Fair Valuation of Embedded Options in Participating Life Insurance Policies

Master's Thesis in Quantitative Finance

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Abstract

Participating life insurance contracts are characterized by a minimum interest rate guarantee and pay dividends to its policyholder based on how well the issuing company is doing. These contracts are endowments that contain implicit options-like features such as minimum interest rate guarantees, stochastic annual surplus participation, terminal bonus and a surrender option. The stable returns are obtained through the combination of guaranteed benefits and non-guaranteed bonuses that are paid to the policyholders. In the existing literature the options are priced under strong assumptions, such as constant interest rates, and only take univariate risk factors, such as stochastic stock prices, into account. In this thesis, we investigate the impact of the term structure, the long-term investment, the price inflation, the mortality, the surrender behavior and the implication of multivariate risk on the fair pricing of the single premium life insurance policy. The fair contract can be priced by applying risk-neutral valuation methods and Monte Carlo simulation.

***Keywords*:** Participating life insurance contract; Risk-neutral valuation; Embedded option, Bonus distribution; Interest rate risk; Inflation risk; Lee-Carter mortality model; Surrender option; Multivariate risk; Monte Carlo simulation.

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This report documents a Master's Thesis in Life Insurance and is written at Watson & Wyatt after 5 years of studying at Erasmus University Rotterdam. The study concentrates on an evaluation of mark-to-market valuation of single premium participating life insurance policies.

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Rotterdam, the Netherlands Keith Wu
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1. Introduction

In a number of European countries, life insurance policies with profit-sharing are designed to provide the policyholder the right to either receive at retirement an assured accumulated fund or a life annuity with a series of regular payments. These products are appealing to the policyholders, not only because they are tax-deferred, but in particular because they offer a certain minimum interest rate guarantee to the insured's account in each policy year. The financial guarantees found in these contracts can be viewed as embedded options. Their popularity along with the recent market turmoil has highlighted the importance of adequately pricing these complex contracts.

The purpose of this work is to get insight into the pricing of minimum rate of return guarantees (MRRGs) embedded in a *participating (*or *profit-sharing)* life insurance policy such as described above. In these policies, the benefits credited to the policyholder are dependent on the performance of a specified investment portfolio, a so-called segregated fund. In practice, the benefits provided by the life insurance providers is usually the guaranteed rate of interest plus some bonus participation (i.e. dividends) entitled to beneficiaries at every policy year. Interest rate guarantees, ancillary bonus features and the possibility of early withdrawals of the contract (i.e. the surrender option) are common examples of implicit option elements. These implicit options can be very valuable and can thus represent a significant risk to the insurance companies issuing these contracts in case of insufficient risk management.

As a result of the significant downturn of the stock market and the decline of interest rates in the 1990's, the issued interest rate guarantees have moved from being far out of the money to being very much in the money. Several life insurance companies, such as *Equitable Life Assurance Society (ELAS)* in UK and *Nissan Mutual Life* in Japan*,* have gone into bankruptcy because they were unable to fulfill the liabilities imposed by minimum rate of return guarantees [see, e.g., Briys and de Varenne (1997)]. The concern over embedded options is also reflected in recent regulatory regimes. Moves towards harmonized accounting standards and more coherent solvency requirements on capital management, financial reporting now requires an evaluation of the market value of insurance liabilities, and thus embedded options, at fair value.[1] With a fair contract we mean a contract for which the premium paid by the policyholder is equal to the value of the contract. As opposed to the shortcomings of traditional deterministic actuarial pricing methods, initiatives such as the Financial Assessment Framework (*Financieel Toetsingskader - FTK*) for pension funds in the Netherlands and the Solvency II for (re)insurance companies in the

[1] Further research into this subject is deemed of common interest to the calculation of Market Consistent Embedded Value (MCEV), Economic Capital (EC), Asset Liability Management (ALM) and Product Pricing & Development.

European Union (Swiss Solvency Test in Switzerland) have been introduced. It is now clear that, in striving for higher returns, not enough adequate attention was paid to the underlying risks embedded in many life insurance products. The weak financial position of many insurance companies and the rise of new accounting and solvency regimes have highlighted the need to secure financial resources and improve risk management practices to meet retirement needs.

The growing interest in the field of an accurate valuation of embedded options in life insurance contracts is witnessed by the very large number of papers devoted to this issue. To our knowledge, however, a comparatively large portion of the academic literature in this area has been analyzing unit-linked or equity-linked guarantees, i.e. policies where the interest rate credited to the insured's account is linked directly and without lags to the return on some reference (equity) portfolio. Since the pioneering work by Brennan and Schwartz (1976) and Boyle and Schwartz (1977) using contingent claims theory, a great prominence has been given in the financial and actuarial literature to the issues of pricing and hedging equity-linked life insurance contracts.[2] Other papers that deal with guaranteed equity-linked contract are Boyle and Hardy (1997), Nielsen and Sandmann (1995, 1996a,b, 2002b), Bacinello and Persson (2002), Schrager and Pelsser (2004), Vellekoop, Vd Kamp and Post (2006) and Castellani et al. (2007). In equity-linked contracts, the minimum return guarantee can be identified as a European put option, and hence the classical Black and Scholes (1973) option pricing formula can be used to determine the value of the financial guarantee.

Participating life insurance contracts are far more complicated than equity-linked contracts. It is the bonus account and associated distribution mechanism which distinguishes what is commonly termed participating contracts from equity-linked contracts. The existence of a bonus account can be visualized as a buffer account on the liability side of the insurer's balance sheet, where investment surplus is set aside in years with good investment performance to be used to cover the annual guarantee in years when the investment return is lower than the guarantee. There are several ways in which this profit-sharing is realized. The interest rate crediting mechanism applied is often referred to as a *portfolio average method* or an *average interest principle* [see Grosen and Jørgensen (2000)], combined with a minimum guarantee.

The valuation of embedded options in insurance products focuses on participating products has been studied by Hipp (1996), Briys and de Varenne (1997), Grosen and Jørgensen (2000, 2002), Miltersen and Persson (2000, 2003), Lindset (2003), Kling, Richter, and Ruß (2006) and Plat and Pelsser (2008). One of closed to ours is the study by Grosen and Jørgensen (2000), who analyze a single premium participating policy in the Black and Scholes (1973) framework, and price by

[2] Extensions of their empirical works are made by Aase and Persson (1997) and Miltersen and Persson (1999) who analyze guarantees lasting for only two periods with stochastic interest rates respectively in a Vasicek (1977) and Heath-Jarrow-Morton (1992) framework.

means of Monte Carlo simulation and binomial lattice approach. The authors show that the typical participating policy can be decomposed into a risk free bond element, a bonus option, and a surrender option. A surrender option is an American-style option that entitles its owner to sell back the contract to the issuer at the surrender value. In case that the investor 'walks away', he/she obtains the account value whereas the reserves remain with the company. Further contributions regarding the fair premium evaluating problem with surrender option is given by Bacinello (2001, 2003a,b), Tanskanen and Lukkarinen (2003), Andreatta and Corradin (2003), Shen and Xu (2005), Bacinello, Biffis and Millossovich (2008).

The value of the embedded option is subject to a wide spectrum of risks, running from pure *financial* risk such as random interest rates, inflation rates and equity risk at one end to pure *actuarial* risk – the risk that the assumptions that actuaries (insurance statisticians) used to price a specific insurance policy may turn out wrong or inaccurate – such as mortality risk and surrender risk. From the literature provided here, very few, if any, of these papers have considered the valuation of participating policies under both actuarial and financial risks and its dependence structure. The downfall of the stock market and interest rates recently has proved to be particularly dangerous for the pricing of insurance contracts. This unrest, also fed by ageing discussion, i.e. the increase of life expectancy and the decline in mortality rates at adult-old ages, encouraged a broadly conducted debate on the pricing and hedging of embedded options and pension guarantees. In the insurance industry, unexpected improvements in life expectancy, the so-called longevity risk, may lead to low-frequency and high-severity losses in businesses which involve cash flows that are contingent on survival, e.g. life annuities. According to Koissi et al. (2006), past mortality projections have consistently underestimated actual improvements in mortality rates and life expectancy at both birth and age 65. Moreover, it should be realized that the existence of any financial incentives to surrender implies that there is an explicit risk of discontinuance. The possibility that clients may choose to exit the contract prematurely may adversely impact the company's financial position. The value of the guarantee contracts is also subject to price inflation. Generally, only inflation-proof pensions can protect pension beneficiaries from financial insufficiency after retirement. Furthermore, multivariate risks involved in stochastic time-series are often neglected when pricing embedded options.

This thesis takes a fundamentally different approach from those studies discussed above. The main contribution with respect to the existing literature is that we consider the embedded options under a more realistic framework. In particularly, we introduce a new approach that investigate the impact of financial risk factors such as interest rate risk, inflation risk, equity risk, and actuarial risk (including mortality risk and surrender risk), on the fair pricing of pension guarantees. In addition, we study the various sources of uncertainty under the presence of multivariate risks. Therefore, our results will give annuity writers better indications on the costs of profit-sharing life insurance contract, which makes our setup more realistic and robust to other studies.

To better understand how fair valuation is feasible, we construct a model for valuation of participating life insurance contracts, by extending the work of Miltersen and Persson (2003). According to Miltersen and Persson (2003), a profit-sharing policy account can be divided into three portions: the insured's account, the insurer's account and a reserve (bonus) account. The authors present closed-form expressions and Monte Carlo simulations to price the option values. In their model, any positive balance on the bonus account at the maturity of the guarantee contract, so-called *terminals bonus,* is rewarded to the policy owner, whereas the negative balance of the bonus account is covered by the insurance provider. However, in the paper proposed by Miltersen and Persson (2003), there has been little focus on the quantification of risks and in particular of the correlation of risks with each other. For instance, an assumption of their model is that the short-term interest rate is known and constant over the lifetime of the contract. Pension liabilities are of a long duration and changes in the short-rate may occur. Because of this, stochastic interest rates may be more appropriate than a constant rate. Furthermore, also mortality risk is not considered in their paper. Since mortality risk has significant financial impact on insurance policies and pension plans, it is now well-accepted fact that stochastic approaches shall be adopted to model mortality risk. Moreover, the authors only consider a single premium policy contract of European-style, whereas policies with surrender option of American-style are more appropriate from a practical perspective. The American type contract has an extra feature compared to the European type contract in that it can be terminated at the investor's discretion at the end of each year. In order to fill this gap, we concentrate on both European and American participating contract with single premium endowment. Additionally, both financial and actuarial risks are incorporated into the empirical study of embedded options. It is worth noting that these two classes of uncertainty can be analyzed separately since we have assumed that financial and demographic factors are independent. We also show how such contracts can be valued using stochastic modeling and Monte Carlo simulation.

