the different principals have different interests, and the agent played the principals and exerts less effort in certain tasks which can result in some delays in the project or even some severer consequences and subsequentially the principals get huge losses because of that. For example, in some projects which require processes, each process needs some small tasks. If the different task is managed by different principals and their interests are not aligned, then it can potentially cause the delay of the project since there is a bidding game between the principals and the agent can exploit this game with the exerted efforts, which yields inefficiency. Another application of these multiple principal situations is in the education field. When people consider different schoolteachers in different subjects as the principal and the student as the agent. In this situation, the externalities still exist, and each subject teacher only cares about the students' grades own their own subject. The stronger the motivation of the subject from the students and the potential reward from that subject, the students can potentially study harder in that subject which they have higher motivations and rewards rather than other subjects they should be studied to improve their grades. This also needs to be noted as a common agency problem like others.

The assumptions of limited liabilities and both principals and agent are important in this paper. Therefore, in terms of risk perception, if all of them are not risk neutral, or both two principals and the agent have different risk perception, the result might change. In the case of limited liabilities, both principals cannot give negative bonuses to the agent. However, in a general form, principals can impose a fine when the performance level of the agent is low. This can also influence the effort choice of agents since the agent will fear about the punishment. Another limitation that needs to be highlighted is whether both principals are going to monitor the effort from the agent. If the principal decides to implement effort monitoring in the multiple principals setting, there is a possibility that through the action from the agent, one principal can get some information of other principals when monitoring. This might subsequentially influence the decision of both principals and the agent. In the collaborative case, this paper assumes that the transaction cost is negligible. When the transaction cost is not negligible, the result will be different because if these costs are high, both principals will not collaborate since the high transaction cost will decrease their profit massively. It is also interesting to check the effect of multiple principals empirically so that behaviors of principals and agents can be assessed accordingly, both non-collaborative and collaborative case.

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8. Appendix

Proof 1 – First Best Option

A - Agent

In this situation, principal can directly contract the effort. This gives the following optimization

$$\max \sum_i y_i + EU_A$$

Then it yields:

$$b_i p_i = EU_A + \theta_1 e_1^2 + \theta_2 e_2^2 + \gamma e_1 e_2$$

Substitute this into principals function we have

$$\sum_i y_i + EU_A = f_1 e_1 + f_2 e_2 - \theta_1 e_1^2 - \theta_2 e_2^2 - \gamma e_1 e_2$$

Took first order condition equal to zero we have

$$\frac{\partial E\pi_1}{\partial e_1} = f_1 - 2\theta_1 e_1 - \gamma e_2 = 0$$

Which gives

$$e_1^{FB} = \frac{f_1 - \gamma e_2}{2\theta_1}$$

Similarly,

$$e_2^{FB} = \frac{f_2 - \gamma e_1}{2\theta_2}$$

Substitute which gives

$$e_1^{FB} = \frac{2f_1\theta_2}{4\theta_1\theta_2} - \frac{\gamma(f_2 - \gamma e_1)}{4\theta_1\theta_2}$$

Then

$$e_1 = \frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}$$

Similarly, for the second effort

$$e_2 = \frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2}$$

e_1, e_2 should be bigger than zero, hence

$$\frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2} \geq 0$$

And

$$\frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2} \geq 0$$

This gives the following expression for task one:

$$\frac{f_1}{f_2} \geq \frac{\gamma}{2\theta_2}$$

And for task two

$$\frac{f_2}{f_1} \geq \frac{\gamma}{2\theta_1}$$

If $f_1 = f_2$ and $\theta_1 = \theta_2$ then for both tasks

$$\gamma \leq 2\theta_1$$

It can also be said if $e_1 \geq e_2$ then

$$2f_1\theta_2 - \gamma f_2 \geq 2f_2\theta_1 - \gamma f_1$$

Which gives

$$\gamma \leq \frac{2(f_1\theta_2 - f_2\theta_1)}{f_2 - f_1}$$

B - Principals

For both principals, since it is complete information, therefore no performance measurement will be enforced which means that $q_i = 1$, and $p_i = 1$ for any e_i and the bonus will be irrelevant.

Therefore, each principal will pay the agent the wage based on agent's level of effort. At the same time, the term $\gamma e_1 e_2$ will be separated the fraction of different tasks from principals. If the agent performs the first best optimal effort, then the agent gets the total wage according to that effort, which will be

$$b_1 = \theta_1 e_1^2 + \frac{\gamma e_1^2 e_2}{e_1 + e_2}$$

$$b_2 = \theta_2 e_2^2 + \frac{\gamma e_1 e_2^2}{e_1 + e_2}$$

Then substitute the effort level of task one gives,

$$b_1 = \theta_1 \left(\frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}\right)^2 + \frac{\gamma \left(\frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}\right)^2 \left(\frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2}\right)}{\left(\frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}\right) + \left(\frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2}\right)}$$

Similarly,

$$b_2 = \theta_2 \left(\frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2}\right)^2 + \frac{\gamma \left(\frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2}\right)^2 \left(\frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}\right)}{\left(\frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}\right) + \left(\frac{2f_2\theta_1 - \gamma f_1}{4\theta_1\theta_2 - \gamma^2}\right)}$$

If $b_1 \geq b_2$, then

$$\theta_1 e_1^2 + \frac{\gamma e_1^2 e_2}{e_1 + e_2} \geq \theta_2 e_2^2 + \frac{\gamma e_1 e_2^2}{e_1 + e_2}$$

Which translates to

$$\frac{\theta_1 e_1^2 (e_1 + e_2) + \gamma e_1^2 e_2}{\theta_2 e_2^2 (e_1 + e_2) + \gamma e_1 e_2^2} \geq 1$$

Rewrite this gives

$$\frac{e_1^2}{e_2^2} \geq \frac{\theta_2 (e_1 + e_2) + \gamma e_1}{\theta_1 (e_1 + e_2) + \gamma e_2}$$

Substitute e_1, e_2

$$\frac{(2f_1\theta_2 - \gamma f_2)^2}{(2f_2\theta_1 - \gamma f_1)^2} \geq \frac{\theta_2 (2f_2\theta_1 - \gamma f_1 + 2f_1\theta_2 - \gamma f_2) + \gamma (2f_1\theta_2 - \gamma f_2)}{\theta_1 (2f_2\theta_1 - \gamma f_1 + 2f_1\theta_2 - \gamma f_2) + \gamma (2f_2\theta_1 - \gamma f_1)}$$

If $f_1 = f_2$ and $\theta_1 = \theta_2$ then $b_1 = b_2 = \left(\theta_1 + \frac{\gamma}{2}\right)\left[\frac{f_1(2\theta_1 - \gamma)}{4\theta_1^2 - \gamma^2}\right]^2$

Proof 2 – Second Best Option: Agent's effort

We have:

$$y_i = f_i e_i$$

$$p_i \in \{0,1\}$$

If $q_i \geq 0$ then $p_i = 1 \ \forall \ e_i$

Then $1 - q_i$ -> $p_i = 1$ with the probability $g_i e_i$

Set minimum wage = 0

Therefore, the utility function of the agent:

$$EU_A = (1 - q_1) b_1 p_1 + (1 - q_2) b_2 p_2 + q_1 b_1 + q_2 b_2 - \theta_1 e_1^2 - \theta_2 e_2^2 - \gamma e_1 e_2$$

