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Forecasting Interest Rates using the (Rotated) Dynamic Nelson-Siegel Model with Shifting Endpoints

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Abstract

This paper has the goal to use the papers van Dijk et al. (2013) and Nyholm (2017) to improve the forecasting accuracy by combining the ideas of the Dynamic Nelson-Siegel model, Rotated Dynamic Nelson-Siegel model and shifting endpoints specifications. Forecasting the government bond yields is of practical importance for, for example, monetary policymakers and investors. We use yield and macroeconomic data. As macroeconomic data, we have inflation and economic growth measured with the consumer price index and industrial production. We consider the Dynamic Nelson-Siegel(DNS) model, the Rotated Dynamic Nelson-Siegel(RDNS) model and Random Walk processes. We expand the RDNS with macro-economic variables and we take account of shifting endpoints for DNS and RDNS. Additionally, we investigate structural breaks. We find that for the short horizon and short maturities the RDNS model, integrated with macroeconomic variables forecasts interest rates more accurately. For the short and middle horizons with long maturity and for the long horizon the DNS model under shifting endpoints with exponential smoothing forecast interest rates more accurately.

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1 Introduction

Until now different methods have been investigated to forecast the interest rates, but no method has been found that has a 'perfect forecast'. It is true that maybe it is not possible to achieve that, but this paper has the goal to use the papers van Dijk et al. (2013) and Nyholm (2017) to improve the forecasting accuracy by combining the ideas of the Dynamic Nelson-Siegel(DNS) model, Rotated Dynamic Nelson-Siegel(RDNS) model and shifting endpoints specifications.

Forecasting the government bond yields is of practical importance for, for example, monetary policymakers and investors. Monetary policymakers need to know what the yields will be in the future to adjust their plans on how they will change the amount of their currency over time. Investors want to know the government bond yields because if it will be high they will invest otherwise they will not. So indirectly a better forecast of the government bond yields is of importance for every citizen because it influences the currency of the country. Both, monetary policymakers and investors, want to divide the yield correctly in the expected future yield and risk premia.

The research Litterman & Scheinkman (1991) concludes that their research strongly suggests that the three factors of the yield curve, namely curvature, level and slope, precisely estimate the cross-section of yields. An example of a term structure model where the fundamental property is a parsimonious factor structure is the Nelson-Siegel model introduced by Nelson & Siegel (1987). The model is expanded by Diebold & Li (2006) to a predicting tool by bringing together the autoregressive specifications for the dynamics of factors level, slope and curvature with the factor depiction of the yields, the Dynamic Nelson-Siegel(DNS) approach. In this approach, an assumption is that the autoregressions are stationary processes with a constant unrestricted mean. Especially for the level factor, the assumption can be unseemly, since the previous acts of interest rates. The paper Duffee (2011) achieves better out-of-sample predictions when we model the first principal components of yields using a random walk. The first principal component is nearly similar to the Nelson–Siegel level factor.

Direct parametrization is of importance because then it is possible to integrate macroeconomic variables within the model in a Taylor-rule-type form as well as attain an expression that uses finite number of standard operations for the term-premia that is implied by the model. Otherwise, it will be not possible to integrate an empirical monetary policy rule where the macroeconomic variables straightly affect the short end of the yield curve. In the traditional DNS modeling set-up direct parametrization is impossible by reason of the short rate would be determined as the aggregate of the first two yield curve factors, also would exclusively be available for $\tau_i = 0$. The DNS model makes direct parametrization of the short rate process itself not possible and it cannot be determined to meet either short-term maturity τ_i, so we look also at the Rotated Dynamic Nelson-Siegel model introduced by Nyholm (2017). The rotation of the DNS factors level, slope and curvature results in the factors short rate, slope with an opposite sign and a similar factor curvature.

We examine for predicting interest rates if it is beneficial to allow for nonstationarity in the Nelson-Siegel factor dynamics. We do not consider the random walk like Duffee (2011), but we look at a shifting endpoint or autoregressive specifications with a time-varying unrestricted mean. Because early literature state that specific macroeconomic variables, especially inflation, need to be modeled as the aggregate of a leisurely changing trend or permanent part and a temporary part (to mention a few, Kozicki & Tinsley (2001), Cogley & Sargent (2005) ,Faust & Wright (2012) and Wright (2013)). Kozicki & Tinsley (2001) is for example stating that ordinary assumptions, the short rate is mean-reverting or consists of a unit root insufficiently, clarify previous shifts in market perceptions of the policy target for inflation.

The main question we answer in this paper is:

"Which model forecasts interest rates more accurately (under shifting endpoints): a *rotated term structure model* or a *traditional term structure model*?"

To get to a conclusion that answers our research question, three sub-questions are answered. They are as follows:

1. Does modeling shifting endpoints using exponential smoothing of interest rates improve forecast accuracy in the rotated and traditional term structure model?

2. Does including macro-economic information to the shifting endpoints improve forecast accuracy in the rotated and traditional term structure model?

3. Are there structural breaks in the model parameters, if yes, when are they and for which parameters?

We control if it is advantageous to impose structural breaks in accordance with previously noticed business cycle dates, to look for when the macroeconomic variables are statistically significant.

We use yield data taken from Liu & Wu (2020). As macro-economic data we have inflation and economic growth, measured with the consumer price index and the industrial production, respectively taken from U.S. Bureau of Labor Statistics (2022) and Board of Governors of the Federal Reserve System (US) (2022). We transformed the macro-economic data by taking the first log difference and standardised the series to have mean zero and variance one.