Different numerical methods are presented to deal with various risk factors existing in the participating contract. The values of the embedded options in this thesis are measured using arbitrage-free option pricing techniques and assuming complete markets. We start by modeling the term structure of interest as a stochastic process using the one-factor Vasicek model (1977). The short inflation rate is assumed to follow an Ornstein–Ulhenbeck process. Provided that the underlying investment portfolio contains additional non-fixed income assets, the equity returns process is modeled by a standard geometric Brownian motion (GBM). Furthermore, the mortality intensity is modeled according to the Lee-Carter mortality model (1992) using Dutch population data with an extension in old-age modeling. With respect to the surrender risk, we perform the powerful Least Square Monte Carlo simulation in order to construct efficient estimator for American-style contract. We also show how to incorporate the multivariate risk into these pricing methods. The methodologies proposed here to value embedded option are all carried out by Monte Carlo simulations.

The analysis of numerical results led to several conclusions. Our numerical studies showed that the insurance company is mainly exposed to the equity risk, interest rate risk and the mortality risk. In particular, it turns out that the risk-neutral value of an insurance contract with stochastic short rates mostly exceeds the value of a contract with a constant or deterministic short rate for a comparable parameter choice. The reason why interest rate risk modeling is essential to insurance companies is because they reflect the expected future returns and at the same time interest rates are also used to discount future cash flows (pension benefits). We also found that the effect of the inflation rate risk on the contract value is relatively small due to the strong mean-reversion characteristic of the inflation process. In other words, the tendency to revert to its long-term inflation rate weakens the impact of inflation rate variability. Moreover, we argue that the multivariate risk modeling is important regarding the pricing of life insurance contracts and embedded options. The way how it affects the contract value crucially depends on the value of the correlation parameter. Furthermore, we show that the mortality risk influences the contract value considerably. This can be attributed to the uncertainty surrounding future mortality rates and life expectancy outcomes. With respect to the surrender option, it is unclear whether the possibility of early withdrawals of the contract positively or negatively impacts the risk-neutral value. It is shown that the surrender effect depends on the parameters used in the model. Additionally, the implication of asset allocation is investigated. We found that the fair values derived from the diversified portfolio are significantly lower when compared to the undiversified portfolio.

The remainder of the thesis is organized as follows. In Chapter 2, we provide an overview of the pension system prevailing in the Netherlands and explain the main principles of the new minimum supervisory requirements laid down in the FTK and Solvency II. We then describe the structure of the insurance contract in a complete market setting and introduce the general pricing framework for single premium policies with guaranteed return embedded in segregated funds in Chapter 3. In Chapter 4, we present the model setup and analyze the impact of financial risks on the fair value of minimum rate of return guarantees in a stochastic framework. This chapter also includes the formulation of the methodology of multivariate risk. In Chapter 5, we provide a detailed empirical study of mortality risk using the original Lee-Carter model. Chapter 6 is devoted to the pricing of surrender options and describes the LSMC methodology. In the final chapter, we summarize our results and give prosperous directions for future research.

2. Public Pension System and Regulatory Rules

In this chapter, we present a brief outline of the Dutch pension system and discuss the current developments in the area of pension reforms in the Netherlands. Furthermore, we motivate the use of market-consistent valuation of pension liabilities. Solvency requirement by regulators are one major reason for it. We end the chapter by providing an overview of various participants and their role with respect to the insurance contract that will be analyzed throughout the thesis.

2.1. Brief overview of the Dutch pension system

As in most developed countries, the Dutch pension system may be characterized in terms of three pillars, namely a flat-rate state pension related to minimum wages, occupational pension schemes which are capital-funded, and individual saving plans.

The first pillar is based on two acts: the General Old Age Pensions Act (*Algemene Ouderdomswet - AOW*) for all residents living in the Netherlands aged 65 and over and the National Survivor Benefits Act (*Algemene Nabestaandenwet - ANW*) for people whose partner has passed away. The AOW is an anti-poverty pillar that is non-contributory, guarantees a decent minimum income at old age, and is primarily dependent on government funding. The AOW contribution is financed by everyone younger than 65 years with a taxable income and is based on a pay-as-you-go (PAYG) framework.[3]

The second pillar (occupational pension) plays a crucial role in the Netherlands and is considered as supplementary to the AOW state pension. More than 90% of the Dutch labor force is covered by an employer-sponsored funded pension plan. The Netherlands' supplementary pension system mainly consists of funded defined benefit (DB) plans. These plans are organized in one of three ways: by company-specific (*Ondernemingspensioenfonds - OPF*), by industry-wide (*Bedrijfstakpensioenfonds - BPF*) or by group pension agreements with insurance companies.[4] OPF are funds set up for the use of accumulating funds only for the purpose of one company or a group of companies, whereas BPF are funds set up for a specific sector of industry, e.g., the Dutch ABP pension fund who covers the pension scheme of government and education workers. The general aim of occupational pension scheme is to supplement employee's future AOW pension to around 70% of the salary. In contrast to the first pillar, the benefits from the second pillar are not flat, but are related in some way to contributions.

[3] In this setup, the total contributions paid by employees are used to pay pensions to current pensioners and no pension capital is set aside for future pension payments.

[4] According to the research of the Dutch Central Bank (*De Nederlandse Bank - DNB*) in 2006 [see Bikker et al. (2009)], two-thirds of the system's total assets and 85% of the plan participants is hold by industry-wide funds. Company-specific funds encompass 30% of the remaining assets and 15% of the plan participants. The market size of group pension agreements is mostly very small.

The third pillar is considered a voluntary savings pillar, available to anyone who is not satisfied with the mandatory state pension system and is willing to supplement the retirement income provided by the first two pillars. Contributions paid into private plans are invested in some way to obtain endowment insurance (lump sum) at retirement or a predetermined periodic payout until the death of the client (life annuity). This third pillar of the Dutch pension system is relatively small.

The present outline focuses in particular on the second pillar since it is closely related to the valuation of interest rate guarantees. We shall later explain the corresponding economic meaning and motivation that leads to market-based values for these products.

2.2. Pension Agreements

Within the second pillar, the nature of pension arrangements agreed between the employer and the employee may be classified as defined-benefit (DB) or defined-contribution (DC) according to how the benefits are determined. The majority of Dutch pension contracts are DB contracts, where the benefit payments upon retirement are predetermined by a fixed formula which usually depends on the member’s salary and the number of years of service. In DB agreements, financial or longevity risks – a change in value caused by the actual mortality rate being lower than the one expected – are typically borne by the scheme sponsor. In contrast, a DC plan will provide a payout at retirement that is dependent upon the amount of money contributed and the performance of the investment vehicles utilized. The investment risks during the accrual stage are therefore borne by the individual.

Retirement arrangements have been predominately DB pension plans, but over the past decade, there has been a shift toward DC arrangements. There are number of factors which have been used to explain the decline of DB schemes. The main motivation behind most DC proposals is to enhance the managing of employees’ costs. In such a plan most of the funding responsibility is shifted from sponsors to employees, reductions in fund values due to stock market crashes are contributed to employees only. In contrast, this will not be possible under a DB scheme where the employer is required to maintain the agreed levels of benefit irrespective of how well the investments are performing. Furthermore, the extra burden related to increased longevity and lower interest rates makes DB plans expensive to maintain. This development, together with increased emphasis on market-consistent valuation in regulatory and accounting rules, has lead to a switch of DB plans to DC plans throughout the world.

Although DC is becoming more fashionable, there are still concerns remaining. A major concern with DC plans is whether all employees have the ability to invest their funds wisely in order to maintain an adequate pension income for future retirement. The credit crisis has dealt a heavy blow to the global financial sector and has highlighted once again that this might not be a socially desirable outcome. To mitigate this problem, insurance companies usually insert into the contract a minimum guarantee that assures the policyholder to receive at maturity at least a predetermined sum [Bodie and Crane (1999)]. For such contracts, realistic valuation techniques and an implementable risk management methodology must be available.

2.3. Regulatory regimes

2.3.1. Financial Assessment Framework (FTK) for pension funds

Since 1 January 2007 the new Dutch Financial Assessment Framework (*Financieel Toetsingskader - FTK*) relating to pension funds has been in place. The FTK is the part of the Pension Act (replacing the old Pension and Savings Funds Act, which goes back to 1952) that lays down the statutory financial requirements for pension funds. This regulation states that it is intended to improve the insight of both the supervised institution as well as the supervisory authority into the institution's financial position and its possible development over the short and medium term. The FTK is built around the principles of mark-to-market valuation, risk-based financial requirements and transparency. The goal behind these initiatives is to protect the policyholder from the consequences of an insolvency of a pension fund. Below, we will outline these principles in more details.

First, according to the new regulatory rules the valuation of liabilities of pension funds should be implemented in a market-consistent way. In the past, assets are mostly valued at market prices; liabilities – as far as they relate to contractual obligations to the insurance client – are measured by established actuarial methods such as a fixed discount rate rather than market-based one. The use of deterministic methods allowed life insurance companies to ignore the sources of variability, and in particular the occurrence of extreme events. For the valuation of the pension liabilities FTK promotes the use of the current nominal term structure of interest rates. More specifically, the FTK prescribes the use of an interest rate swap curve. Second, the FTK manages the risks trough consistent strengthening of the solvency requirements. The pension funds are subject to a series of stress tests that are expected to encourage more investments in fixed income (see Appendix A).[5] Third, the FTK is driven by a growing awareness of the need for transparency, corporate governance and the safe guaranteeing of the assets in the pension fund.

[5] As aforementioned, a pension fund must have sufficient own funds to meet its obligations. The FTK compromises two conditions on the solvency. First, the solvency should be at least 5% of the liabilities. If not, a recovery period of one year is allowed. Second, a confidence of 97.5% that the value of the fund's investments will not less than the level of the technical provisions within a period of a year is required. A recovery period of 15 years is allowed to meet the requirement.