First order condition for effort one

$$(1 - q_1) b_1 g_1 - 2\theta_1 e_1 - \gamma e_2 = 0$$

Optimal effort one:

$$e_1^* = \frac{(1 - q_1) b_1 g_1 - \gamma e_2}{2\theta_1}$$

Using the same method, we have:

$$e_2^* = \frac{(1 - q_2) b_2 g_2 - \gamma e_1}{2\theta_2}$$

Substitute effort of task two we have:

$$e_1 = \frac{(1 - q_1) b_1 g_1}{2\theta_1} - \frac{\gamma b_2 g_2 (1 - q_2) - \gamma^2 e_1}{4\theta_1 \theta_2}$$

$$= \frac{2\theta_2(1-q_1)b_1g_1 - [\gamma b_2g_2(1-q_2) - \gamma^2 e_1]}{4\theta_1\theta_2}$$

Therefore, we have

$$4\theta_1\theta_2 e_1 - \gamma^2 e_1 = (1-q_1)(2\theta_2 b_1 g_1) - (\gamma b_2 g_2)(1-q_2)$$

$$e_1 = \frac{(1-q_1)2\theta_2 b_1 g_1 - \gamma b_2 g_2 (1-q_2)}{4\theta_1\theta_2 - \gamma^2}$$

While

$$4\theta_1\theta_2 > \gamma^2$$

Same we also have for the effort in second task

$$e_2 = \frac{(1-q_2)2\theta_1 b_2 g_2 - \gamma b_1 g_1 (1-q_1)}{4\theta_1\theta_2 - \gamma^2}$$

For the effort for task one, this situation needs to be satisfied:

$$(1-q_1)2\theta_2 b_1 g_1 \geq \gamma b_2 g_2 (1-q_2)$$

Which means the asking bonus for the agent in task one will be:

$$b_1 \geq \frac{\gamma b_2 g_2 (1-q_2)}{2\theta_2 g_1 (1-q_1)}$$

Which can also be written

$$\frac{b_1}{b_2} \geq \frac{\gamma g_2 (1-q_2)}{2\theta_2 g_1 (1-q_1)}$$

Same thing applies to the effort for task two

$$(1-q_2)2\theta_1 b_2 g_2 \geq \gamma b_1 g_1 (1-q_1)$$

And the asking bonus for the effort for task two

$$b_2 \geq \frac{\gamma b_1 g_1 (1-q_1)}{2\theta_1 g_2 (1-q_2)}$$

Which can also be written

$$\frac{b_2}{b_1} \geq \frac{\gamma g_1 (1-q_1)}{2\theta_1 g_2 (1-q_2)}$$

In the form of the externalities for each task respectively:

$$\gamma \leq \frac{(1-q_1)2\theta_2 b_1 g_1}{b_2 g_2 (1-q_2)}$$

And

$$\gamma \leq \frac{(1-q_2)2\theta_1 b_2 g_2}{b_1 g_1 (1-q_1)}$$

Proof 3 – Situation for Principals

For the first principal we have:

$$E\pi_1 = f_1 e_1 - b_1 p_1$$

Then:

$$E\pi_1 = f_1\left(\frac{(1-q_1)2\theta_2 b_1 g_1 - \gamma b_2 g_2(1-q_2)}{4\theta_1\theta_2 - \gamma^2}\right) - [b_1 q_1 + b_1(1-q_1)g_1\frac{(1-q_1)2\theta_2 b_1 g_1 - \gamma b_2 g_2(1-q_2)}{4\theta_1\theta_2 - \gamma^2}]$$

And:

$$\frac{\partial E\pi_1}{\partial b_1} = \frac{(1-q_1)2\theta_2 g_1 f_1}{4\theta_1\theta_2 - \gamma^2} - [q_1 + \frac{4(1-q_1)^2 g_1^2\theta_2 b_1 - \gamma b_2 g_1 g_2(1-q_2)(1-q_1)}{4\theta_1\theta_2 - \gamma^2}]$$

Set the first order condition to zero we have

$$\frac{(1-q_1)2\theta_2 g_1 f_1}{4\theta_1\theta_2 - \gamma^2} = q_1 + \frac{4b_1\theta_2 g_1^2(1-q_1)^2 - \gamma(1-q_1)(1-q_2)b_2 g_1 g_2}{4\theta_1\theta_2 - \gamma^2}$$

Then ->

$$(1-q_1)2\theta_2 g_1 f_1 = (4\theta_1\theta_2 - \gamma^2)q_1 + 4b_1\theta_2 g_1^2(1-q_1)^2 - \gamma(1-q_1)(1-q_2)b_2 g_1 g_2$$

Which gives

$$b_1^* = \frac{(1-q_1)2\theta_2 g_1 f_1 - (4\theta_1\theta_2 - \gamma^2)q_1 + \gamma(1-q_1)(1-q_2)b_2 g_1 g_2}{4\theta_2 g_1^2(1-q_1)^2}$$

While

$$4\theta_2 g_1^2(1-q_1)^2 > 0$$

Using the same method, it can be illustrated

$$b_2^* = \frac{(1-q_2)2\theta_1 g_2 f_2 - (4\theta_1\theta_2 - \gamma^2)q_2 + \gamma(1-q_1)(1-q_2)b_1 g_1 g_2}{4\theta_1 g_2^2(1-q_2)^2}$$

While

$$4\theta_1 g_2^2(1-q_2)^2 > 0$$

Substitute the bonus level of principal two which gives

$$b_1 = \frac{(1-q_1)2\theta_2 g_1 f_1}{4\theta_2 g_1^2(1-q_1)^2} - \frac{(4\theta_1\theta_2 - \gamma^2)q_1}{4\theta_2 g_1^2(1-q_1)^2} + \frac{\gamma(1-q_1)(1-q_2)g_1 g_2[(1-q_2)2\theta_1 g_2 f_2 - (4\theta_1\theta_2 - \gamma^2)q_2 + \gamma(1-q_1)(1-q_2)b_1 g_1 g_2]}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2}$$

Then:

$$b_1 = \frac{[(1-q_1)2\theta_2 g_1 f_1 - (4\theta_1\theta_2 - \gamma^2)q_1] \times 4\theta_1 g_2^2 (1-q_2)^2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} + \frac{2\theta_1\gamma(1-q_1)(1-q_2)^2 g_1 g_2^2 f_2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} - \frac{\gamma(1-q_1)(1-q_2)g_1 g_2 \times (4\theta_1\theta_2 - \gamma^2)q_2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} + \frac{\gamma^2(1-q_1)^2(1-q_2)^2 b_1 g_1^2 g_2^2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2}$$

Which has

$$\left(\frac{16\theta_1\theta_2 - \gamma^2}{16\theta_1\theta_2}\right) b_1 = \frac{[(1-q_1)2\theta_2 g_1 f_1 - (4\theta_1\theta_2 - \gamma^2)q_1] \times 4\theta_1 g_2^2 (1-q_2)^2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} + \frac{2\theta_1\gamma(1-q_1)(1-q_2)^2 g_1 g_2^2 f_2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} - \frac{\gamma(1-q_1)(1-q_2)g_1 g_2 \times (4\theta_1\theta_2 - \gamma^2)q_2}{16\theta_1\theta_2 g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2}$$