We consider the following methods: (1) Dynamic Nielson-Siegel method(DNS), (2)Rotated Dynamic Nielson-Siegel(RDNS) method, integrated with macroeconomic variables and allow for exogenously determined structural breaks in selected parameters of the model, (3) Random Walk process for interest rates of each maturity, (4) Random Walk process for the short rate, slope and curvature factors in the RDNS model and (5)Random Walk process for the level, slope and curvature factors in the DNS model. We look at two different specifications for the shifting endpoints (1) Exponential smoothing of interest rates and (2) Exponential smoothing of realized inflation and economic growth. We find the best approach using the root mean square prediction errors(RMSPE).

We find that for the short horizon and 3 and 12 months maturities the RDNS model, integrated with macroeconomic variables and for the short and middle horizon with 10 years maturity and the long horizon the DNS model under shifting endpoints with exponential smoothing forecast interest rates more accurately. For the short and middle horizons with maturities of 3 and 5 years, our benchmark the Random Walk processes forecast interest rates more accurately.

Our work contributes to the literature by investigating whether there is an improvement in the forecasting performance when allowing for shifting endpoints in the RDNS model and comparing the forecasting performances of the RDNS model and the DNS model both allowed and not allowed for shifting endpoints. There is previous research on these models.

Firstly, van Dijk et al. (2013) investigated shifting endpoints using the DNS model. They found that it is possible to contribute benefits in the out-of-sample forecast correctness when permitting shifting endpoints in yield curve factors. Additionally, we investigate the RDNS model, integrated with macroeconomic variables and allow for exogenously determined structural breaks in all parameters.

Secondly, Nyholm (2017) investigated the RDNS model, integrated with macroeconomic variables and allow for exogenously determined structural breaks in selected parameters(Time-decay parameter, constant parameter, autoregression parameter and the macroeconomic parameters). They found that only for the period 1990 to 2002 the macroeconomic variables are statistically significant and when allowing parameters to vary over the sample there is little profit. But they did not look at shifting endpoints and only looked at the period 1990 to 2014. We have more recent data and investigate the RDNS model, integrated with macroeconomic variables and allow for exogenously determined structural breaks for all parameters, so also for the moving average parameter.

The rest of this paper is structured as follows. First, we present the data we use in our analysis in Section 2. In Section 3, we illustrate which models we use in our research and which methods and tests we employ to arrive at our results. Afterward, we present and discuss our results in Section 4. We finish this paper with a conclusion in which we summarise our research and results.

2 Data

The yield data is taken from Liu & Wu (2020), they constructed a new zero-coupon yield curve that can capture the variation of the raw data better. So it improves for example the data sets of Fama & Bliss (1987) and Gürkaynak et al. (2007) which are used often in previous literature. The macro-economic data, inflation and economic growth, measured with the consumer price index and the industrial production, are respectively taken from U.S. Bureau of Labor Statistics (2022) and Board of Governors of the Federal Reserve System (US) (2022). CPIAUCSL and INDPRO are respectively the names of the variables. We transformed the data by taking the first log difference and standardised the series to have mean zero and variance one. We look at the time from January 1985 until December 2021. Our first prediction using an expanding window is for January 1994. We estimate the model parameters in all occasions using data starting from January 1985. The frequency of the data is monthly. In total, we have 444 observations.

Table 1 shows for each maturity the mean, standard deviation, minimum value, maximum value and autocorrelation of the yields and for the three factors level, slope and curvature simple empirical proxies. We can spot the in general used stylized facts of the yield curve, which are

1. In general the yield curve is concave and upward sloping,
2. The volatility of yields drops with maturity. Additionally, the statistics of the proxies demonstrate that the slope is more persistent than the curvature as well as that the level is the most persistent.
3. Yields are persistent, with long-term yields representing marginally greater auto correlations compared to short-term yields

Table 1: Descriptive Statistics Yield

Maturity	Mean	Stdv	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$
3	4.52	3.52	0.01	15.95	0.989	0.861	0.652
6	4.67	3.58	0.03	16.13	0.990	0.870	0.669
9	4.79	3.59	0.05	16.11	0.990	0.876	0.685
12	4.87	3.59	0.06	15.96	0.991	0.882	0.700
15	4.94	3.59	0.09	15.90	0.991	0.887	0.714
18	5.01	3.59	0.11	15.94	0.991	0.891	0.727
21	5.06	3.58	0.12	15.91	0.992	0.894	0.737
24	5.11	3.55	0.12	15.72	0.992	0.897	0.746
30	5.20	3.51	0.11	15.54	0.992	0.902	0.761
36	5.30	3.48	0.12	15.57	0.992	0.905	0.771
48	5.48	3.40	0.17	15.48	0.993	0.909	0.785
60	5.62	3.32	0.23	15.20	0.993	0.912	0.795
72	5.75	3.27	0.31	14.99	0.993	0.914	0.801
84	5.85	3.21	0.38	14.95	0.993	0.913	0.804
108	6.01	3.11	0.49	14.94	0.993	0.916	0.808
120(level)	6.07	3.04	0.53	14.94	0.993	0.912	0.808
Slope	1.55	1.37	-3.76	4.36	0.948	0.485	-0.096
Curvature	-0.37	0.97	-2.69	2.77	0.928	0.631	0.367

Note: We outline descriptive statistics for US Treasury yields over the period August 1971 up to December 2021. The proxy for slope is the yield of the maturity of 120 months less the yield of the maturity of 3 months, for level, the yield of the maturity of 120 months, and for curvature, twice the yield of the maturity of 24 months less the yields of the maturities of 3 and 120 months. For all maturities separately, we report the mean, standard deviation (SD), minimum, maximum and the jth order autocorrelation coefficients $\hat{\rho}(j)$ for j = 1, 12 and 30. Additionally, we add stats for empirical proxies for the level, slope and curvature factors of the yield curve. The maturity is considered in months.