The financial crisis that broke out in September 2008 has unfavorable consequences for the Dutch pension funds. Many pension funds faced a funding ratio deficit, as their portfolio was not build to protect against extreme shocks. The required coverage ratios imposed by the FTK came under pressure. Pension liabilities (e.g. pension guarantees) which are valued on a mark-to-market basis have been severely influenced by the low interest rates. Together with the poor investment performances, the matching of asset and liabilities becomes more complicated and is likely to drive pension funds to favor fixed income over corporate equities. The fear of falling behind the 105% ratio is expected to push Dutch pension funds to increase pension contributions and (or) to reduce pension benefits. From these perspectives, it is thus crucial that risk management must be secured more tightly in the governance structure of the enterprise.

2.3.2. Solvency II for (re)insurance companies

In the past few years, we have observed several failures of financial companies, for instance Barings Bank (1995), HIH Insurance Australia (2001) and Lehman Brothers (2008). It is therefore essential that risks are known, specified and controlled by the management.

New European regulations for insurance companies, known as the Solvency II guidelines, will govern the capital requirements of (re)insurance companies in the European Union. The aim of Solvency II is to improve insurance regulation and supervision by introducing a more advanced risk based economic approach. Solvency II bases supervision more on the concept of 'exposure to risk' and therefore offers more securities and a more intelligent system than the old regime, i.e. Solvency I.[6]

The new regulatory rules show strong affinity with the Basel II of the banking sector and local initiatives like the Swish Solvency Test. Similar to the banking supervisory system, Basel II, The European Union Solvency II system is based on a three-pillar approach. The first pillar includes quantitative requirements such as assets, liabilities and capital valuation. The second pillar focuses on the supervisors and their review process such as risk management as well as company's internal control. The third pillar addresses supervisory reporting and public disclosure of financial and other information by insurance companies. The latter case will promote the market discipline and greater transparency, which will help to ensure the stability of insurers and reinsurers. For a complete overview and discussions of Solvency II we refer to Sandström (2007).

[6] Solvency II should solve a number of serious shortcomings of the current Solvency I regulations. For example, under Solvency I, only liability-driven risk is taken into account and also in a rather simplified way. No distinction has been made between different levels of asset allocation regarding the solvability buffer: the required capital for a 70% equities and 30% bonds asset allocation is the same as for a 30% equities and 70% bonds asset allocation, while the corresponding balance sheet risks are obviously completely different.

2.4. Outlook, pension scheme reform and policy debates

Like other countries, the Netherlands has adapted and reformed its pension system during the last few decades. In this subsection we review some the most remarkable pension reforms observed in the Netherlands: (i) shift from final to average earnings contracts; (ii) absence of complete indexation; (iii) potential increment of the AOW retirement age.[7]

As already stated, many companies in the United States, Europe and Japan have shifted from defined-benefit (DB) to defined-contribution (DC) schemes. In contrast, Dutch pension plans have mainly preserved their DB character in recent years, although they have switched from 'final-pay' to 'average-wage' schemes. The average-wage plans may be better viewed as hybrid DB-DC schemes. In the postwar period, workers are in general eligible for a pension covering 70% of the last earned income after 35-40 years of paid work (final-pay scheme). Nowadays, this standard has been modified. A total pension based on average-wage scheme is considered sufficient. In the case of an average-wage scheme, individuals accrue pension rights annually based on the salary earned in each year of their pensionable service (rather than the final salary scheme). According to a survey of the Dutch Central Bank (DNB), the majority of the contracts (56.1%) were final-pay contracts in 2001, whereas in 2005 the percentage of active participants with a final-pay and an average-wage scheme are respectively 10.6 and 74.3.

Another important development is that the common practice of granting full indexation (correction for pension rights for inflation) by pension funds is stopped. In the Netherlands, indexation of pension benefits to either wage or price inflation – depending on the terms of the pension contract – has long been considered a guaranteed right. Due to adverse economic circumstances and international accounting rules, many pension funds became severely underfunded.[8] This has resulted in grave uncertainties in the indexation of many pension schemes. In order to recover, pension contributions were increased and indexation cuts are bound to be implemented. Bikker and Vlaar (2007) analyze the conditional indexation in DB pension plans in the Netherlands and provide recommendations regarding the various types of indexation and contribution policies. Without full indexation, the accrued pension benefit at retirement may therefore not reach the 70% target of final or average salary schemes. At this moment, the vast majority (85%) of pension fund members build up pension according to a conditional average-wage DB type.

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[7] In March 2009, the Prime Minister Jan Peter Balkenende of the Netherlands made a formal proposal to raise the legal retirement age from 65 to 67. Till today, no official agreement has been reached between the Dutch government and the respective labour unions and employee associations.

[8] Low interest rates imply lower expected future returns, raising the discounted costs of future pension benefits. Under the previous regulatory regime, effective until end-2006, the maximum allowed actuarial interest rate, used to calculate the levels of contributions and the funding ratio of the fund, remained constant at 4%.

More recently, the Dutch government has announced a proposal to raise the official 'AOW retirement age' from 65 to 67 in an effort to improve state finances and to cope with the contraction of the working population which finances the Dutch state pension. This issue has caused extensive discussions among politicians, unions for employees and public opinions. In this thesis, the valuation of profit-sharing contracts is based on a pensionable age 65. Any deviations from the legal retirement age should be easily implemented by a minor modification of the model.

2.5. Review of pension participants

Occupational pension schemes are mainly arranged by pension funds or life insurance companies. The participation structure of various pension participants subject to regulatory supervision is presented in the following figure.

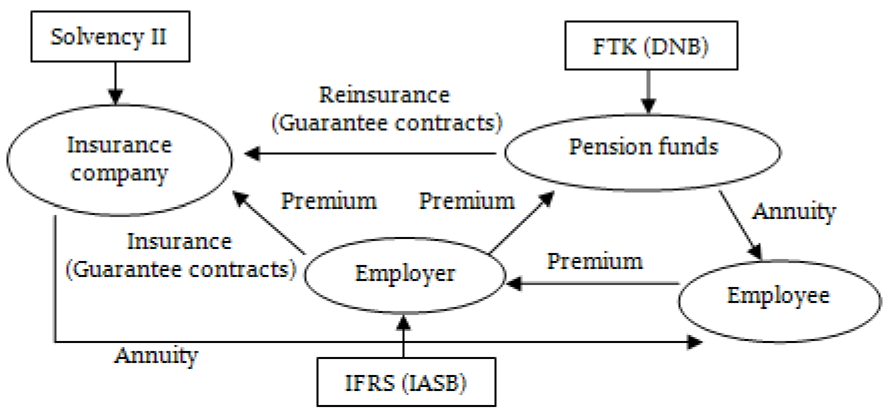

Fig. 1. This figure provides a general overview of the linkage between the different pension participants and its connections with (international) financial regulation regimes. Annuity provided to employees is only available after the retirement.

We start by analyzing the role of pension funds and insurance companies in more details. The main task of a pension fund is to organize the investment of employees' retirement funds contributed by the employer and employees. At the retirement age, accrued pension benefits are provided to beneficiaries. Both industry and company pension funds are faced with two major decisions: 1) provide pension plans and bear the various risks involved; or 2) transfer the risks to a third party by issuing an insurance contract (i.e. reinsurance).[9] For the latter, extra risk premiums are required by the (re)insurance company for bearing the risks. On the other hands, guarantee

[9] An insurance contract is characterized by the agreement that one party (e.g. reinsurance company) accepts significant insurance risk from another party (e.g. insurance company or pension fund).

contracts may also directly implied by insurance companies without involvement of a pension fund.

As a result of the recent economic recession, higher contributions are required from active pension or life insurance plan participants, indexation cuts are bound to be implemented, or even defined benefits to beneficiaries may be reduced. Pension funds are allowed to implement these measures whenever a compelling situation arises. For those reasons, pension funds are authorized to invest greater capital amounts in high-risk assets, such as equity share, real estate and mortgages. Insurance companies, on the other hand, do not have the ability to implement these measures in case of insolvency. Disastrous consequences of mismatching the risk associated with pension guarantees (implicit options) are thus critical issues for insurance companies. The objective of this thesis is to conduct further research into the valuation approach of minimum of return guarantees, in order to assess the risk profile of embedded options and mitigate potentially large losses from financial guarantee contract for particularly insurance companies.

As discussed in Section 2.3, the European Commission has set up a new framework for the supervision of the insurance sector titled Solvency II and the statutory financial requirements for pension funds is presented in the FTK. On top of that, the market-oriented approach has been also promoted by the international accountancy standards (IFRS), as compiled by the International Accounting Standard Board (IASB), to encourage financial institutions (employers) to value their balance sheets at current market prices. At this moment, the insurance liabilities are reported based on their book value, where economic assumptions are often not directly linked to the financial market. The main drawback remains the inability to value the costs of options and guarantees.

Finally, we mention that the supervision and intervention measures should be implemented in a coordinated fashion and at international level. This would demand an improved mutual harmonization of the European supervision and the associated local authorities such as the Dutch Central Bank (DNB) and the Netherlands Authority for the Financial Markets (AFM). DNB and AFM are responsible for monitoring the financial position of pension providers and assess whether they are financially sound and able to fulfill their future obligations.

3. Insurance Contract and Embedded Options

In the previous chapter we have reviewed the Dutch pension system and motivated the use of market-based valuation of liabilities. In this chapter we discuss the primary characteristics of minimum rate of return guarantees in Section 3.1. We then describe the linkage between minimum interest rate guarantees and embedded options in Section 3.2. Section 3.3 is devoted to the description of the insurance contract that we analyze in this thesis. In Section 3.4, we present closed-form solutions for the initial market value of insurance accounts under the assumption of deterministic interest rates. We then provide the fair pricing principle in Section 3.5. We treat numerical examples in the final section of this chapter.

3.1. Background on minimum rate of return guarantees

Minimum rate of return guarantees (MRRG) are important elements of many insurance contracts offered in today's financial market. The main characteristic of MRRGs is to provide benefits contingent on survival or death of individuals. These guarantee contracts are embedded in various life insurance products in the Netherlands, such as equity-linked (unit-linked) products and profit-sharing (participating) contracts.