Next

$$b_1 = \frac{(1-q_1)(1-q_2)^2 8\theta_1\theta_2 g_1 g_2^2 f_1}{(16\theta_1\theta_2 - \gamma^2) g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} - \frac{(4\theta_1\theta_2 - \gamma^2)q_1 \times 4\theta_1 g_2^2 (1-q_2)^2}{(16\theta_1\theta_2 - \gamma^2) g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} + \frac{2\theta_1\gamma(1-q_1)(1-q_2)^2 g_1 g_2^2 f_2}{(16\theta_1\theta_2 - \gamma^2) g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2} - \frac{\gamma(1-q_1)(1-q_2)g_1 g_2 \times (4\theta_1\theta_2 - \gamma^2)q_2}{(16\theta_1\theta_2 - \gamma^2) g_1^2 g_2^2 (1-q_1)^2(1-q_2)^2}$$

Then:

$$b_1^* = \frac{g_2(1-q_2)}{g_1(1-q_1)}\{\frac{(1-q_1)(1-q_2)g_1 g_2 \theta_1(8\theta_2 f_1 + 2\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2) g_1 g_2^2 (1-q_1)(1-q_2)^2} - \frac{(4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1 q_2]}{(16\theta_1\theta_2 - \gamma^2) g_1 g_2^2 (1-q_1)(1-q_2)^2}\}$$

By using the same way, the bonus level of principal two

$$b_2^* = \frac{g_1(1-q_1)}{g_2(1-q_2)}\{\frac{(1-q_1)(1-q_2)\theta_2 g_1 g_2(8\theta_1 f_2 + 2\gamma f_1)}{(16\theta_1\theta_2 - \gamma^2) g_1^2 g_2 (1-q_1)^2(1-q_2)} - \frac{(4\theta_1\theta_2 - \gamma^2)[q_2 4\theta_2 g_1(1-q_1) + \gamma(1-q_2)g_2 q_1}{(16\theta_1\theta_2 - \gamma^2) g_1^2 g_2 (1-q_1)^2(1-q_2)}\}$$

Since the agent is protected by the limited liability, therefore $b_i \geq 0$

Which gives:

$$(1-q_1)(1-q_2)g_1 g_2 \theta_1(8\theta_2 f_1 + 2\gamma f_2) \geq (4\theta_1\theta_2 - \gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1 q_2]$$

$$(1-q_1)(1-q_2)\theta_2 g_1 g_2(8\theta_1 f_2 + 2\gamma f_1) \geq (4\theta_1\theta_2 - \gamma^2)[q_2 4\theta_2 g_1(1-q_1) + \gamma(1-q_2)g_2 q_1]$$

If it is symmetric, meaning that $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$, it will become

$$b_1^* = b_2^* = \frac{\theta_1^2 f_1(8 + 2\gamma)}{(16\theta_1^2 - \gamma^2) g_1(1-q_1)} - \frac{(4\theta_1^2 - \gamma^2)q_1[4\theta_1 + \gamma]}{(16\theta_1^2 - \gamma^2) g_1^2 (1-q_1)^2}$$

Proof 4 – Bidding principals

This situation, $b_1 \geq b_2$ and $\frac{b_1}{b_2} \geq 1$

$$\frac{\{\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2f_1+2\gamma f_2)}{(16\theta_1\theta_2-\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}-\frac{(4\theta_1\theta_2-\gamma^2)[q_14\theta_1g_2(1-q_2)+\gamma(1-q_1)g_1q_2]}{(16\theta_1\theta_2-\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}\}}{\{\frac{(1-q_1)(1-q_2)\theta_2g_1g_2(8\theta_1f_2+2\gamma f_1)}{(16\theta_1\theta_2-\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}-\frac{(4\theta_1\theta_2-\gamma^2)[q_24\theta_2g_1(1-q_1)+\gamma(1-q_2)g_2q_1]}{(16\theta_1\theta_2-\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}\}}\geq 1$$

Which gives

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2f_1+2\gamma f_2)-(4\theta_1\theta_2-\gamma^2)[q_14\theta_1g_2(1-q_2)+\gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)\theta_2g_1g_2(8\theta_1f_2+2\gamma f_1)-(4\theta_1\theta_2-\gamma^2)[q_24\theta_2g_1(1-q_1)+\gamma(1-q_2)g_2q_1]}\times\frac{(16\theta_1\theta_2-\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}{(16\theta_1\theta_2-\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}\geq 1$$

This gives

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2f_1+2\gamma f_2)-(4\theta_1\theta_2-\gamma^2)[q_14\theta_1g_2(1-q_2)+\gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)\theta_2g_1g_2(8\theta_1f_2+2\gamma f_1)-(4\theta_1\theta_2-\gamma^2)[q_24\theta_2g_1(1-q_1)+\gamma(1-q_2)g_2q_1]}\geq\frac{g_2(1-q_2)}{g_1(1-q_1)}$$

Since we also need that

$$\frac{b_1}{b_2}\geq\frac{\gamma g_2(1-q_2)}{2\theta_2g_1(1-q_1)}$$

Which means that

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2f_1+2\gamma f_2)-(4\theta_1\theta_2-\gamma^2)[q_14\theta_1g_2(1-q_2)+\gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)\theta_2g_1g_2(8\theta_1f_2+2\gamma f_1)-(4\theta_1\theta_2-\gamma^2)[q_24\theta_2g_1(1-q_1)+\gamma(1-q_2)g_2q_1]}\times\frac{(16\theta_1\theta_2-\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}{(16\theta_1\theta_2-\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}\geq\frac{\gamma g_2(1-q_2)}{2\theta_2g_1(1-q_1)}$$

This gives the result of:

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2f_1+2\gamma f_2)-(4\theta_1\theta_2-\gamma^2)[q_14\theta_1g_2(1-q_2)+\gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)\theta_2g_1g_2(8\theta_1f_2+2\gamma f_1)-(4\theta_1\theta_2-\gamma^2)[q_24\theta_2g_1(1-q_1)+\gamma(1-q_2)g_2q_1]}\geq\frac{\gamma[g_2(1-q_2)]^2}{2\theta_2[g_1(1-q_1)]^2}$$

Therefore,

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2f_1+2\gamma f_2)-(4\theta_1\theta_2-\gamma^2)[q_14\theta_1g_2(1-q_2)+\gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)\theta_2g_1g_2(8\theta_1f_2+2\gamma f_1)-(4\theta_1\theta_2-\gamma^2)[q_24\theta_2g_1(1-q_1)+\gamma(1-q_2)g_2q_1]}\geq\frac{g_2(1-q_2)}{g_1(1-q_1)}\geq\frac{\gamma[g_2(1-q_2)]^2}{2\theta_2[g_1(1-q_1)]^2}$$

Then,

$$\frac{g_2(1-q_2)}{g_1(1-q_1)}\geq\frac{\gamma[g_2(1-q_2)]^2}{2\theta_2[g_1(1-q_1)]^2}$$