In figure 1 the United States yields for the 3 months, 1 year, 3 years and 10 years maturities are shown. With our shifting-endpoint specification, we are trying to allow the low-frequency patterns we can see in figure 1. For each maturity, the yields move upwards through the 1970s and downwards from around 1980, in agreement with the expectations of long-run inflation and shifts in inflation. Also, the standardised macroeconomic variables with a zero mean and unit variance, which we are using are shown.

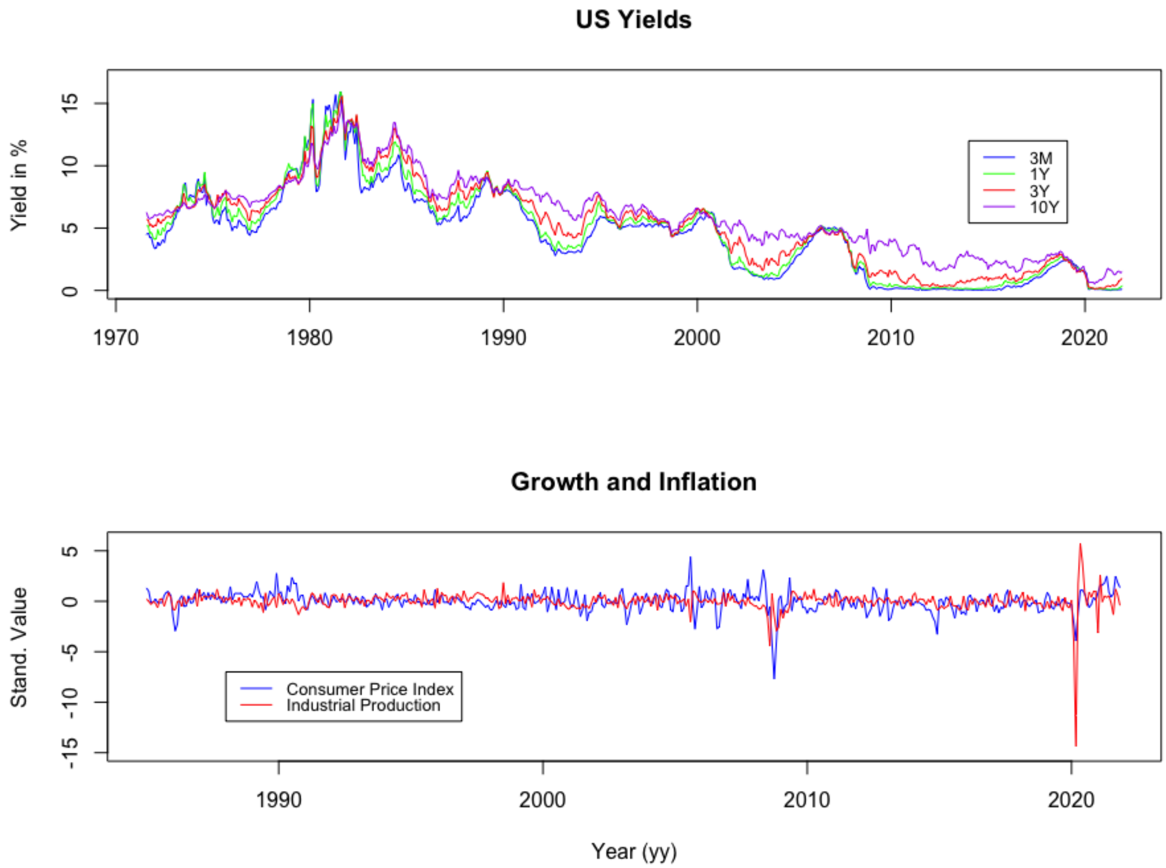

Figure 1: US Data

Note: This figure shows for the 3 months, 1 year, 3 and 10 years maturities in the upper panel the monthly US Treasury yields from August 1971 until December 2021. The bottom panel presents the inflation rate (CPI) and industrial production (IP), which are the macroeconomic variables. We have standardised these macro series with a subtraction of the empirical mean as well as dividing all observations separately by the empirical standard deviation.

3 Methodology

In Section 3.1 we discuss the model specification of the Dynamic Nelson-Siegel model, in Section 3.2 we discuss the model specification of the Rotated Dynamic Nelson-Siegel model, in Section 3.3 we look at the shifting endpoint model and in Section 3.4 we determine for which business cycle dates we investigate the structural breaks.

3.1 Method 1: Dynamic Nielson-Siegel

The first method we are going to look at is the DNS model. In van Dijk et al. (2013) they investigate forecasting interest rates with shifting endpoints using this model. For a maturity of τ periods and time t express $y_t(\tau)$ as the continuously compounded yield to maturity on a zero coupon bond. Then they look at the following three-factor model for the yield curve

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau}\right) + \epsilon_t(\tau). \tag{1}$$

The beta's β_{1t},β_{2t} and β_{3t} can be explained as latent dynamic factors. The loading on β_{1t} may be considered as a level factor because it does not depend on the maturity τ. The loading on β_{2t} may be viewed as a slope factor because it begins at 1 for $\tau = 0$ as well as decays monotonically to zero as the maturity increases. The loading on β_{3t} may be considered as a curvature factor because it is equivalent to infinity and positive in the middle as well as zero at maturities zero.

The parameter λ_t controls the exponential decay rate, so how quick the loading β_{2t} declines to zero as well as the maturity at which the loading on β_{3t} reaches its highest value. They assume that λ_t is constant and equal to 0.0609, because in this way at a maturity of 30 months it maximizes the curvature factor loading, following previous research.

The assumptions of the disturbance $\epsilon_t(\tau)$ is that it has a mean zero and a variance independent over time and across maturities(white noise). The disturbances denote the measurement error.