Under a defined benefit (DB) pension arrangement, the value of an equity-linked contract is directly linked to the performance of a portfolio of assets associated with the contract, such as a mutual fund, a certain stock, a stock index or a foreign currency. The guaranteed minimum is then binding only at the expiration of the contract. The return of a profit-sharing contract is determined periodically until the termination of the contract. The profit-sharing insurance contract stands in contrast to equity-linked products in that interest is credited to the policy periodically according to some bonus distribution mechanism which smoothes past returns on the life insurance company's assets. Intuitively, profit-sharing contracts are more valuable than equity-linked contracts [see Finkelstein et al. (2003)].

In order to gain a better understanding of the valuation of the two types of guarantees, we refer to a pure endowment policy as an example. With respect to a policy with term T, the contract holding period, $[0, T]$, is defined as sub-periods $0 = t_0 \leq t_i < t_n = T$, for $i = 0, \dots, n-1$. For convenience we assume the initial investment for both cases is 1.

In an equity-linked product, the benefit payable is determined by comparing the accumulated investment returns with the accumulated guaranteed returns at end of each contract period. For the holding period $[0, T]$, the benefit payable can be written as:

$$\mathrm{EL(t)} = \exp\left\{\max\left[\sum_{i=0}^{n-1} \delta_i\,, r_G T\right]\right\}, \tag{3.1}$$

where δ_i denotes the rate of return of the investment portfolio during the i-th policy period $[t_{i-1}, t_i)$, and r_G denotes guaranteed rate of return (usually 3% or 4% in the Netherlands).

In a profit-sharing policy, the benefit payable is adjusted according to the achieved investment return adjusted with some bonus distribution mechanism, i.e. δ_t^*, and the level of minimum guaranteed return for each period. These types of guarantees are often referred to as 'cliquet-style' guarantee, where guaranteed amounts are re-set periodically. The payoffs of a profit-sharing contract are given by:

$$\mathrm{PS(t)} = \exp\left\{\sum_{i=0}^{n-1} \max[\delta_t^*, r_G(t_{i+1} - t_i)]\right\}. \tag{3.2}$$

Note that the reader might not find such contracts in the market given by expressions (3.1) and (3.2) due to the simple nature of the contract. In this thesis, we limit our attention to the case of profit-sharing policies. We refer the reader to Brennan and Schwartz (1976) and Boyle and Schwartz (1977) for considerations on equity-linked guarantees.

Why study profit-sharing policies instead of equity-linked policies?

The focus of this thesis is on participating life insurance contract embedded in segregated investments. This is motivated by several reasons. First, a large body of the existing literature is dedicated to the valuation of options embedded in equity-linked insurance products, whereas to our knowledge there has been little focus on profit-sharing policies, despite this being one of the most common life insurance contract sold in Europe. Second, in terms of market size, participating life insurance products are considered to be the most important modern life insurance products in major insurance markets around the world. Third, participating contracts are considered more risky since the risk from investments is borne by the insurer. In an equity-linked plan, the investment risk is however switched back to the policyholder. With the current bearish equity markets and fair value calculations at the center of attention, it should be realized that these options are in or at least at the money. Finally, the existence of the surplus distribution mechanism makes the calculations of financial guarantee contract even more complicated. Indeed, many insurance company executives around the world have indicated that they regard the pricing of guarantees and options embedded in insurance contracts as the most important and difficult financial challenge they face.

Purpose of a pension guarantee

Guarantees in insurance contracts are offered for many reasons. We can indentify three potential purposes served by minimum rate of return guarantees: (i) pension guarantees increase competiveness by providing the retirees with additional certainty; (ii) financial regulations require guaranteed minimum cash values – so that no one falls below a particular income threshold after retirement; (iii) requirements to receive favorable tax treatment (either for the insurer or for the policyholder).

Various types of profit-sharing policies

The tariffs rate of Dutch insurers was based on the technical interest rate of 4% until August of 1999. Since it is clear that a better return can be made in the market than the interest basis of 4%, many insurance contracts offer profit provisions. The current technical interest rate in the Netherlands equals 3%.

One of the most common forms of profit-sharing in the Netherlands is based on a moving average of the so-called u-yield. The u-yield is the average of six u-yield-parts, where the subsequent u-yield-parts are weighted average of an effective return on a basket of government bonds with maturity varies between 2 and 15 years. Contracting the u-yield, the building blocks of the t-yield provide a less accurate estimate of effective return on investments. The t-yield compromises the weighted average return on government bonds with maturities of at least 7 years. From this point of view, the u-yield is more representative for the effective return on investments and is therefore preferred (source: Pensioengids 2008).

Broadly speaking, there exist various types of profit-sharing provisions in the Dutch life insurance market:

1) Profit-sharing based on TL-discount or UL-discount
2) Profit-sharing based on TL-discount or UL-discount with continuation discount
3) Profit-sharing based on excess return u-yield
4) Profit-sharing of the so-called segregated funds, where regular and terminal bonuses are given though the life of the product, based on the return of the underlying investment portfolios

Throughout the thesis, we will focus on the fair valuation of minimum rate of return guarantees with profit-sharing based on segregated investments. We refer the reader to Appendix B for a further explanation regarding the different types of profit-sharing insurance contract existing in the Netherlands.

In other parts of the world, examples of life insurance policies include guaranteed minimum death benefits (GMDB) as well as guaranteed minimum living benefits (GMLB) in the US, and guaranteed annuity options (GAO) in the UK. The pricing of GMDB and GMLB has been tackled by several authors, e.g., we refer to Milevsky and Salisbury (2002) and Bauer, Kling and Ruß (2006). With respect to the valuation of GAOs, see Boyle and Hardy (2003), Pelsser (2003), Ballotta and Haberman (2003) and Biffis and Millossovich (2006) for more information.

In the next section, we set out in more detail what we mean by the additional cost of guarantees and to explain the link between investment guarantees and embedded option in a general context. Before any further developments, we also draw the attention on the fact that the product pricing is coordinated in such way that is consistent with the principles of the Dutch pension system and the regulatory rules as discussed in Chapter 2.

3.2. Understanding the costs of guarantees and the link with embedded options

A barrier to market consistent valuation of pension guarantees is the lack of transparency in costs of insurance products. Provided that an insurer promises to pay a certain quantity, the valuation of the pension benefits is usually straightforward. Problems arise when a guarantee or option is issued by the greater of two values, say, A and B, where it is not certain which of the two is greater at the expiration date. Suppose that premiums paid or the accumulation of premiums is invested in an equity-linked fund, then it is uncertain which will be the greater amount and therefore the final benefit. In order to provide an adequate pricing approach, it is reasonable to look at the different cost components and verify which one reflects the uncertainty and thus the value of guarantee. The cost of the guarantee contract consists of two components: realized benefit (B) and additional cost (A-B) which is related to the uncertainty around the performance of the equity index-tracking fund.

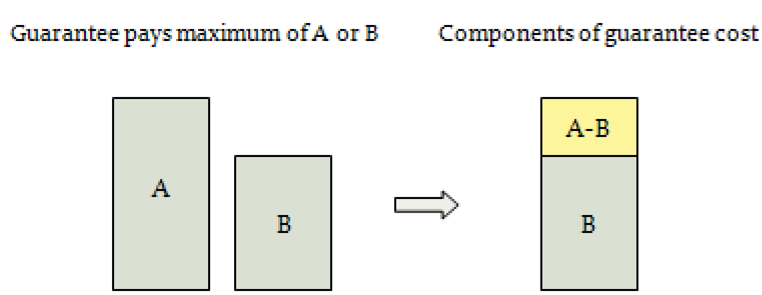

Fig. 2. The cost of the guarantee consists of the cost of the normal benefit B (realized benefit), the guaranteed minimum (A) and the guarantee cost (A-B).

Suppose that the guarantee contract is valued as the greater of A and B, where A is the minimum guaranteed rate and B is the rate of return of the investment portfolio. In this case, the insurance contract is very similar to a put option with a fixed strike price A. The extra guarantee benefit is therefore equivalent to the payoff of the put option (i.e. A-B). For clarity we will illustrate the option-like representation of participating benefits by the following example:

> *"Consider a European put option with a strike price of €4.00 and a maturity of 12 months. This is very similar to an insurance guarantee that has promised to invest the policyholder's premium in the relevant stock and has guaranteed to provide a benefit to the policyholder in 12 month's time, based on a minimum stock price of €4.00. If the stock price drops below €4.00 (say €3.00) after 12 months, then exercise of the put option would yields €1.00 and provides sufficient benefit to the policyholder. If the stock price cumulated above €4.00, then the put option becomes worthless. However, the insurance company would be able to provide the promised benefit of €4.00 plus the remaining profit after selling the stock."*

The option-like representation described above is presented in a rather simplified way. With respect to the fair valuation of profit-sharing contract, the pricing of embedded option becomes difficult due to the existence of the bonus policy as mentioned earlier. On top of that, the endowments can be cashed in early and the holder then receives a lump sum (i.e. surrender value) instead of any future death or maturity benefits. The surrender value is determined by the insurance company depending on how long the policy has been running and how much has been paid in to it. In life insurance contracts this is labeled the surrender option or American-style option. A contract without a surrender option is called European. Put differently, the European-style contract is defined as the contract which pays off at the expiration date. In this thesis, we consider both the American and European types of the MRRG with profit-sharing.

3.3. Contract design

In this section, we develop a theoretical framework to analyze fair single premium participating life insurance contracts with a minimum guarantee. A fair contract is a contract where the initial market value of capital inflow (premiums) equals the initial market value of the capital outflow (benefits). We wish to study a savings plan which yields a guaranteed interest rate plus a participation in the positive excess return of a given investment portfolio.