Which means,

$$\frac{\gamma g_2(1-q_2)}{2\theta_2g_1(1-q_1)}\leq 1$$

Similarly, when $b_2 \geq b_1$, it will be

$$\frac{\gamma g_1(1-q_1)}{2\theta_1 g_2(1-q_2)} \leq 1$$

Proof 5 – Agent Effort

Replace the bonus level back to the effort level we have

$$\begin{aligned} e_1 = {} & \frac{(1-q_1)g_1 2\theta_2}{4\theta_1\theta_2-\gamma^2} \times \frac{g_2(1-q_2)}{g_1(1-q_1)} \{\frac{(1-q_1)(1-q_2)g_1 g_2\theta_1(8\theta_2 f_1+2\gamma f_2)}{(16\theta_1\theta_2-\gamma^2)g_1 g_2^2(1-q_1)(1-q_2)^2} \\ & - \frac{(4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2)+\gamma(1-q_1)g_1 q_2]}{(16\theta_1\theta_2-\gamma^2)g_1 g_2^2(1-q_1)(1-q_2)^2}\} \\ & - \frac{\gamma g_2(1-q_2)}{4\theta_1\theta_2-\gamma^2} \times \frac{g_1(1-q_1)}{g_2(1-q_2)} \{\frac{(1-q_1)(1-q_2)\theta_2 g_1 g_2(8\theta_1 f_2+2\gamma f_1)}{(16\theta_1\theta_2-\gamma^2)g_1^2 g_2(1-q_1)^2(1-q_2)} \\ & - \frac{(4\theta_1\theta_2-\gamma^2)[q_2 4\theta_2 g_1(1-q_1)+\gamma(1-q_2)g_2 q_1]}{(16\theta_1\theta_2-\gamma^2)g_1^2 g_2(1-q_1)^2(1-q_2)}\} \end{aligned}$$

Which yield

$$\begin{aligned} e_1 = {} & \frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{16\theta_1\theta_2^2 f_1}{(16\theta_1\theta_2-\gamma^2)} - \frac{(4\theta_1\theta_2-\gamma^2)q_1 \times 8\theta_1\theta_2}{(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)} + \frac{4\theta_1\theta_2\gamma f_2}{(16\theta_1\theta_2-\gamma^2)} \right. \\ & \left. - \frac{\gamma 2\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-\gamma^2)g_2(1-q_2)}\right] - \frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{8\theta_1\theta_2\gamma f_2}{(16\theta_1\theta_2-\gamma^2)} \\ & - \frac{(4\theta_1\theta_2-\gamma^2)q_2 \times 4\theta_2\gamma}{(16\theta_1\theta_2-\gamma^2)g_2(1-q_2)} + \frac{2\theta_2\gamma^2 f_1}{(16\theta_1\theta_2-\gamma^2)} - \frac{\gamma^2(4\theta_1\theta_2-\gamma^2)q_1}{(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)}] \end{aligned}$$

Next

$$\begin{aligned} e_1 = {} & \frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{16\theta_1\theta_2^2 f_1 - 8\theta_1\theta_2\gamma f_2}{(16\theta_1\theta_2-\gamma^2)} - \frac{(4\theta_1\theta_2-\gamma^2)q_1(8\theta_1\theta_2-\gamma^2)}{(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)} + \frac{4\theta_1\theta_2\gamma f_2 - 2\theta_2\gamma^2 f_1}{(16\theta_1\theta_2-\gamma^2)} \\ & + \frac{2\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-\gamma^2)g_2(1-q_2)}] \end{aligned}$$

Which gives

$$\begin{aligned} e_1 = {} & \frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{2\theta_2(8\theta_1\theta_2 f_1 - 4\theta_1\gamma f_2 + 2\theta_1\gamma f_2 - \gamma^2 f_1)}{(16\theta_1\theta_2-\gamma^2)} - \frac{(4\theta_1\theta_2-\gamma^2)q_1(8\theta_1\theta_2-\gamma^2)}{(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)} \\ & + \frac{3\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-\gamma^2)g_2(1-q_2)}] \end{aligned}$$

Thus

$$\begin{aligned} e_1^* = {} & \frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2-\gamma^2)} - \frac{(4\theta_1\theta_2-\gamma^2)q_1(8\theta_1\theta_2-\gamma^2)}{(16\theta_1\theta_2-\gamma^2)g_1(1-q_1)} \\ & + \frac{3\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-\gamma^2)g_2(1-q_2)}] \end{aligned}$$

Similarly,

$$e_2^* = \frac{1}{4\theta_1\theta_2 - \gamma^2}\{\frac{[2\theta_1(8\theta_1\theta_2 f_2 - \gamma^2 f_2 - 2\theta_2\gamma f_1)]}{(16\theta_1\theta_2 - \gamma^2)} - \frac{(4\theta_1\theta_2 - \gamma^2)q_2(8\theta_1\theta_2 - \gamma^2)}{(16\theta_1\theta_2 - \gamma^2)g_2(1-q_2)} + \frac{3\gamma\theta_2(4\theta_1\theta_2 - \gamma^2)q_2}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)}\}$$

And both effort levels are bigger and equal than zero

Proof 6 – Agent actions

A – Non-symmetric case

Comparing the effort with the first best option for the task one

$$e_1^* = \frac{1}{4\theta_1\theta_2 - \gamma^2}[\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)} - \frac{(4\theta_1\theta_2 - \gamma^2)q_1(8\theta_1\theta_2 - \gamma^2)}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)} + \frac{3\gamma\theta_2(4\theta_1\theta_2 - \gamma^2)q_2}{(16\theta_1\theta_2 - \gamma^2)g_2(1-q_2)}]$$

Denote the first best option of effort as e_1^{FB} and second-best option as e_1^* and then compare the following inequality

$$e_1^* \le e_1^{FB}$$

Therefore, this gives

$$\frac{1}{4\theta_1\theta_2 - \gamma^2}\{\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)} - [\frac{(4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)}]\} \le \frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}$$

Which gives

$$\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)} - [\frac{(4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)}]\} \le 2f_1\theta_2 - \gamma f_2$$

This yield,

$$[2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)]g_1(1-q_1)g_2(1-q_2) - (4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)] \le (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)$$

Then,

$$g_1(1-q_1)g_2(1-q_2)[2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2) - (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)] - (4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)] \le 0$$

Which gives this ratio

$$\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2) - (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)}{(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)} \le \frac{(4\theta_1\theta_2 - \gamma^2)}{g_1(1-q_1)g_2(1-q_2)}$$

Set $\gamma = 0$, then compare the effort level of second-best option and first best option from task one, which gives

$$\frac{1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)} \leq \frac{1}{2\theta_1}$$

Set $q_1, q_2 = 0$, then for task one

$$e_1 = \frac{1}{4\theta_1\theta_2 - \gamma^2} \frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)}$$

Compare the effort level of first best option and second-best option:

$$\frac{2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)} \leq 2f_1\theta_2 - \gamma f_2$$

Which gives

$$2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2) \leq (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)$$

And then

$$\gamma^3 - 12\theta_1\theta_2\gamma f_2 - 16\theta_1\theta_2^2 f_1 \geq 0$$

To access the effect of limited liabilities and the externalities when the corresponding variables change, then set the following functions derived from the inequality

$$\begin{aligned}[2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)]g_1(1-q_1)g_2(1-q_2) - (4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 + \gamma^2)q_1 g_2(1-q_2) \\ - 3\gamma\theta_2 q_2 g_1(1-q_1)] \leq (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)\end{aligned}$$

This gives

$$\begin{aligned}F(q_1, q_2, \gamma) = {} & [2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)]g_1(1-q_1)g_2(1-q_2) \\ & - (4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)]\end{aligned}$$

And

$$G(q_1, q_2, \gamma) = (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)$$

Take the following first order conditions

$$\begin{cases} F'_{q_1}(q_1, 0, 0) = -16\theta_1\theta_2^2 g_1 g_2 f_1 - 32\theta_1^2\theta_2^2 g_2 \ (Set\ q_2, \gamma = 0) \\ F'_{q2}(0, q_2, 0) = -16\theta_1\theta_2^2 g_1 g_2 f_1 \ (Set\ q_1, \gamma = 0) \\ F'_{\gamma}(0,0,\gamma) = [-4\theta_2 f_1\gamma - 2\theta_1\theta_2 f_2]g_1 g_2 \ \ (Set\ q_1, q_2 = 0) \end{cases}$$