Finally, the model has separate univariate first-order autoregressive processes as specifications of the dynamics of the factors level, slope and curvature. These are given by

$$\beta_{j,t+1} = \mu_j + \phi_j(\beta_{jt} - \mu_j) + \eta_{j,t+1}, \tag{2}$$

for j=1,2,3 where the error term $\eta_{j,t+1}$ have a mean zero and a variance σ_j^2 and are assumed to be serially and mutually independent at all time periods.

Using Ordinary Least Squares(OLS) and fitting for each month t the model 1 to the cross-section of yields, the betas can be estimated. The ϕ is also estimated using OLS and equation 2.

The approach to obtain forecasts of future yields for the DNS model is generating forecasts of the factors $\beta_{j,t+h}$ by iterating for the desired forecast horizon of h periods equation 2 and then achieve using equation 1 multi-step forecasts of the interest rates.

3.2 Method 2: Rotated Dynamic Nielson-Siegel

The second method we are going to look at is, proposed by Nyholm (2017), the RDNS model, integrated with macroeconomic variables and allow for exogenously decided structural breaks in selected parameters of the model. In the RDNS model, the Nielson-Siegel factors are rotated from $[Level, -Slope, Curvature]$ to $[ShortRate, Slope, Curvature]$. Where the standard form of the RDNS model, obtained by pre-multiplying equation 1 with A and incorporate this into the dynamics in equation 2 by using the transformed factors gamma, is given by

$$y_t(\tau) = G \cdot \gamma_t + \epsilon_t(\tau), \tag{3}$$

$$\gamma_{j,t+1} = m_j + \phi_j(\gamma_{j,t} - m_j) + z_{j,t+1}, \tag{4}$$

where

$$G = H \cdot A^{-1}, \tag{5}$$

$$\gamma_{j,t+1} = A_j \cdot \beta_{t+1}, \tag{6}$$

$$m_j = A_j \cdot \mu, \tag{7}$$

$$z_{j,t} = A_j \cdot v_t, \tag{8}$$

With v_t an error term, A the rotation matrix selected to suchlike degree that the correct factor explanation appear and I is the identity matrix, where $I = A^{-1} \cdot A$ and A_j is the j-th row of A. β_t is a vector with the three factors, $\beta_t = (\beta_{1t}, \beta_{2t}, \beta_{3t})'$. H is representing the parameterization of the factor loading matrix of the DNS model and is defined as

$$H = \begin{bmatrix} 1 & \dfrac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} & \dfrac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau} \end{bmatrix}, \tag{9}$$

where λ_t is the same as in the DNS model.

In the paper they set the key-maturity(τ_s), corresponding to the short-rate factor, to 3 months, because it reflects the position of monetary policy when escaping institutional features belonging to the money market that can affect rates noticed at very short maturities. Then they parameterize A by

$$A_{(1,1:3),\tau_s} = \left[1 \quad \frac{1-e^{-\lambda_t\tau_s}}{\lambda_t\tau_s} \quad \frac{1-e^{-\lambda_t\tau_s}}{\lambda_t\tau_s} - e^{-\lambda_t\tau_s}\right], \tag{10}$$

$$A_{(2,1:3),\tau_s} = \left[0 \quad -\left(\frac{1-e^{-\lambda_t\tau_s}}{\lambda_t\tau_s}\right) \quad -\left(\frac{1-e^{-\lambda_t\tau_s}}{\lambda_t\tau_s} - e^{-\lambda_t\tau_s}\right)\right], \tag{11}$$

$$A_{(3,1:3),\tau_s} = \left[0 \quad 1-\left(\frac{1-e^{-\lambda_t\tau_s}}{\lambda_t\tau_s}\right) \quad 1-\left(\frac{1-e^{-\lambda_t\tau_s}}{\lambda_t\tau_s} - e^{-\lambda_t\tau_s}\right)\right]. \tag{12}$$

It applies that the RDNS model (without extensions such as macro variables or shifting endpoints) is equivalent to the standard DNS model. The expanded RDNS model, where we can integrate the macroeconomic variables has the same observation equation as the standard form, but it has the following state equation

$$\gamma_{j,t} = m_j + \phi_j(\gamma_{j,t-1} - m_j) + Q_j \cdot M_t + W_j \cdot z_{t-1} + z_{j,t}, \tag{13}$$

where Q consists of the coefficients considering the association among the yield curve factors and inflation and industrial production, the macroeconomic variables $M = [CPI, IP]'$, and W consists of the MA(1) coefficients. We compare the expanded RDNS model with the RDNS model with macro-based shifting endpoints. We can estimate both in R (R Core Team (2020)) with the *arima* function of the *stats* package.

3.3 Shifting endpoint model

Now we look at the shifting endpoint model, this is specification 14 for the DNS model and specification 15 for the RDNS model with respectively a time-varying unrestricted mean $\mu_{j,t+1}$ and $m_{j,t+1}$

$$\beta_{j,t+1} = \mu_{j,t+1} + \phi_j(\beta_{jt} - \mu_{j,t}) + \eta_{j,t+1}, \tag{14}$$

$$\gamma_{j,t+1} = m_{j,t+1} + \phi_j(\gamma_{j,t} - m_{j,t+1}) + z_{j,t+1}. \tag{15}$$

We consider two possibilities to estimate $\mu_{j,t+1}$ and $m_{j,t+1}$.