We consider the pension plan of a single representative participant aged x at time zero entering a *single premium* participating life insurance policy and maturing T years after as presented in Fig. 3. Let P_0 be the initial amount deposit by the investor of age x in an account A at the inception of the contract. We assume no additional payments are done after the inception of the contract. The contract is linked to the segregated fund, e.g. a basket of stocks and bonds, for an investment horizon of T years. In a DB plan, policyholders insured with minimum guarantees are eligible to either receive an assured accumulated fund (i.e. lump sum) or a predefined annuity after the retirement date (i.e. An). The annuities are intended to provide the annuitant with a steady stream of income after the retirement date upon to death. Since we are interested in the fair valuation of embedded options, we focus rather on the discounted lump sum value instead of requiring the annuity value. Under these contracts, the insurer is obliged to pay a specified amount of benefit to the beneficiary of the insured client if he/she dies (*death benefit*) or surrenders (*surrender value*) within the term of the contract. More precisely, we assume that in the case of death during the t-th year of contract, $0 < t < T$, a death benefit is assigned to the beneficiary at the end of the year. The amount of the death benefit is equal to the accrued policyholder's account at the end of the year of death. The contract can also be terminated depending on the policyholder's discretion before time T. In that case, the insured is eligible to receive the surrender value reflecting his/her past contributions to the policy, minus any costs and charges incurred by the insurance company. Note that surrender can only occur at integral times.

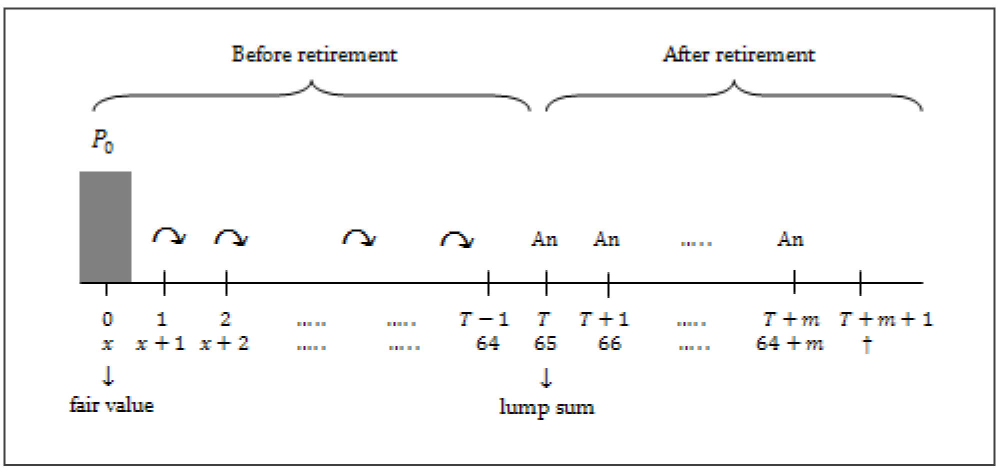

Fig. 3. The policyholder deposits an initial amount P_0 in an account A at the inception of the contract and invested for T years. Policyholders insured with minimum guarantees are eligible to either receive an assured accumulated fund (i.e. lump sum) or a predefined annuity (i.e. An) after the retirement age (i.e. 65). The fair value (or the present value) of the insurance contract is determined at time 0.

Next, we present the valuation framework in which participating life insurance contracts can be valued and analyzed in a simplified risk environment. We build upon the methodology proposed by Miltersen and Persson (2003), who analyze the valuation of a *single premium* life insurance policy with and without a surplus distribution mechanism. Along the lines of Miltersen and Persson (2003), the instantaneous interest rate is assumed to be deterministic and constant during the investment horizon. The return of risky asset is governed by a standard geometric Brownian motion.

The modeling of the insurance business and its associated risks in this thesis has necessarily required a number of simplifying assumptions. In the basic valuation framework we crucially rely on the assumption that the effect of mortality, surrender and multivariate risk are not presented. More specifically, the policyholder is assumed to be alive until the age of retirement (death probabilities are set equal to zero within the contract period); the possibility of early withdrawals of the policy is not permitted; no attention is paid to the dependence structures among various risk factors. Unlike Miltersen and Persson (2003), we consider price inflation by assuming that the annual inflation rate equals the expected inflation $\bar{\pi}$. As we proceed, we will relax some of these assumptions when the valuation framework becomes more realistic. For the ease of exposition, we also neglect all types of costs, and assume the market is complete and perfectly liquid.[10] Moreover, the effects of financial regulations and management decisions concerning asset allocation or surplus distribution are not considered in this research.[11]

The financial characteristics of the contracts between the insurance company (or pension fund) and the investor are defined as follows. The insurance provider promises the investor an annual rate of return on the account A in year t equal to

$$r_G + \alpha[\delta_t - r_G]^+, \tag{3.3}$$

where r_G denotes the guaranteed rate (also referred to as technical interest rate) which is assumed to be constant during the contract period, δ_t is the return gained from segregated investments portfolio in year t, and α is the fraction of the positive excess rate of return which is credit to the investor's account. The + superscript presents the positive part of $\delta_t - r_G$, i.e. $\max(\delta_t - r_G, 0)$. Recall that $\alpha = 0$ corresponds to the extreme situation where surplus is never distributed (all surplus remains at the insurance company). This special case resembles an equity-linked policy including an annual minimum rate of return guarantee, but without a surplus distribution mechanism.

The insurer will make a loss on these policies if the guaranteed payout is greater than the asset share. Therefore the insurer needs to charge the policyholder for providing these guarantees. Following Miltersen and Persson (2003), the insurer is eligible to receive a fraction of the obtained gain from investments to cover the costs made by the insurance company. We

[10] A financial market is said to be complete if the equivalent martingale measure is unique and all contingent claims can be hedged perfectly and priced uniquely (e.g. Balck and Scholes (1973) and the CRR model proposed by Cox et al (1979)).

[11] For more details on the evaluation of the management decision on the embedded option value, we refer the reader to the article of Kling, Rochter and Ruß (2006).

model this by adding a fraction β of the excess rate of returns to the insurer's account, denoted by C. The parameter $\beta \in [0,1]$ determines the share of the positive surplus that is distributed to the insurer and hence can be associated with the required risk premium by insurer. The benefits is defined as

$$\beta[\delta_t - r_G]^+. \tag{3.4}$$

Further note that we are also subject to the constraint $\alpha + \beta \leq 1$. When $\alpha + \beta < 1$, the remaining share of the excess investment return, i.e. $1 - (\alpha + \beta)$, is attributed to the reserve account (R).

One major feature of these investment plans is the participation in surplus distribution which is arranged between the policyholder and the insurer. Typically, the insurer employs a specified rule of surplus distribution, namely *average interest principle*, to credit interest at or above a specified guaranteed rate to the policyholders every period. The surplus distribution mechanism with a reserve account R serves as a buffer to protect the policy reserve from unfavorable fluctuations in the asset base. The reserve account can be used to back up the insured's account A in case that the realized annual rate of return on assets is below the base rate. In particular, the bonus account can become negative, but the insurer has to consolidate a negative balance at the end of the insurance period. Any further non-guaranteed bonus is paid to the policyholders at the expiration known as a terminal bonus.

In fact, the participating life insurance contract can be viewed as two components, namely the terminal insured's account ($A_T + R_T^+$) and the terminal insurer's account ($C_T + R_T^-$). The insured's account is decomposed into a fixed guaranteed payment (bond), a bonus option value which gives the insured client a fraction of the surplus, and a terminal bonus which depends on the investment performance over years (i.e. profit-sharing contract without surrender). The insurer's account consists of a fraction of the investment surplus (i.e. risk premium) and the possibly loss on the terminal bonus account at expiry. To derive the American version of the participating life insurance contract, the surrender option is then given by the difference between the value for the nonsurrenderable participating contract and the value for the American-style contract.

In order to obtain the fair value of the contract, we need to specify the three accounts (i.e. A, R and C) separately. At the end of year t, the balance of the insured's account A is simply the previous year's balance accrued with the guaranteed minimum rate of return r_G, and the fraction of positive excess rate of return given by $\alpha(\delta_t - r_G)^+$. Hence we have

$$A_t = A_{t-1} e^{r_G + \alpha(\delta_t - r_G)^+} = P_0 e^{\sum_{i=1}^{t}(r_G + \alpha(\delta_i - r_G)^+)}. \tag{3.5}$$

The balance of the insurer's account C is denoted by

$$C_t = C_{t-1} + A_{t-1}\left(e^{\beta(\delta_t - r_G)^+} - 1\right). \tag{3.6}$$

The last term of Equation (3.6) represents the share, i.e. β, of the excess return obtained in year t that is credited to the insurer. The expression in (3.6) can be rewritten as

$$C_t = \sum_{i=1}^{t}\left(e^{\beta(\delta_t - r_G)^+} - 1\right)A_{i-1}. \tag{3.7}$$

The remaining amount is credited to the bonus reserve account R which is residually determined as

$$R_t = Pe^{\sum_{i=1}^{t}\delta_i} - A_t - C_t. \tag{3.8}$$

In case that the bonus reserve account is negative at date T, the insurer is obligated to cover the potential deficit on the account R. Then, the accumulated value in account C represents the profit obtained by the insurer for issuing the minimum guarantee contract. It is obvious that the insurer may either lose or win money. To further illustrate the balance account, we consider the next example:

Table 1
Example of distributions between account for a given scenario

Year (t)	Return (δ_t)	Excess return ($\delta_t - r_G$)	Investment account (P_t)	Insured's account (A_t)	Reserve account (R_t)	Insurer's account (C_t)
0	–	–	100.00	100.00	0.00	0.00
1	15%	12%	116.18	109.42	3.72	3.05
2	5%	2%	122.14	113.88	4.66	3.59
3	−5%	−8%	116.18	117.35	−4.76	3.59
4	10%	7%	128.40	125.23	−2.50	5.67
5	20%	17%	156.83	140.49	5.23	11.10

Numerical example of the distribution of returns between the different accounts with the following parameter values: minimum guaranteed rate $r_G = 0.03$, profit-sharing rate insured $\alpha = 0.50$, profit-sharing rate insurer $\beta = 0.25$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$ and $T = 5$.

The first year's realized rate of return, 15%, is distributed as follows: account A at $t = 1$ is attributed with the amount, $A_0 e^{r_G + \alpha(\delta_t - r_G)^+} = 100e^{(0.03+0.5(0.15-0.03)^+} \approx 109.42$, account C is attributed with the amount $C_0 + A_0\left(e^{\beta(\delta_t - r_G)^+} - 1\right) = 0 + 100\left(e^{0.25(0.15-0.03)^+} - 1\right) \approx 3.05$ and reserve account R is credited with $P_0 e^{\sum_{i=1}^{1}\delta_i} - A_1 - C_1 = 100e^{0.15} - 109.42 - 3.05 \approx 3.72$.