And

$$\begin{cases} G'_{q_1}(q_1, 0, 0) = -32 g_1 g_2 \theta_1 \theta_2^2 f_1 \ (Set\ q_2, \gamma = 0) \\ G'_{q2}(0, q_2, 0) = -32 g_1 g_2 \theta_1 \theta_2^2 f_1 \ (Set\ q_1, \gamma = 0) \\ G'_{\gamma}(0,0,\gamma) = g_1 g_2(-16\theta_1\theta_2 f_2 + 3f_2\gamma^2 - 4\gamma f_1\theta_2) \ (Set\ q_1, q_2 = 0) \end{cases}$$

For $G'_{\gamma}(0,0,\gamma)$, when $\gamma = \frac{2\theta_2 f_1}{3f_2}$, $G''\left(0,0,\frac{2\theta_2 f_1}{3f_2}\right) = 0$

since $\frac{f_1}{f_2} \geq \frac{\gamma}{2\theta_2}$ and $\frac{f_2}{f_1} \geq \frac{\gamma}{2\theta_1}$ needs to be satisfied, then evaluate $G'_\gamma\left(0,0,\frac{2\theta_2 f_1}{f_2}\right)$

$$G'_\gamma\left(0,0,\frac{2\theta_2 f_1}{f_2}\right) = \left[-16\theta_1\theta_2 f_2 + 3\left(\frac{2\theta_2 f_1}{f_2}\right)^2 f_2 - 4\left(\frac{2\theta_2 f_1}{f_2}\right) f_1\theta_2\right] g_1 g_2$$

$$G'_\gamma\left(0,0,\frac{2\theta_2 f_1}{f_2}\right) = \left[-16\left(\frac{2\theta_2 f_1}{f_2}\right)\frac{f_1}{2f_2}\theta_2 + \frac{4\theta_2^2 f_1^2}{f_2}\right] g_1 g_2$$

$$G'_\gamma\left(0,0,\frac{2\theta_2 f_1}{f_2}\right) = -\frac{12\theta_1^2 f_1^2}{f_2} g_1 g_2 < 0$$

Which means that the function $G'_\gamma(0,0,\gamma)$ is always smaller than zero from $\gamma \in (0, \frac{2\theta_2 f_1}{f_2})$

B – Symmetric case

Consider the symmetric case, this will be

$$\frac{1}{4\theta_1\theta_2 - \gamma^2}\left\{\frac{[2\theta_2(8\theta_1\theta_2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_2)]}{(16\theta_1\theta_2 - \gamma^2)} - \left[\frac{(4\theta_1\theta_2 - \gamma^2)[(8\theta_1\theta_2 - \gamma^2)q_1 g_2(1-q_2) - 3\gamma\theta_2 q_2 g_1(1-q_1)]}{(16\theta_1\theta_2 - \gamma^2)g_1(1-q_1)g_2(1-q_2)}\right]\right\} \leq \frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}$$

Since that $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$, this becomes

$$\frac{1}{4\theta_1^2 - \gamma^2}\left\{\frac{[2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1)]}{(16\theta_1^2 - \gamma^2)} - \left[\frac{(4\theta_1^2 - \gamma^2)[(8\theta_1^2 - \gamma^2)q_1 g_1(1-q_1) - 3\gamma\theta_1 q_1 g_1(1-q_1)]}{(16\theta_1^2 - \gamma^2)g_1(1-q_1)g_1(1-q_1)}\right]\right\} \leq \frac{2f_1\theta_1 - \gamma f_1}{4\theta_1^2 - \gamma^2}$$

And then

$$\left\{\frac{[2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1)]}{(16\theta_1^2 - \gamma^2)} - \left[\frac{(4\theta_1^2 - \gamma^2)[(8\theta_1^2 - \gamma^2)q_1 g_1(1-q_1) - 3\gamma\theta_1 q_1 g_1(1-q_1)]}{(16\theta_1^2 - \gamma^2)g_1(1-q_1)g_1(1-q_1)}\right]\right\} \leq 2f_1\theta_1 - \gamma f_1$$

Which gives

$$\frac{[2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1)]}{(16\theta_1^2 - \gamma^2)} - \frac{q_1[(4\theta_1^2 - \gamma^2)(8\theta_1^2 - \gamma^2) - 3\gamma\theta_1]}{(16\theta_1^2 - \gamma^2)g_1(1-q_1)} \leq 2f_1\theta_1 - \gamma f_1$$

Set $q_1 = 0$ yields

$$2\theta_1(8\theta_1^2 f_1 - \gamma^2 f_1 - 2\theta_1\gamma f_1) \leq (2f_1\theta_1 - \gamma f_1)(16\theta_1^2 - \gamma^2)$$

Which gives

$$\gamma^3 - 12\theta_1^2\gamma + 16\theta_1^3 \geq 0$$

Set the function

$$H(\gamma) = \gamma^3 - 12\theta_1^2\gamma + 16\theta_1^3$$

Then take the derivative

$$H'(\gamma) = 3\gamma^2 - 12\theta_1^2$$

And $\gamma = 2\theta_1$, $H'(\gamma) = 0$

Also, the denominator of the effort function $4\theta_1^2 - \gamma^2 \neq 0$ and $16\theta_1^2 - \gamma^2 \neq 0$ need to be satisfied, therefore,

$$\gamma \neq 2\theta_1$$

$$\gamma \neq 4\theta_1$$

Recall that in Proof 3, the effort level must be larger than zero which means

$$\frac{\gamma g_2(1-q_2)}{2\theta_2 g_1(1-q_1)} \leq 1$$

Since it is the symmetric case, this becomes

$$\frac{\gamma}{2\theta_1} \leq 1$$

Rewrite gives

$$\gamma \leq 2\theta_1$$

Hence, for both functions $H(\gamma)$,$H'(\gamma)$, their domain will be $\gamma \in (0, 2\theta_1)$ and the inequality $H(\gamma) > 0$ always holds in $\gamma \in (0, 2\theta_1)$ which means the first best effort cannot be achieved.

C – No Externalities

Using the effort levels and bonus levels mentioned before and set $\gamma = 0$ which gives the first best effort,