3.3.1 Exponential smoothing of interest rates

The first approach is to generate the $\mu_{j,t+1}$ and $m_{j,t+1}$ by the exponential smoothing recursion

$$\mu_{j,t+1} = \alpha\beta_{jt} + (1-\alpha)\mu_{jt}, \tag{16}$$

$$m_{j,t+1} = \alpha\gamma_{j,t} + (1-\alpha)m_{j,t+1}, \tag{17}$$

for t= 2,3,..., with decay parameter 0< α <1 and beginning with $\mu_{j1} = \beta_{j1}$ and $m_{j1} = \gamma_{j1}$. Substituting 16 in 14 and 17 in 15 results

$$\beta_{j,t+1} = w_j\beta_{jt} + (1 - w_j)\mu_{j,t} + \eta_{j,t+1}, \tag{18}$$

$$\gamma_{j,t+1} = w_j\gamma_{j,t} + (1 - w_j)m_{j,t} + z_{j,t+1}, \tag{19}$$

with $w_j = \phi_j + \alpha$ To get multi-step forecasts of $\beta_{j,t}$ and $\gamma_{j,t}$, respectively the equations 16, 18 and 17, 19 can be iterated forwards, that are

$$\hat{\beta}_{j,t+1} = w_j\hat{\beta}_{jt} + (1 - w_j)\hat{\mu}_{j,t}, \tag{20}$$

$$\hat{\gamma}_{j,t+1} = w_j\hat{\gamma}_{j,t} + (1 - w_j)\hat{m}_{j,t+1}. \tag{21}$$

Firstly, we practice for only the unconditional mean exponential smoothing of the level factor for the DNS model and the short rate factor for the RDNS model. Then we use these to forecast the other factors of the models. Secondly, we apply exponential smoothing to permit shifting endpoints of each factor of both models. In van Dijk et al. (2013) they set the smoothing parameter α for monthly data equal to 0.1, which is adequately near for yearly series to 0.7 . We also follow previous research and assume a constant α equal to 0.1.

3.3.2 Exponential smoothing of realized inflation and economic growth

When we apply exponential smoothing of realized inflation and growth data, we express real-time exponentially smoothed realized inflation and industrial production by π_t^{ES} and ψ_t^{ES}, respectively in month t and consider the regressions

$$\beta_{1t} = \theta_{0,1} + \theta_{1,1}\pi_t^{ES} + \xi_{1t}, \tag{22}$$

$$\gamma_{1t} = \lambda_{0,1} + \lambda_{1,1}\pi_t^{ES} + \epsilon_{1t}, \tag{23}$$

$$\beta_{2t} = \theta_{0,2} + \theta_{1,2}\psi_t^{ES} + \xi_{2t}, \tag{24}$$

$$\gamma_{2t} = \lambda_{0,2} + \lambda_{1,2}\psi_t^{ES} + \epsilon_{2t}. \tag{25}$$

Then the level factor and short rate factor can be forecasted with equations 14 and 15 respectively, setting $\mu_{1t} = \theta_{0,1} + \theta_{1,1}\pi_t^{ES}$ and $m_{1t} = \theta_{0,2} + \theta_{1,2}\psi_t^{ES}$and projecting that μ_{1t} and m_{1t} will stay constant at its end-of-sample value. With 2 and 4 the other two factors can be forecasted.

3.4 Structural breaks

We impose these structural breaks exogenously and split the period into four ranges: the first part begins at the start of the data sample in January 1994 until November 2002, which describes the economic growth experienced in the 1990s up to the dot com boom. The second part begins from the expansionary period until the time that the sovereign and banking crisis had started taking hold globally, so from around December 2002 until November 2008. The third part from December 2008 to February 2019, ten months before the start of the COVID-19 crisis, demonstrates a low growth and low inflationary economic environment. The fourth part from March 2019 until the end of the sample, with firstly strongly negative growth and afterward a recovery phase.

4 Results

In this section we discuss our results by using tables 3 and 4. In table 2 the distinct used prediction methods and their labels are shown. The predictions for interest rates of each maturity are made at horizons $h = 6, 12, 24$ months ahead.

Label	Description
DNS	Dynamic Nelson-Siegel method
RDNS	Rotated Dynamic Nelson-Siegel method
Expanded RDNS	RDNS method integrated with macroeconomic variables
RWY	Random Walk process for interest rates of all maturities
RW	Random Walk process for the factors level*, slope and curvature
RZI	Smooth realized inflation for the level* factor alone
ESLSC	Exponential smoothing for the factors level*, slope and curvature
ESL	Exponential smoothing for the level* factor alone
RZIG	RZI and smooth realized GDP growth for slope factor

*For the RDNS method we transform the level factor to the short rate factor

Table 2: The labels of the forecasting methods. We specify the prediction method we have used in this paper and give each its abbreviation.

4.1 Dynamic Nelson-Siegel

From the first part of table 3, we can see that the Random Walk processes more accurately forecast than the Dynamic Nelson Siegel model, because the RMSPE is less. For instance, at the 6 months forecast horizon, both of the Random Walk processes give a reduction of 0-7% in out-of-sample RMSPE, compared to DNS, which confirms the statement of Duffee (2011). The forecasting capability of the yield curve increases when imposing nonstationarity for the level factor (in our situation the factors slope and curvature also). Additionally, adding shifting endpoints in general results in more accurate forecasts for the DNS method. Additionally, both Random Walk processes forecast more accurately than the Exponential Smoothing methods for the short horizon. For the middle and long horizons, the Exponential Smoothing methods forecast more accurately than the Random Walk processes. Furthermore, the Random Walk processes forecast more accurately than realized macro exponential smoothing methods. In addition, RW forecasts are about equal to RWY. Therefore, the factor format of the Nelson–Siegel structure does not contribute gains, when nonstationarity of the yield curve is required. Moreover, ESL more accurately forecasts for middle and long horizons than ESLSC and is about equal for the short horizon. On the other hand, RZIG more accurately forecasts the yields than RZI. Finally, exponential smoothing of yield factors methods more accurately forecast than the realized macro exponential smoothing methods.