The distribution of the second year's return is similar. However, the realized rate of return obtained in the third year is negative (i.e. -5%) and the minimum rate of return guarantee is triggered. Account A is credited with the amount $A_2 e^{r_G+\alpha(\delta_t-r_G)^+} = 113.88e^{(0.03+0.5(-0.05-0.03)^+} \approx 117.35$, account C is credited with the amount $C_2 + A_2\left(e^{\beta(\delta_t-r_G)^+} - 1\right) = 3.59 + 113.88\left(e^{0.25(-0.05-0.03)^+} - 1\right) \approx 3.59$ since no profit is shared with the insurer, account R at $t = 3$ becomes $P_0 e^{\sum_{i=1}^{3}\delta_i} - A_3 - C_3 = 116.18 - 117.35 - 3.59 \approx -4.76$, where the amount 9.43 ($= R_2 - R_3$) is subtracted from reserve account R to compensate the loss on investments.

Subsequently, positive returns of 15 and 20% are obtained in year four and five respectively, therefore we end up with a positive reserve account. Given that the contract matures at date five, the investor now receives the balance on account A and R, in total 145.72, whereas the insurer receives the balance of 11.10 of the insurer account. It is also possible that account R is negative at the expiration date. In that case, the insurer has to cover all the negative balance of the reserve account.

3.4. Closed-form solution for insurance accounts

All agents are assumed to operate in a continuous time frictionless economy with a perfect financial market, so that tax effects, transaction costs and short-sales constraints and other imperfections are neglected.

At this point, we ignore the issue of asset allocation and assume that the participating fund consisting of stocks only with a single premium paid at the inception of the contract (i.e. stock weight $w = 1$). We will look into these more complex arrangements in Chapter 4 after the basic setup. The annual continuous compounded rate of return δ_t, is normally distributed and independent over different years. The risk-neutral probability measure $\mathbb{Q}$ is used to price derivative securities such as one embeds this insurance contract. The total market value of asset P evolves according to a geometric Brownian motion, i.e. with dynamics along the lines of Black and Scholes (1973). Existence of this measure also implies that the financial market is arbitrage free. Note that there are no dividends payments on the assets included in the benchmark portfolio. The dynamics of δ_t is given by

$$\delta_t = r - \frac{1}{2}\sigma^2 + \sigma(W_t - W_{t-1}), \tag{3.9}$$

where $r - \frac{1}{2}\sigma^2$ is the drift term, σ is the volatility of the rate of return on the benchmark portfolio and $W = \{W_t, t \geq 0\}$ is a standard one dimension Wiener process under the probability measure $\mathbb{Q}$. Furthermore, r denotes the instantaneous interest rate which is assumed to be constant and deterministic during the life of the contract.

3.4.1. *The fair value of account A*

The present value of account A from Equation (3.5) with a fixed guaranteed rate r_G, deterministic interest rate r and constant volatility σ is defined as

$$\begin{aligned}
\Pi\left(0,\frac{A_T}{P_0}\right) &= \Pi\left(0, e^{\sum_{i=1}^{T}(r_G+\alpha(\delta_i-r_G)^+)}\right) \\
&= E^Q\left[e^{-rT}e^{\sum_{i=1}^{T}(r_G+\alpha(\delta_i-r_G)^+)}\right] \\
&= E^Q\left[e^{-rT}e^{\sum_{i=1}^{T}\left(r_G\vee(\alpha\delta_i+(1-\alpha)r_G)\right)}\right] \\
&= E^Q\left[e^{-rT}e^{\sum_{i=1}^{T}\left((\alpha r_G\vee\alpha\delta_i)+(1-\alpha)r_G\right)}\right] \qquad (3.10)\\
&= E^Q\left[e^{-rT}e^{(1-\alpha)\sum_{i=1}^{T}r_G}e^{\sum_{i=1}^{T}(\alpha r_G\vee\alpha\delta_i)}\right] \\
&= e^{(1-\alpha)\sum_{i=1}^{T}r_G}\prod_{i=1}^{T}E^Q\left[e^{-r}\left(e^{(\alpha r_G\vee\alpha\delta_i)}\right)\right] \\
&= e^{(1-\alpha)\sum_{i=1}^{T}r_G}\prod_{i=1}^{T}E^Q\left[e^{-r}\left(e^{\alpha r_G}\vee e^{\alpha\delta_i}\right)\right],
\end{aligned}$$

where $\Pi\left(0,\frac{A_T}{P_0}\right)$ presents the fair value of $\frac{A_T}{P_0}$ at date zero and $\vee$ denotes the max operator, i.e., $X\vee Y=\max(X,Y)$.[12] In order to evaluate $E^Q\left[e^{-r}\left(e^{\alpha r_G}\vee e^{\alpha\delta_i}\right)\right]$, we can rewrite this as

$$\begin{aligned}
E^Q\left[e^{-r}\left(e^{\alpha r_G}\vee e^{\alpha\delta_i}\right)\right] &= e^{-r}E^Q\left[\left(e^{\alpha\delta_i}-e^{\alpha r_G}\right)^+ + e^{\alpha r_G}\right] \\
&= e^{-r}E^Q\left[\left(e^{\alpha\delta_i}-e^{\alpha r_G}\right)^+\right] + e^{\alpha r_G - r}. \qquad (3.11)
\end{aligned}$$

The first term of Equation (3.11) can been seen as a European call option on a modified underlying security with payoff $e^{\alpha\delta_i}$ at the maturity of the option and strike price $e^{\alpha r_G}$. Under the objective martingale measure $\mathbb{Q}$, the value of this modified underlying security is

$$e^{-r}E^Q\left[e^{\alpha\delta_i}\right] = e^{-r}e^{\alpha\left(r-\frac{1}{2}\sigma^2\right)+\frac{1}{2}\alpha^2\sigma^2} = e^{(\alpha-1)\left(r+\frac{1}{2}\alpha\sigma^2\right)}$$

and its volatility is $\alpha\sigma$. Applying the Black and Scholes (1973) formula we can evaluate

[12] In this thesis, the fair value at time 0 is presented as

$$\Pi(0,Z_T) = E^Q\left[e^{-\int_0^T r_s ds}Z_T\right],$$

where $\Pi(0,Z_T)$ presents the fair value of Z_T at time 0, $E^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, $\mathbb{Q}$, given the information at date zero, Z_T is a (stochastic) payoff at date T, and r_s is the instantaneous short term interest rate process.

$$e^{-r}E^{Q}\left[\left(e^{\alpha\delta_i}-e^{\alpha r_G}\right)^{+}\right]=e^{(\alpha-1)\left(r+\frac{1}{2}\alpha\sigma^2\right)}\Phi\left(\frac{r-r_G-\frac{1}{2}\sigma^2+\alpha\sigma^2}{\sigma}\right)-e^{\alpha r_G-r}\Phi\left(\frac{r-r_G-\frac{1}{2}\sigma^2}{\sigma}\right), \qquad (3.12)$$

where Φ denotes the cumulative standard normal distribution. Finally, use Equations (3.11) and (3.12) to get the last expression of (3.10)

$$\Pi\left(0,\frac{A_T}{P_0}\right)=\prod_{i=1}^{T}\left(e^{(1-\alpha)\left(r_G-r-\frac{1}{2}\alpha\sigma^2\right)}\Phi\left(\frac{r-r_G-\frac{1}{2}\sigma^2+\alpha\sigma^2}{\sigma}\right)+e^{\alpha r_G-r}\Phi\left(\frac{r_G-r+\frac{1}{2}\sigma^2}{\sigma}\right)\right), \qquad (3.13)$$

which is a closed-form solution for the present value of the insured's account A.

3.4.2. *The fair value of account C*

In order to determine the fair value of the insurer's account C and to verify whether the pension guarantees are profitable given the chosen parameters, we derive the closed-form expression for the account C (see Equation (3.7)) at date $t \leq T$ as

$$\begin{aligned}\Pi\left(0,\frac{C_T}{P_0}\right)&=\sum_{i=1}^{t}\Pi\left(0,\left(e^{\beta(\delta_i-r_G)^{+}}-1\right)\frac{A_{i-1}}{P_0}\right)\\&=\sum_{i=1}^{t}\Pi\left(i-1,\left(e^{\beta(\delta_i-r_G)^{+}}-1\right)\right)e^{-r(t-i+1)}\,\Pi\left(0,\frac{A_{i-1}}{P_0}\right)\\&=\pi e^{-rt}\sum_{i=1}^{t}\pi_A(i-1)e^{r(i-1)}, \qquad (3.14)\end{aligned}$$

where $\pi_A(t)=\Pi\left(0,\frac{A_t}{P_0}\right)=\Pi\left(0,e^{\sum_{i=1}^{t}(r_G+\alpha(\delta_i-r_G)^{+})}\right)$ which has the closed-form expression (3.13) with $\pi_A(0)=1$ and

$$\begin{aligned}\pi&=\Pi\left(t-1,e^{\beta(\delta_t-r_G)^{+}}-1\right)\\&=\Pi\left(0,e^{\beta(\delta_1-r_G)^{+}}-1\right)\\&=e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)-\beta r_G}\Phi\left(\frac{r-r_G-\frac{1}{2}\sigma^2+\beta\sigma^2}{\sigma}\right)-e^{-r}\Phi\left(\frac{r-r_G-\frac{1}{2}\sigma^2}{\sigma}\right).\end{aligned}$$

The above derivation can be found using the linearity of the market value operator, the law of iterated expectations,[13] and the independence and identical distribution of annual returns over different years. The expression for π can be derived by a minor modification of Equation (3.13). For more details regarding the derivation we refer to the article of Miltersen and Persson (2003).

[13] The law of iterated expectation is given by $E(X|I_1)=E(E(X|I_2)|I_1)$, where the value of I_1 is determined by that of I_2.

As opposed to numerical solutions, the application of the analytical solutions here is only valid under certain conditions, i.e. deterministic interest rate, stochastic equity process, no occurrence of inflation, mortality, surrender and multivariate risk. When the valuation setup becomes stochastic, we will resort to sophisticated mathematical methods such as the Monte Carlo simulation instead of analytical solutions to provide fair calculations of the insurance policy. However, a vast number of trial runs are required for Monte Carlo approach to be accurate. The length of computation time may be problematic when the size of the evaluation problem is large. This practical issue is one research direction that may be explored as a continuation of this investigation.