$$e_1^{FB} = \frac{f_1}{2\theta_1}$$

$$e_2^{FB} = \frac{f_2}{2\theta_2}$$

The second-best option effort

$$e_1 = \frac{(1-q_1)b_1 g_1}{2\theta_1}$$

$$e_2 = \frac{(1-q_2)b_2 g_2}{2\theta_2}$$

And then

$$e_1 = \frac{f_1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)}$$

$$e_2 = \frac{f_2}{4\theta_2} - \frac{q_2}{2g_2(1-q_2)}$$

The bonus level offered by principals

$$b_1^* = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$$

$$b_2^* = \frac{(1-q_2)g_2 f_2 - 2\theta_2 q_2}{2g_2^2(1-q_2)^2}$$

If symmetric then for the first best option

$$e_1^{FB} = e_2^{FB} = \frac{f_1}{2\theta_1}$$

The second-best option effort level

$$e_1 = e_2 = \frac{(1-q_1)b_1 g_1}{2\theta_1}$$

And the bonus level offered by principals

$$b_1^* = b_2^* = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$$

Proof 7 – Collaborative Principals

A – No Externalities

When $\gamma = 0$

$$e_1^{FB} = \frac{f_1}{2\theta_1}$$

$$e_2^{FB} = \frac{f_2}{2\theta_2}$$

The second-best option effort

$$e_1 = \frac{(1-q_1)b_1 g_1}{2\theta_1}$$

$$e_2 = \frac{(1-q_2)b_2 g_2}{2\theta_2}$$

And then

$$e_1 = \frac{f_1}{4\theta_1} - \frac{q_1}{2g_1(1-q_1)}$$

$$e_2 = \frac{f_2}{4\theta_2} - \frac{q_2}{2g_2(1-q_2)}$$

The bonus level offered by principals

$$b_1^* = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$$

$$b_2^* = \frac{(1-q_2)g_2 f_2 - 2\theta_2 q_2}{2g_2^2(1-q_2)^2}$$

If symmetric then for the first best option

$$e_1^{FB} = e_2^{FB} = \frac{f_1}{2\theta_1}$$

The second-best option effort level

$$e_1 = e_2 = \frac{(1-q_1)b_1 g_1}{2\theta_1}$$

And the bonus level offered by principals

$$b_1^* = b_2^* = \frac{(1-q_1)g_1 f_1 - 2\theta_1 q_1}{2g_1^2(1-q_1)^2}$$

B – Bonus Level

When $\gamma > 0$

$$\begin{aligned} E\pi = f_1\left(\frac{(1-q_1)2\theta_2 b_1 g_1 - \gamma b_2 g_2(1-q_2)}{4\theta_1\theta_2 - \gamma^2}\right) \\ - \left[b_1 q_1 + b_1(1-q_1)g_1 \frac{(1-q_1)2\theta_2 b_1 g_1 - \gamma b_2 g_2(1-q_2)}{4\theta_1\theta_2 - \gamma^2}\right] \\ + f_2\left(\frac{(1-q_2)2\theta_1 b_2 g_2 - \gamma b_1 g_1(1-q_1)}{4\theta_1\theta_2 - \gamma^2}\right) \\ - \left[b_2 q_2 + b_2(1-q_2)g_2 \frac{(1-q_2)2\theta_1 b_2 g_2 - \gamma b_1 g_1(1-q_1)}{4\theta_1\theta_2 - \gamma^2}\right] \end{aligned}$$

$$\begin{aligned} E\pi'_{b_1} = \frac{(1-q_1)2\theta_2 g_1 f_1}{4\theta_1\theta_2 - \gamma^2} - \frac{f_2\gamma g_1(1-q_1)}{4\theta_1\theta_2 - \gamma^2} - \left[q_1 + \frac{4(1-q_1)^2 g_1^2\theta_2 b_1 - \gamma b_2 g_1 g_2(1-q_2)(1-q_1)}{4\theta_1\theta_2 - \gamma^2}\right] \\ - (\frac{-\gamma g_1(1-q_1)b_2(1-q_2)g_2}{4\theta_1\theta_2 - \gamma^2}) \end{aligned}$$

$$\frac{g_1(1-q_1)(2\theta_2 f_1 - \gamma f_2)}{4\theta_1\theta_2 - \gamma^2} - q_1 = \frac{4(1-q_1)^2 g_1^2\theta_2 b_1 - 2\gamma b_2 g_1 g_2(1-q_2)(1-q_1)}{4\theta_1\theta_2 - \gamma^2}$$

Which gives

$$b_1^C = \frac{g_1(1-q_1)(2\theta_2 f_1 - \gamma f_2) - q_1(4\theta_1\theta_2 - \gamma^2) + 2\gamma b_2 g_1 g_2(1-q_2)(1-q_1)}{4(1-q_1)^2 g_1^2\theta_2}$$

Similarly,

$$b_2^C = \frac{g_2(1-q_2)(2\theta_1 f_2 - \gamma f_1) - q_2(4\theta_1\theta_2 - \gamma^2) + 2\gamma b_1 g_1 g_2(1-q_2)(1-q_1)}{4(1-q_2)^2 g_2^2\theta_1}$$

Substitute the bonus level of principal two which gives

$$b_1^C = \frac{g_1(1-q_1)(2\theta_2 f_1 - \gamma f_2)}{4\theta_2 g_1^2(1-q_1)^2} - \frac{(4\theta_1\theta_2-\gamma^2)q_1}{4\theta_2 g_1^2(1-q_1)^2} + \frac{2\gamma(1-q_1)(1-q_2)g_1 g_2[g_2(1-q_2)(2\theta_1 f_2-\gamma f_1)-(4\theta_1\theta_2-\gamma^2)q_2+2\gamma(1-q_1)(1-q_2)b_1 g_1 g_2]}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2}$$

Then:

$$b_1^C = \frac{[g_1(1-q_1)(2\theta_2 f_1-\gamma f_2)-(4\theta_1\theta_2-\gamma^2)q_1]\times 4\theta_1 g_2^2(1-q_2)^2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} + \frac{2\gamma(2\theta_1 f_2-\gamma f_1)(1-q_1)(1-q_2)^2 g_1 g_2^2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} - \frac{2\gamma(1-q_1)(1-q_2)g_1 g_2\times(4\theta_1\theta_2-\gamma^2)q_2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} + \frac{4\gamma^2(1-q_1)^2(1-q_2)^2 b_1 g_1^2 g_2^2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2}$$

Which has

$$\left(\frac{16\theta_1\theta_2-4\gamma^2}{16\theta_1\theta_2}\right)b_1 = \frac{[(1-q_1)g_1(2\theta_2 f_1-\gamma f_2)-(4\theta_1\theta_2-\gamma^2)q_1]\times 4\theta_1 g_2^2(1-q_2)^2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} + \frac{2\gamma(2\theta_1 f_2-\gamma f_1)(1-q_1)(1-q_2)^2 g_1 g_2^2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} - \frac{2\gamma(1-q_1)(1-q_2)g_1 g_2\times(4\theta_1\theta_2-\gamma^2)q_2}{16\theta_1\theta_2 g_1^2 g_2^2(1-q_1)^2(1-q_2)^2}$$

Then:

$$b_1^C = \frac{(1-q_1)(1-q_2)^2 4\theta_1(2\theta_2 f_1-\gamma f_2)g_1 g_2^2}{(16\theta_1\theta_2-4\gamma^2)g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} - \frac{(4\theta_1\theta_2-\gamma^2)q_1\times 4\theta_1 g_2^2(1-q_2)^2}{(16\theta_1\theta_2-4\gamma^2)g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} + \frac{2\gamma(2\theta_1 f_2-\gamma f_1)(1-q_1)(1-q_2)^2 g_1 g_2^2}{(16\theta_1\theta_2-4\gamma^2)g_1^2 g_2^2(1-q_1)^2(1-q_2)^2} - \frac{2\gamma(1-q_1)(1-q_2)g_1 g_2\times(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-4\gamma^2)g_1^2 g_2^2(1-q_1)^2(1-q_2)^2}$$

Then:

$$b_1^C = \frac{g_2(1-q_2)}{g_1(1-q_1)}\{\frac{(1-q_1)(1-q_2)g_1 g_2 4\theta_1(2\theta_2 f_1-\gamma f_2+4\gamma\theta_1 f_2-2\gamma^2 f_1)}{(16\theta_1\theta_2-4\gamma^2)g_1 g_2^2(1-q_1)(1-q_2)^2} - \frac{(4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2)+2\gamma(1-q_1)g_1 q_2]}{(16\theta_1\theta_2-4\gamma^2)g_1 g_2^2(1-q_1)(1-q_2)^2}\}$$