For the short horizon and maturities of 3,12,36 and 60 months RWY is the most accurate, for maturity 120 ESL and ESLSC are more accurate. For the middle horizon and 3, 12 and 120 months maturities ESL is the most accurate and for maturities 36 and 60 RWY is the most accurate. For the long horizon and maturities, 3,12 and 36 ESL is the most accurate and for 60 and 120 maturities, ESLSC is the most accurate.

4.2 Rotated Dynamic Nelson-Siegel

From the second part of table 3 firstly, we can see that expanding the RDNS results in the short and middle horizons for more accurate forecasts than the RDNS for the maturities 3 and 12 months. For the other horizons and maturities expanding the RDNS results in less accurate forecasts. Also, expanded RDNS forecasts more accurately for 3 and 12 months maturities and less accurately for 36,60 and 120 months maturities for the short and middle horizons than the Random Walk processes. For the long horizon, the Random Walk processes forecast more accurately than the expanded RDNS. Additionally, expanded RDNS forecasts are in general more accurate than ESL for all horizons. Furthermore, expanded RDNS forecasts more accurately than ESLSC for the middle horizon. For the long horizon less accurately, except 120 months maturity. For the short horizon for 3, 12 and 120 months maturities expanded RDNS forecasts more accurately than ESLSC. For the long horizon, RZI forecasts more accurately than expanded RDNS and RZIG forecasts less accurately, except for 3 and 12 months maturities. For the short and middle horizons the RZI and RZIG forecast more accurately for 36,60 and 120 months maturities than expanded RDNS.

For all horizons, the Random Walk processes forecast in general more accurately than the RDNS, except for the 3 and 12 months maturities for the long horizon and 3 months maturity of the short horizon for RW. Adding shifting endpoints does result in more accurate forecasts than the RW and RWY for the short and middle horizons. But for large horizon, it has more accurate forecasts, except for the 3 and 12 months maturities. RDNS has more accurate forecasts than ESL and ESLSC in the middle horizon. In the long horizon, RDNS has a more accurate forecast than ESL. For ESLSC in the long horizon, RDNS forecasts are less accurate, except for 3 months maturity. In the short horizon, ESL and ESLSC forecast more accurately than RDNS for 3 and 12 months maturities and less accurately for 36,60 and 120 months maturities. RDNS more accurately forecasts than RZI in the short horizon. In the middle and long horizons, RZI more or equal accurately forecast than RDNS. RDNS more or equal accurately forecasts than RZIG in all horizons. In the short horizon for 3 and 12 months maturities, ESL and ESLSC more accurately forecast than Random Walk processes. In the middle and long hoziron, ESL and ESLSC less accurately forecast than the Random Walk processes. In the short and middle horion, RZI and RZIG less accurately forecast than RW and RWY. In the long horizon, RZI and RZIG for maturities 3 and 12 months more accurately forecast and for maturities 36, 60 and 120 months maturities less accurately forecast than RW and RWY. ESL in general more accurately forecasts than ESLSC. RZI more or equal accurately forecasts than RZIG. RZI forecasts more accurately than ESL and ESLSC, except for the 3, 12 and 36 months maturities in the short horizon. RZIG forecasts more accurately than ESL and ESLSC in the middle horizon. In the long horizon RZIG forecasts more accurately than ESL and only for maturities 3 and 120 months in the long horizon than ESLSC. RZIG forecasts more accurately than ESL for only 36,60 and 120 months maturities and for only 60 and 120 months maturities than ESLSC.

For the short horizon and 3 and 12 months maturities expanded RDNS and for 36 and 60 months RW and RWY are the most accurate, for the maturity of 120 months RWY and RDNS are more accurate. For the middle horizon and 3 and 12 months maturities expanded RDNS, for maturities 36 RWY is the most accurate and for maturities 60 and 120 RW and RWY are the most accurate. For the long horizon and 3 months maturity RDNS, RZI and RZIG are the most accurate. For 12 months maturity RZI, for 36 and 60 months maturities RWY is the most accurate and for 120 months maturity RW is the most accurate.

4.3 Comparison DNS with RDNS

We can see that RDNS more accurately forecasts than DNS with 1%-29% difference. Only for the long horizon and 3 months maturity DNS more accurately forecasts than RDNS. We can see that the Random Walk processes forecast the same after the rotation. That is expected for RWY process, which is just depending on the yield data because the transformation does not affect the yields data. Also it is expected for RW process, because the yields for the RW process are calculated using $G \cdot \gamma_t = H \cdot A^{-1} \cdot A \cdot \beta_t = H \cdot \beta_t$. So, the transformation has no affect. For the long horizon shifting endpoint models less accurately forecast after the rotation. For 3,12 an 36 months maturities for the short horizon the rotation makes the forecasts more accurate and for 60 and 120 months makes the forecasts less accurate. For the middle horizon ESL less accurately forecasts after rotation. For the middle horizon, ESLSC less accurately forecasts after rotation, except for maturities 3 and 12 months. Transformation makes for the middle horizon the RZI method forecasts more accurate. After transformation, the RZIG is equally accurate for 12 and 60 months, for 3 and 120 months less accurate and for 36 months more accurate for the middle horizon.

For the short horizon and maturities, 3 and 12 months expanded RDNS and for 36 and 60 months, RW and RWY are the most accurate, for maturity 120 ESL and ESLSC of DNS are more accurate. For the middle horizon and maturities, 3 and 12 months expanded RDNS, for maturities 36 RWY is the most accurate and for 60 months maturity RW and RWY, for 120 months maturity ESL of DNS are the most accurate. For the long horizon and 3,12 and 36 months maturities, ESL of DNS is the most accurate. For 60 and 120 months maturities, ESLSC of DNS is the most accurate.