3.5. Fair pricing principle

Since we are interested in the fair valuation of minimum rate of return guarantees, we first need to define the fair pricing principle. This can be represented in the following way

$$\Pi(0, \{A_T + R_T^+\} - \{C_T + R_T^-\}) = P_0. \tag{3.15}$$

Equation (3.15) can be interpreted as the fair pricing condition of the profit-sharing life insurance contract, where $\{A_T + R_T^+\}$ presents the accumulated account value of the policyholder at time T and $\{C_T + R_T^-\}$ denotes the insurer's account at time T. For the contract fulfilling the fair pricing principle, the fair value must equal the initial investment P_0 in the case of a single premium policy.

Based on the fair pricing principle, given in (3.15), it is oblivious that $\Pi(0, C_T) - \Pi(0, R_T^-) = 0$. More specifically, C_T represents the risk premium for which the specific loss incurred by the insurer at the maturity, R_T^-, is compensated. Hence, this leads to the condition $P_0 = \Pi(0, A_T) + \Pi(0, R_T^+)$.

3.6. Numerical examples

We present results from the numerical analysis of the simplified model for a single premium life insurance policy. As discussed earlier, we have derived closed-form solutions for the initial market values of the final balances of the accounts A and C under the risk-neutral framework. However, it is more interesting to determine the initial market value of the total cashflow to the investor $A_T + R_T^+$ and the total cashflow to the insurer $C_T + R_T^-$. Since no similar analytic solutions are available for R_T^+ and R_T^-, we have to resort to numerical methods. To illustrate these points, we look at some fair combinations that are affected by the changes and discuss their possible causes. We call a contract fair if the contract's risk-neutral value equals the single premium paid.

3.6.1. Example 1: Illustrations of contract description based on one sample path

First we show some plots of simulations runs of the model to give the reader a feel how things work. Fig. 4 plots the comparison between the various policy interest rates and the market returns obtained over a 20 year period resulting from a simulation run of our model. The plot clearly illustrates the smoothing that takes place in the policy interest rate. These rates are bounded below by the guaranteed rate of interest (r_G) of 3% which comes into effect in certain years. The existence of a reserve account can be associated with a sort of 'hedge' construction and is thus essential for contracts with profit-sharing.

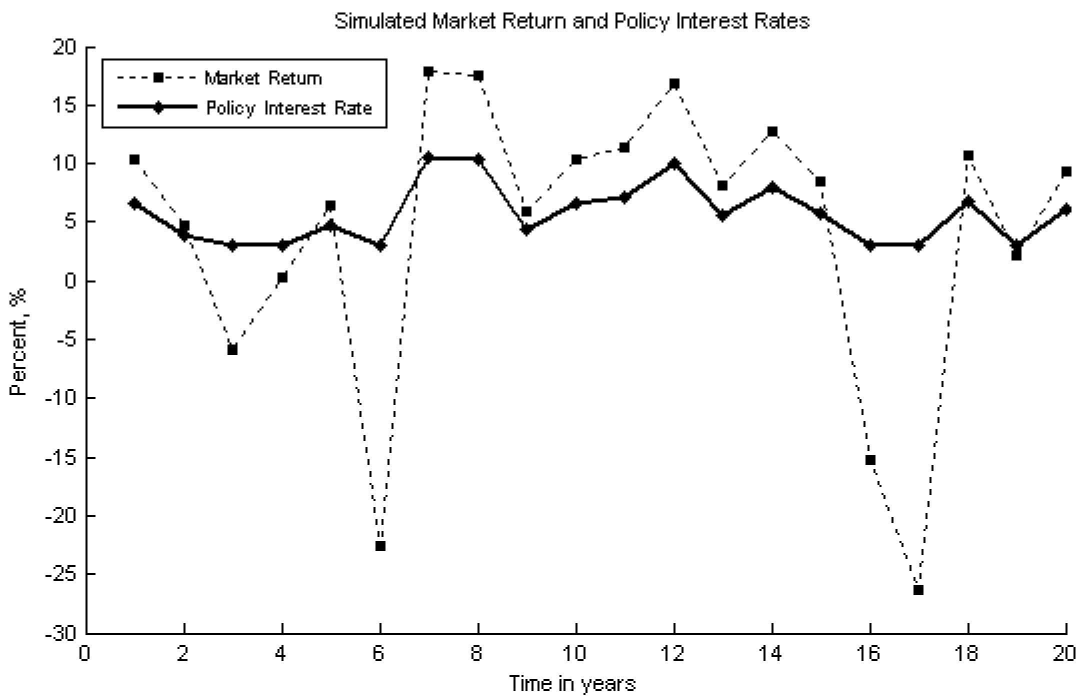

Fig. 4. Sample path of market returns and policy interest rates entitled to the insured with the following parameter values: minimum guaranteed rate $r_G = 0.03$, profit-sharing rate insured $\alpha = 0.50$, profit-sharing rate insurer $\beta = 0.25$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 20$, $r = 0.10$ and $\sigma_s = 0.15$.

It is also interesting to look at the evolution of several insurance accounts to gain additional understanding, especially when the market condition is adverse. Fig. 5 presents the development of four different accounts during an investment horizon of 20 years: (i) *Minimum Guaranteed Account (MG)*; (ii) *Insured's Account (A);* (iii) *Insurer's Account (C)* and (iv) *Reserve Account (R)*. The *Initial Guaranteed Amount* denotes the amount of money which is guaranteed by the insurer. The policyholder receives r_G per year for the entire period of the contract and the value of the guaranteed account at maturity can thus be associated with a bond element which is determined by $e^{(r_G T)} \cdot P_0$. The *Insured's Account (A)* is given by the *Minimum Guaranteed Account (MG)* plus the profits shared during the life of the contract. The pace of increment of account A depends on the fraction of excess return α entitled to the policyholder. As it can be seen from Fig. 5, the obtained value for the *Insurer's Account* (C) is negative at the expiration date, but is positive for all periods prior to maturity. This can be motivated by the fact that the terminal surplus at the final date is below zero. Otherwise stated, the *Reserve Account (R)* at expiry is negative. For this particular scenario, the insurer has to cover the loss obtained by account R due to poor investment performances.

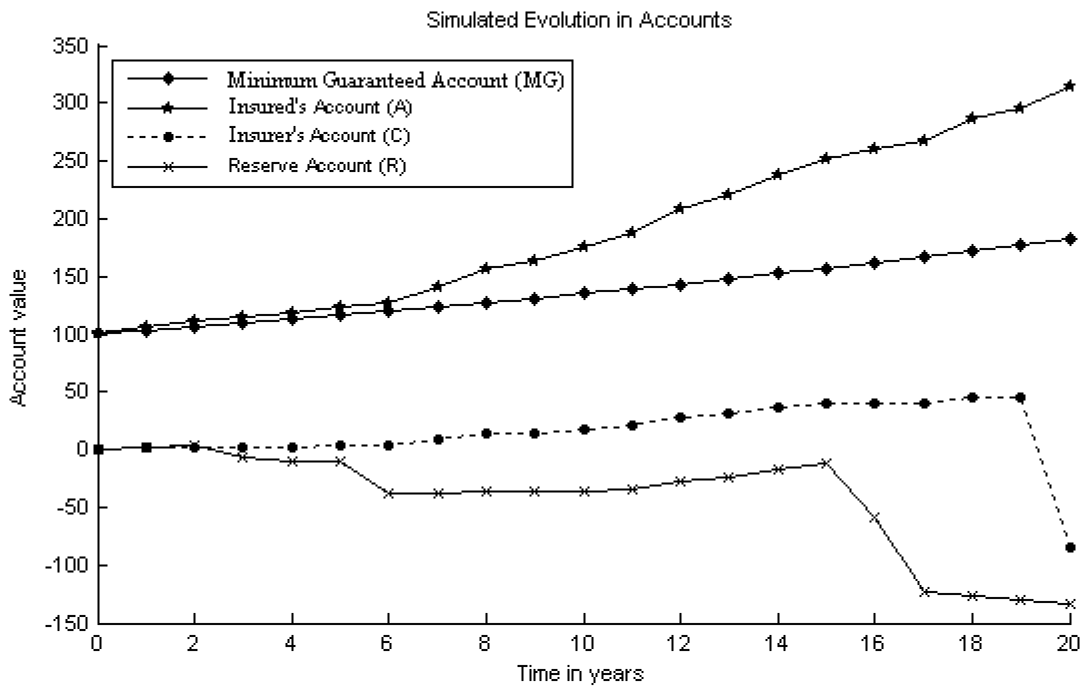

Fig. 5. Sample path of evolution in different policy accounts with the following parameter values: minimum guaranteed rate $r_G = 0.03$, profit-sharing rate insured $\alpha = 0.50$, profit-sharing rate insurer $\beta = 0.25$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 20$, $r = 0.10$ and $\sigma_s = 0.15$.

3.6.2. Example 2: Fair parameter combinations

Having demonstrated some essential characteristics of the contract considered we now turn to their valuation. In the footsteps of Miltersen and Persson (2003), we have implemented a numerical simulation algorithm in order to calculate the expectation under the equivalent martingale measure, $\mathbb{Q}$,

$$\frac{\Pi(0, X_T)}{P_0} = e^{-rT} E^Q \left[\frac{X_T(\Theta)}{P_0} \right], \tag{3.16}$$

where X_T denotes the market value of the cash flow at time T and $\Theta = \{P_0, \delta_t, r_G, \alpha, \beta\}$ represents the set of parameters included in the model framework. In our second example we seek to find reasonable parameter combinations of (α, β, r_G) for our insurance contract. As our example, we look at the negative terminal reserve account R_T^-.

To obtain suitable combinations of profit-sharing rates for insured αs, profit-sharing rates for insurer βs and minimum guaranteed rates $r_G s$ with respect to $\Pi(0, R_T^-)/P_0$, we employ numerical search algorithm for given α and r_G to find the fair β. In order to quantify the term $\Pi(0, R_T^-)/P_0$, we use the closed-form solution as defined in Section 3.4 to calculate

$$\frac{\Pi(0, C_T)}{P_0} = \frac{\Pi(0, R_T^-)}{P_0}. \tag{3.17}$$

$\frac{\Pi(0,R_T^-)}{P_0}$ can be interpreted as the fair percentage up-front premium the investor will have to pay for issuing the insurance policy.