By using the same way, the bonus level of principal two

$$b_2^C = \frac{g_1(1-q_1)}{g_2(1-q_2)}\{\frac{(1-q_1)(1-q_2)4\theta_2 g_1 g_2(2\theta_1 f_2-\gamma f_1+4\gamma\theta_2 f_1-2\gamma^2 f_2)}{(16\theta_1\theta_2-4\gamma^2)g_1^2 g_2(1-q_1)^2(1-q_2)} - \frac{(4\theta_1\theta_2-\gamma^2)[q_2 4\theta_2 g_1(1-q_1)+2\gamma(1-q_2)g_2 q_1]}{(16\theta_1\theta_2-4\gamma^2)g_1^2 g_2(1-q_1)^2(1-q_2)}\}$$

If $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$, this gives $b_1^C = b_2^C = \frac{4\theta_1(2\theta_1 f_1 - \gamma f_1 + 4\gamma\theta_1 f_1 - 2\gamma^2 f_1)}{(16\theta_1^2 - 4\gamma^2)g_1(1-q_1)} - \frac{(4\theta_1^2-\gamma^2)(q_1 4\theta_1 + 2\gamma q_1)}{(16\theta_1^2-4\gamma^2)g_1^2(1-q_1)^2}$

Comparing collaborative bonuses with non-collaborative bonuses

$$b_1^* \leq b_1^C$$

Which can be written as

$$\frac{b_1^*}{b_1^C} \leq 1$$

And

$$\frac{\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2)}{(16\theta_1\theta_2 - \gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2} - \frac{(4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1q_2]}{(16\theta_1\theta_2-\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}}{\frac{(1-q_1)(1-q_2)g_1g_2 4\theta_1(2\theta_2 f_1 - \gamma f_2 + 4\gamma\theta_1 f_2 - 2\gamma^2 f_1)}{(16\theta_1\theta_2 - 4\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2} - \frac{(4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2) + 2\gamma(1-q_1)g_1q_2]}{(16\theta_1\theta_2-4\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}} \leq 1$$

Then

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2) - (4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)g_1g_2 4\theta_1(2\theta_2 f_1 - \gamma f_2 + 4\gamma\theta_1 f_2 - 2\gamma^2 f_1) - (4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2) + 2\gamma(1-q_1)g_1q_2]} \times \frac{16\theta_1\theta_2 - 4\gamma^2}{16\theta_1\theta_2 - \gamma^2} \leq 1$$

Which gives

$$\frac{(1-q_1)(1-q_2)g_1g_2\theta_1(8\theta_2 f_1 + 2\gamma f_2) - (4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2) + \gamma(1-q_1)g_1q_2]}{(1-q_1)(1-q_2)g_1g_2 4\theta_1(2\theta_2 f_1 - \gamma f_2 + 4\gamma\theta_1 f_2 - 2\gamma^2 f_1) - (4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2) + 2\gamma(1-q_1)g_1q_2]} \leq \frac{16\theta_1\theta_2 - \gamma^2}{16\theta_1\theta_2 - 4\gamma^2}$$

Note that when $\gamma = 0$, then both side of inequality is equal to one

If symmetric,

$$\frac{\theta_1^2 f_1(8+2\gamma)g_1(1-q_1)}{(16\theta_1^2-\gamma^2)g_1(1-q_1)} - \frac{(4\theta_1^2-\gamma^2)q_1[4\theta_1+\gamma]}{(16\theta_1^2-\gamma^2)g_1(1-q_1)} \leq \frac{4\theta_1(2\theta_1 f_1 - \gamma f_1 + 4\gamma\theta_1 f_1 - 2\gamma^2 f_1)g_1(1-q_1)}{(16\theta_1^2-4\gamma^2)g_1(1-q_1)} - \frac{(4\theta_1^2-\gamma^2)(q_1 4\theta_1 + 2\gamma q_1)}{(16\theta_1^2-4\gamma^2)g_1(1-q_1)}$$

Then

$$\frac{\theta_1^2 f_1(8+2\gamma)g_1(1-q_1) - (4\theta_1^2-\gamma^2)q_1[4\theta_1+\gamma]}{4\theta_1(2\theta_1 f_1 - \gamma f_1 + 4\gamma\theta_1 f_1 - 2\gamma^2 f_1)g_1(1-q_1) - (4\theta_1^2-\gamma^2)(q_1 4\theta_1 + 2\gamma q_1)} \leq \frac{16\theta_1^2-\gamma^2}{16\theta_1^2-4\gamma^2}$$

C – Effort level

I – Collaborative effort level

Replace the bonus level back to the effort level we have

$$e_1=\frac{(1-q_1)g_1 2\theta_2}{4\theta_1\theta_2-\gamma^2}\times\frac{g_2(1-q_2)}{g_1(1-q_1)}\{\frac{(1-q_1)(1-q_2)g_1g_2 4\theta_1(2\theta_2 f_1-\gamma f_2+4\gamma\theta_1 f_2-2\gamma^2 f_1)}{(16\theta_1\theta_2-4\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}$$
$$-\frac{(4\theta_1\theta_2-\gamma^2)[q_1 4\theta_1 g_2(1-q_2)+2\gamma(1-q_1)g_1q_2]}{(16\theta_1\theta_2-4\gamma^2)g_1g_2^2(1-q_1)(1-q_2)^2}\}$$
$$-\frac{\gamma g_2(1-q_2)}{4\theta_1\theta_2-\gamma^2}$$
$$\times\frac{g_1(1-q_1)}{g_2(1-q_2)}\{\frac{(1-q_1)(1-q_2)4\theta_2 g_1g_2(2\theta_1 f_2-\gamma f_1+4\gamma\theta_2 f_1-2\gamma^2 f_2)}{(16\theta_1\theta_2-4\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}$$
$$-\frac{(4\theta_1\theta_2-\gamma^2)[q_2 4\theta_2 g_1(1-q_1)+2\gamma(1-q_2)g_2q_1]}{(16\theta_1\theta_2-4\gamma^2)g_1^2g_2(1-q_1)^2(1-q_2)}\}$$

Which yield

$$e_1=\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{16\theta_1\theta_2^2 f_1-16\gamma^2\theta_1\theta_2 f_1}{(16\theta_1\theta_2-4\gamma^2)}-\frac{(4\theta_1\theta_2-\gamma^2)q_1\times 8\theta_1\theta_2}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}+\frac{32\gamma\theta_1\theta_2 f_2-4\theta_1\theta_2\gamma f_2}{(16\theta_1\theta_2-4\gamma^2)}\right.$$
$$\left.-\frac{4\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}\right]-\frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{8\theta_1\theta_2\gamma f_2-8\gamma^3\theta_2 f_2}{(16\theta_1\theta_2-4\gamma^2)}$$
$$-\frac{(4\theta_1\theta_2-\gamma^2)q_2\times 4\theta_2\gamma}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}+\frac{16\gamma^2\theta_2^2 f_1-4\theta_2\gamma^2 f_1}{(16\theta_1\theta_2-4\gamma^2)}$$
$$-\frac{2\gamma^2(4\theta_1\theta_2-\gamma^2)q_1}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}]$$