4.4 Structural Breaks

In table 4 we can see the RDNS model integrated with macroeconomic variables and allow for structural breaks in all parameters. It is clear that the AR coefficient for the whole period and for the periods with structural breaks stays significant between 0.9 and 1.

Also, we can see that the MA coefficient for when we do not allow for structural breaks is significant for the coefficient of the short rate and slope factors. When we allow for structural breaks, for the period 1994-2002 this is the same. But it is different for 2002-2008 where only the coefficient of the slope factor is significant, 2008-2019 neither of the coefficients are significant and in 2019-2021 only the coefficient for the short rate factor is significant.

Additionally, we can see that the coefficient of the macroeconomic variable consumer price index is insignificant for the periods 1994-2002, 2008-2019 and 2019-2021, but significant for the slope factor in 2002-2008 when allowing for structural breaks. When we do not allow it, the coefficient of the macroeconomic variable consumer price index is significant for the slope and curvature factors.

Moreover, it appears that the coefficient of the macroeconomic variable industrial production is insignificant for 1994-2002 and 2008-2019 when allowing for structural breaks. But it is significant for the curvature factor in 2002-2008 and the short rate factor in 2019-2021 when allowing for structural breaks. When we look at not allowing for structural breaks we can see that the coefficient of the macroeconomic variable industrial production is only significant for the short rate factor.

So we can see, that allowing for structural breaks show that the significance level of our coefficient of the parameters change over the periods.

Table 3: Out-of-sample root mean square prediction errors of alternative interest rate forecast

Forecast horizon:	h = 6 months					h = 12 months					h = 24 months				
Maturity (months):	3	12	36	60	120	3	12	36	60	120	3	12	36	60	120
DNS	0.77	0.78	0.80	0.77	0.63	1.26	1.25	1.21	1.13	0.90	1.89	1.90	1.86	1.74	1.42
RW	0.74	0.74	0.73	0.71	0.63	1.26	1.22	1.08	0.97	0.82	1.96	1.89	1.53	1.27	0.93
RWY	0.72	0.74	0.73	0.71	0.62	1.24	1.23	1.07	0.97	0.82	1.94	1.90	1.52	1.26	0.95
ESL	0.76	0.77	0.78	0.75	0.61	1.20	1.19	1.10	1.02	0.80	1.57	1.55	1.41	1.27	0.94
ESLSC	0.78	0.78	0.76	0.72	0.61	1.35	1.31	1.16	1.03	0.79	1.94	1.81	1.48	1.25	0.89
RZI	0.77	0.79	0.80	0.78	0.63	1.27	1.27	1.21	1.12	0.88	1.81	1.81	1.71	1.55	1.19
RZIG	0.76	0.77	0.79	0.76	0.63	1.22	1.23	1.17	1.09	0.87	1.70	1.70	1.62	1.49	1.16
RDNS	0.74	0.75	0.76	0.74	0.62	1.25	1.23	1.13	1.04	0.83	1.91	1.85	1.66	1.49	1.13
Expanded RDNS	0.67	0.70	0.75	0.74	0.64	1.18	1.17	1.16	1.11	0.95	1.96	1.87	1.68	1.53	1.22
RW	0.74	0.74	0.73	0.71	0.63	1.26	1.22	1.08	0.97	0.82	1.96	1.89	1.53	1.27	0.93
RWY	0.72	0.74	0.73	0.71	0.62	1.24	1.23	1.07	0.97	0.82	1.94	1.90	1.52	1.26	0.95
ESL	0.69	0.72	0.78	0.79	0.79	1.34	1.40	1.43	1.43	1.44	1.94	1.96	1.84	1.69	1.39
ESLSC	0.69	0.73	0.76	0.75	0.72	1.34	1.30	1.21	1.16	1.13	1.94	1.85	1.65	1.51	1.28
RZI	0.75	0.76	0.77	0.75	0.63	1.25	1.23	1.12	1.03	0.83	1.91	1.84	1.62	1.44	1.07
RZIG	0.74	0.75	0.77	0.75	0.63	1.25	1.23	1.16	1.09	0.88	1.91	1.86	1.72	1.58	1.24

Note: In the upper panel the forecasts based on the estimated betas of the DNS model and in the lower panel the forecasts based on the estimated gammas of the RDNS model can be seen. We report the values in percentage points for the time between January 1994 and December 2021. The prediction evaluation task is recursive, using in each case data beginning from January 1985 for model approximation. In Table 2 the labels can be found of the given forecast method.

Table 4: RDNS model integrated with macroeconomic variables and allow for structural breaks in all parameters

		Intercept	AR coefficients	MA coefficients	CPI coefficients	IP coefficients
1994 to 2021	Short Rate Factor	2.27(0.10)	0.99(0.00)	0.23(0.00)	0.02(0.07)	0.02(0.00)
	Slope Factor	2.18(0.00)	0.97(0.00)	0.23(0.00)	-0.05(0.00)	0.00(0.76)
	Curvature Factor	-2.09(0.01)	0.96(0.00)	0.01(0.89)	0.07(0.05)	-0.01(0.78)
1994 to 2002	Short Rate Factor	3.23(0.02)	0.99(0.00)	0.38(0.00)	0.02(0.36)	0.00(0.97)
	Slope Factor	2.42(0.02)	0.97(0.00)	0.31(0.00)	-0.03(0.33)	0.00(0.90)
	Curvature Factor	-1.21(0.24)	0.93(0.00)	0.11(0.28)	0.12(0.15)	-0.03(0.75)
2002 to 2008	Short Rate Factor	1.90(0.10)	0.98(0.00)	0.09(0.32)	0.03(0.11)	-0.01(0.76)
	Slope Factor	3.30(0.03)	0.97(0.00)	0.33(0.01)	-0.08(0.01)	-0.01(0.81)
	Curvature Factor	-2.92(0.01)	0.90(0.00)	0.24(0.28)	0.13(0.05)	0.22(0.01)
2008 to 2019	Short Rate Factor	1.38(0.12)	1.00(0.00)	0.11(0.25)	-0.01(0.10)	0.01(0.31)
	Slope Factor	2.64(0.00)	0.97(0.00)	0.09(0.40)	-0.02(0.45)	0.04(0.34)
	Curvature Factor	-3.45(0.05)	0.98(0.00)	-0.13(0.23)	0.00(0.93)	0.00(0.98)
2019 to 2021	Short Rate Factor	1.09(0.24)	0.97(0.00)	0.42(0.01)	0.03(0.51)	0.03(0.00)
	Slope Factor	0.85(0.14)	0.92(0.00)	0.23(0.40)	0.05(0.48)	-0.03(0.12)
	Curvature Factor	-1.34(0.01)	0.85(0.00)	0.20(0.18)	-0.20(0.06)	0.01(0.59)

Note: This table presents parameter approximations determined from the Rotated Dynamic Nelson-Siegel method, integrated with macroeconomic variables and allow structural breaks in all model parameters.

5 Conclusion

In this research, we tried to answer the following research question.

> "Which model forecasts interest rates more accurately (under shifting endpoints): a *rotated term structure model* or a *traditional term structure model*?"

Firstly, we have found for the DNS framework that adding shifting endpoints in general results in more accurate forecasts for the DNS method. The difference between the Random Walk process for only interest rates of all maturities and the Random Walk process for all three factors curvature, level and slope is little. For the long horizon including shifting endpoints with exponential smoothing give an improvement in Root Mean Square Prediction Errors relative to stationary and random walk benchmarks. The Random Walk processes forecast more accurately than the other methods, for the short horizon.

Secondly, we have found for the RDNS framework expanding the RDNS by integrating it with macroeconomic variables is improving the forecasting accuracy for the short and middle horizon for the maturities 3 and 12 months relative to the non-expanded RDNS. For the short horizon and maturities, 3 and 12 months maturities expanded RDNS and for 36 and 60 months, RW and RWY are the most accurate, for maturity 120 RWY and RDNS are more accurate. For the middle horizon and 3 and 12 months maturities expanded RDNS, for 36 months maturity RWY is the most accurate and for maturities 60 and 120 months, RW and RWY are the most accurate. For the long horizon and 3 months maturity RDNS, RZI and RZIG are the most accurate. For 12 months maturity RZI, for 36 and 60 months maturities RWY is the most accurate and for 120 months maturity RW is the most accurate.

Thirdly, we have found that RDNS in general more accurately forecasts than DNS. The Random Walk processes forecast the same after the rotation. For the long horizon shifting endpoints less accurately forecast after the rotation. For 3,12 and 36 months maturities for the short horizon the rotation makes the forecasts more accurate and for 60 and 120 months makes the forecasts less accurate. For the middle horizon ESL less accurately forecasts after rotation. For the middle horizon, ESLSC less accurately forecasts after rotation, except for maturities 3 and 12 months. Transformation makes for the middle horizon the RZI method forecasts more accurate. RZIG forecasts equal accurate for 12 and 60 months, for 3 and 120 months less accurate and for 36 months more accurate for the middle horizon after transformation.

Finally, we have found that the macroeconomic variable coefficient for the consumer price index was only significant for the slope factor in 2002 to 2008 period and the coefficient for industrial production was significant for the curvature factor from 2002 to 2008 and the short rate factor in 2019 to 2021.

We can conclude that for the short horizon and 3 and 12 months maturities the rotated dynamic Nelson-Siegel model, integrated with macroeconomic variables and for the short and middle horizon with 10 years maturity and the long horizon the traditional Nelson-Siegel model under shifting endpoints with exponential smoothing forecast interest rates more accurately. For the short and middle horizons with maturities of 3 and 5 years our benchmark the Random Walk processes forecast interest rates more accurately.

In the future, there can be research about whether the results for exogenously determined structural breaks are the same when testing for structural breaks and using the found ones. Also comparing the predictive accuracy of the different forecast methods using the Diebold-Mariano statistic is an option. Bootstrapping could be used to establish the significance of the Diebold-Mariano statistic. Additionally, using a rolling window instead of the used expanding window is an option. For the rolling window, the length of the data window is fixed. There can also be research on using a different exponential smoothing step, α. We based our choice on previous literature.

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A Brief explanation of R code

In the zip-file Code and Data the following files can be found:

- Yield Analyze.R, this is the code for analyzing the yield data by calculating the means, standard deviations, minimums, maximums and autocorrelations. Also, it is the code for plotting the yield data and the macroeconomic data.
- Dynamic Nelson Siegel.R, this is the code for estimating the betas of the DNS model and using them to forecast using the DNS, RW, RWY, ESL, ESLSC, RZI, RZIG models with horizons h=6, 12 and 24.
- Rotated Dynamic Nelson Siegel.R, this is the code for estimating the gamma's of the RDNS model and using them to forecast the interest rates using the RDNS, expanded RDNS, RW, RWY, ESL, ESLSC, RZI, RZIG models with horizons h=6, 12 and 24.
- LW_monthly_DNS.xlsx is the file with the yields starting from January 1985.
- LW_monthly_RDNS.xlsx is the file with the yields starting from January 1970.
- LW_monthly_yield_analyze.xlsx is the file with the yields starting from August 1971.
- LW_monthly_macro.xlsx is the file with the yields and the dates in the correct format in the 19'th column.
- INDPRO.xls is the file with the standardised and non-standardised industrial production data.
- CPIAUCSL.xls is the file with the standardised and non-standardised consumer price index data.