Then

$$e_1=\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{16\theta_1\theta_2^2 f_1-16\gamma^2\theta_1\theta_2 f_1+32\gamma\theta_1\theta_2 f_2-4\theta_1\theta_2\gamma f_2}{(16\theta_1\theta_2-4\gamma^2)}\right.$$
$$\left.-\frac{8\theta_1\theta_2\gamma f_2-8\gamma^3\theta_2 f_2+16\gamma^2\theta_2^2 f_1-4\theta_2\gamma^2 f_1}{16\theta_1\theta_2-4\gamma^2}\right]$$
$$-\frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{(4\theta_1\theta_2-\gamma^2)q_1\times 8\theta_1\theta_2}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}-\frac{2\gamma^2(4\theta_1\theta_2-\gamma^2)q_1}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}$$
$$+\frac{4\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}-\frac{(4\theta_1\theta_2-\gamma^2)q_2\times 4\theta_2\gamma}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}]$$

Which gives

$$e_1=\frac{1}{4\theta_1\theta_2-\gamma^2}\left[\frac{16\theta_1\theta_2^2 f_1-16\gamma^2\theta_1\theta_2 f_1+32\gamma\theta_1\theta_2 f_2-4\theta_1\theta_2\gamma f_2}{(16\theta_1\theta_2-4\gamma^2)}\right.$$
$$\left.-\frac{8\theta_1\theta_2\gamma f_2-8\gamma^3\theta_2 f_2+16\gamma^2\theta_2^2 f_1-4\theta_2\gamma^2 f_1}{16\theta_1\theta_2-4\gamma^2}\right]$$
$$-\frac{1}{4\theta_1\theta_2-\gamma^2}[\frac{(4\theta_1\theta_2-\gamma^2)q_1\times 8\theta_1\theta_2}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}-\frac{2\gamma^2(4\theta_1\theta_2-\gamma^2)q_1}{(16\theta_1\theta_2-4\gamma^2)g_1(1-q_1)}$$
$$+\frac{4\gamma\theta_2(4\theta_1\theta_2-\gamma^2)q_2}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}-\frac{(4\theta_1\theta_2-\gamma^2)q_2\times 4\theta_2\gamma}{(16\theta_1\theta_2-4\gamma^2)g_2(1-q_2)}]$$

Then

$$e_1 = \frac{1}{4\theta_1\theta_2 - \gamma^2}\left[\frac{8\gamma^3\theta_2 f_2 - (16\theta_1\theta_2 + 16\theta_2^2 - 4\theta_2)f_1\gamma^2 + 20\theta_1\theta_2 f_2\gamma + 16\theta_1\theta_2^2 f_1}{16\theta_1\theta_2 - 4\gamma^2} - \frac{(4\theta_1\theta_2 - \gamma^2)q_1(8\theta_1\theta_2 - 2\gamma^2)}{(16\theta_1\theta_2 - 4\gamma^2)g_1(1-q_1)}\right]$$

Using the same way

$$e_2 = \frac{1}{4\theta_1\theta_2 - \gamma^2}\left[\frac{8\gamma^3\theta_1 f_1 - (16\theta_1\theta_2 + 16\theta_1^2 - 4\theta_1)f_2\gamma^2 + 20\theta_1\theta_2 f_1\gamma + 16\theta_1^2\theta_2 f_2}{16\theta_1\theta_2 - 4\gamma^2} - \frac{(4\theta_1\theta_2 - \gamma^2)q_2(8\theta_1\theta_2 - 2\gamma^2)}{(16\theta_1\theta_2 - 4\gamma^2)g_2(1-q_2)}\right]$$

II – Non symmetric case

Evaluate $e_1^C \le e^{FB}$ gives

$$\frac{1}{4\theta_1\theta_2 - \gamma^2}\left[\frac{8\gamma^3\theta_2 f_2 - (16\theta_1\theta_2 + 16\theta_2^2 - 4\theta_2)f_1\gamma^2 + 20\theta_1\theta_2 f_2\gamma + 16\theta_1\theta_2^2 f_1}{16\theta_1\theta_2 - 4\gamma^2}\right] \le \frac{2f_1\theta_2 - \gamma f_2}{4\theta_1\theta_2 - \gamma^2}$$

Rewrite the inequality then

$$8\gamma^3\theta_2 f_2 - (16\theta_1\theta_2 + 16\theta_2^2 - 4\theta_2)f_1\gamma^2 + 20\theta_1\theta_2 f_2\gamma + 16\theta_1\theta_2^2 f_1 \le (2f_1\theta_2 - \gamma f_2)(16\theta_1\theta_2 - 4\gamma^2)$$

Then

$$8\gamma^3\theta_2 f_2 - (16\theta_1\theta_2 + 16\theta_2^2 - 4\theta_2)f_1\gamma^2 + 20\theta_1\theta_2 f_2\gamma + 16\theta_1\theta_2^2 f_1 \le 32f_1\theta_1\theta_2^2 - 8\gamma^2 f_1\theta_2 - 16\theta_1\theta_2\gamma f_2 + 4f_2\gamma^3$$

Which gives

$$(8\theta_2 f_2 - 4f_2)\gamma^3 - (16\theta_1\theta_2 + 16\theta_2^2 - 12\theta_2)f_1\gamma^2 + 36\theta_1\theta_2 f_2\gamma - 16\theta_1\theta_2^2 f_1 \le 0$$

Set $M(\gamma) = (2\theta_2 f_2 - f_2)\gamma^3 - (4\theta_1\theta_2 + 4\theta_2^2 - 3\theta_2)f_1\gamma^2 + 9\theta_1\theta_2 f_2\gamma - 4\theta_1\theta_2^2 f_1$

Then $M'(\gamma) = 3(2\theta_2 f_2 - f_2)\gamma^2 - 2(4\theta_1\theta_2 + 4\theta_2^2 - 3\theta_2)f_1\gamma + 9\theta_1\theta_2 f_2$

III – Symmetric case

If $q_1 = q_2$, $g_1 = g_2$, $\theta_1 = \theta_2$ and $f_1 = f_2$

The effort level becomes

$$e_1^C = \frac{1}{4\theta_1^2 - \gamma^2}\left[\frac{8\gamma^3\theta_1 f_2 - (16\theta_1^2 + 16\theta_1^2 - 4\theta_1)f_1\gamma^2 + 20\theta_1^2 f_2\gamma + 16\theta_1^3 f_1}{16\theta_1^2 - 4\gamma^2} - \frac{(4\theta_1^2 - \gamma^2)q_1(8\theta_1^2 - 2\gamma^2)}{(16\theta_1\theta_2 - 4\gamma^2)g_1(1-q_1)}\right]$$

then the function $M(\gamma) \to N(\gamma)$ when set $q_1 = 0$ which can be written in

$$N(\gamma) = [(2\theta_1 - 1)\gamma^3 - (8\theta_1^2 - 3\theta_1)\gamma^2 + 9\theta_1^2\gamma - 4\theta_1^3]f_1$$

Since $\gamma \in (0, 2\theta_1)$

Then evaluate this inequality

$$N(0)N(2\theta_1) < 0$$

Which gives

$$(-4\theta_1^3 f_1)(-16\theta_1^4 f_1 + 18\theta_1^3 f_1) < 0$$

Then

$$64\theta_1^2 - 72\theta_1 < 0$$

Which yields

$$\theta_1 \in (0, \frac{8}{9})$$

This means

$$\forall\, \theta_1 \in \left(0, \frac{8}{9}\right);\ \exists\, \gamma \in (0, 2\theta_1), N(\gamma) = 0$$