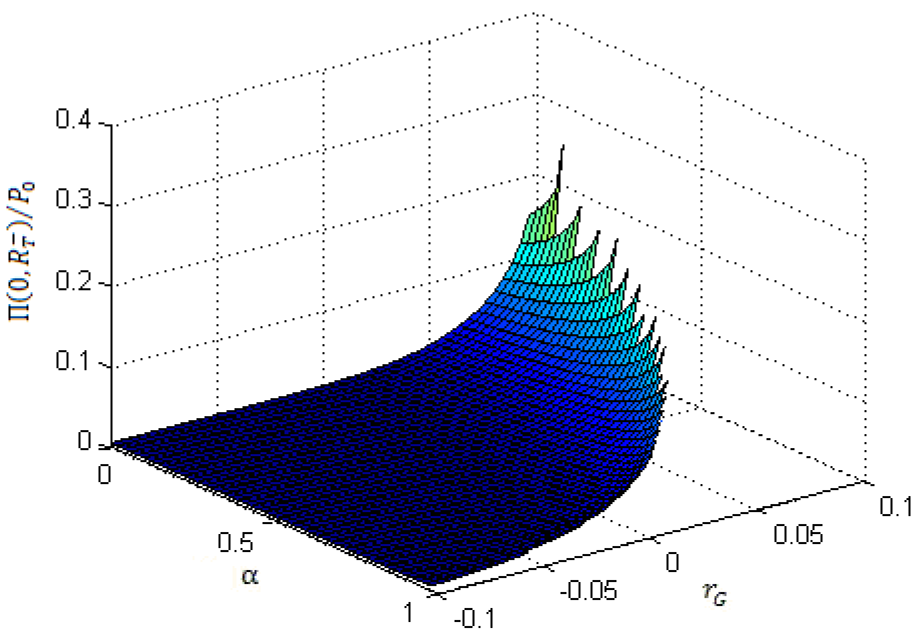

Fig. 6. Fair single premium life insurance contract parameter combinations of profit-sharing rate insured α, minimum guaranteed rate r_G and $\frac{\Pi(0,R_T^-)}{P_0}$ for $\sigma = 10\%$ and $r = 10\%$, and $T = 5$.

Fig. 6 plots the fair combinations $(\alpha^*, \beta^*, r_G^*)$ for a 5-year contract fulfilling the fair pricing principle. For example, when we look at the case where $\alpha = 0.5$ and $r_G = 0.03$, we observe that the fair percentage up-front premium the investor will have to pay $\left(\frac{\Pi(0,X_T)}{P_0}\right)$ is relatively low (i.e. $\pm 2\%$). Although β is not directly observable in the graph, it is implicitly included in the numerical search algorithm. Based on the numerical illustration in Fig. 6, several interesting statements can be formulated. First, we observe that some of the parameter combinations are simply not feasible, for instance, for extremely large value of α ($> \pm 0.98$) and $r_G (> \pm 5\%)$, fair pricing cannot be obtained. Another essential observation is that the cost level of the annual minimum rate of return guarantee at the inception of the contract is increasing with respect to r_G. As expected, the higher the base rate r_G, the higher the minimum guaranteed value. In some circumstances, the fair percentage up-front premium even reaches ±25% which is quite unusual. It is thus very likely that the insurer will end up with a huge loss of fees in such case.

The values presented here are only for illustrative purposes. One would expect that the parameter combination with negative r_G is not rational and hence not realistic from practical point of view. In the remaining part of the thesis, we limit our attention to the case of $r_G = 3\%$ since this is the current standard of minimum guaranteed rate issued in the Dutch life insurance market.

3.6.3. Example 3: Fair value analysis

In the previous section we conducted empirical analysis on the fair parameter combination of our product. The main objective now is to determine the fair value of the European participating contract with single endowment. More specifically, we are interested in the present value of the initial market value of the total cash flow for both the policyholder and the insurance provider.

To evaluate the fair premium needed for each specific insurance account, we consider a single premium contract of a policyholder enters the contract at the age of 25. The basic values for the parameters used are: minimum guaranteed rate $(r_G) = 0.03$, stock weight $(w) = 1$, initial investment $(P_0) = 100$, initial insured's account $(A_0) = 100$, initial reserve account $(R_0) = 0$, initial insurer's account $(C_0) = 0$, $T = 40$, $\sigma_S = 0.15$, long-term inflation rate $(\bar{\pi}) = 0.024$ and simulation runs = 50000. In addition, we investigate the financial impact for different levels of profit-sharing rates for insured αs, profit-sharing rates for insurerβs and interest rate r on the fair price of the minimum rate of return guarantee. To be more specific, these set of parameters include: $\alpha = \{0.10, 0.20, 0.30\}$, $\beta = \{0.30, 0.40, 0.50\}$ and $r = \{0.06, 0.08, 0.10\}$. In what follows, all experiments are carried out with Monte Carlo simulations.

In Table 2, we provide the results of the fair price of the European participating contract for different parameter choices. The risk-neutral value of the insurance contract is determined by discounting the sum of the insured's account $(A_T + R_T^+)$ and the insurer's account $(C_T + R_T^-)$. Note that the initial premium of the contract $P_0 = 100$ and thus the market value of the contract should be equal to this value if the fair pricing condition is satisfied. However, since real contracts often include commissions and fees for different services and options, it is preferred that the fair contract value is less than the initial premium P_0.

From Table 2, it can be observed that the fair value of the European contract for different levels of profit-sharing rates of insured αs and profit-sharing rates of insurer βs (i.e. the last three columns) is decreasing with respect to the interest rates. Of course, high interest rates imply higher expected returns and lower discounted costs of future pension liabilities. In addition, no fair pricing principle can be established for $r = 6\%$ for all possible contract setting. We also look at the impact of changes in the fraction of profit shared with the policyholder α and the fraction of profit shared with the insurer β. We observe that when α is increasing, then, as expected, the insured's account value becomes higher in most cases. In fact, high rates of profit-sharing attributed to the policyholder (αs) often lead to lower reserve account R and the ability to maintain the required minimum guarantee r_G becomes doubtful. Similarly, the amount of the insurer's account is increasing with the rate of profit-sharing of the insurance company (β), since more risk premiums are attributed to the insurer for larger value of βs.

Table 2
Fair values of a single premium European life insurance contract for different levels of interest rates (r), profit-sharing rate insured (α) and profit-sharing rate insurer (β). Assumptions used are: 1) stochastic equity return; 2) deterministic interest rate; 3) deterministic inflation rate; 4) no mortality risk; 5) no surrender risk.

Interest rate scenario	Ps-rate insured (α)	Insured's account ($A_T + R_T^+$) Ps-rate insurer (β)			Insurer's account ($C_T + R_T^-$) Ps-rate insurer (β)			European contract Ps-rate insurer (β)		
		0.30	0.40	0.50	0.30	0.40	0.50	0.30	0.40	0.50
$r = 10\%$	0.10	91.07	87.96	85.01	8.82	11.91	15.02	82.25	76.05	70.00
	0.20	90.54	86.78	84.21	9.33	12.79	16.11	81.21	73.99	68.10
	0.30	90.85	87.88	84.85	8.72	12.23	15.84	82.13	75.66	69.01
$r = 8\%$	0.10	93.70	89.41	86.09	6.41	10.43	13.75	87.29	78.98	72.34
	0.20	97.95	94.26	91.59	1.89	5.20	8.87	96.06	89.07	82.71
	0.30	107.07	103.90	102.30	−7.61	−4.42	−1.20	114.69	108.32	103.50
$r = 6\%$	0.10	122.07	119.04	116.83	−22.59	−19.60	−16.57	144.67	138.64	133.39
	0.20	140.76	139.08	136.89	−41.31	−38.20	−36.19	182.07	177.28	173.08
	0.30	168.45	166.45	164.93	−67.58	−66.42	−65.12	236.03	232.87	230.04

The table reports the fair values of the terminal insured's account ($A_T + R_T^+$) and the terminal insurer's account ($C_T + R_T^-$) under the risk-neutral measure $\mathbb{Q}$. The present value of the insurance policy is defined as $\Pi(0, X_T) = e^{-(r-\bar{\pi})T} E_Q[X_T]$, where X_T presents the market value of the insurance account at expiry date, r is the constant short rate of interest and $\bar{\pi}$ describes the long-term inflation rate. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, long-term inflation rate $\bar{\pi} = 0.0240$ and scenario $= 50000$.

According to the fair pricing principle as given in (3.15), our goal is find a fair contract in which the conditions $\Pi(0, C_T) - \Pi(0, R_T^-) = 0$ and $P_0 = \Pi(0, A_T) + \Pi(0, R_T^+)$ are satisfied, or more equivalently $\Pi(0, \{A_T + R_T^+\} - \{C_T + R_T^-\}) = P_0$. Given the financial assumptions we have made earlier in this chapter and the results presented in Table 2, the most appropriate fair parameter combination satisfying the two conditions would be $\{r^* = 8\%,\ \alpha^* = 0.20,\ \beta^* = 0.30\}$, with a European contract value of 96.06.

Finally, in order to gain additional insights with respect to the fair valuation of interest rate guarantees we decompose the contract value into their basic components, i.e. the risk-free bond element (MG), the insured's account (A), the bonus insured (A-G), the positive terminal bonus (R+), the insurer's account (C) and the negative terminal bonus (R-). The characteristic of implicit option elements for $\beta = \{0.30, 0.40, 0.50\}$ is illustrated from Table E1 to Table E3 in Appendix E.

In the interpretation of these tables, it can be noticed that the fair risk-free bond element (MG) is fixed for different values of profit-sharing rate attributed to the insured αs and the insurer βs, and rises tremendously if the market interest rate drops towards. For instance, the change in interest rate from 10% to 8% in Table E1 results in a change in the bond value from 15.88 to 35.35. The bonus option of the insurer (A-G) appears to be increasing with the profit-sharing rate α which is quite intuitive. We also see that the terminal bonus R+ becomes more valuable due to the increment in the level of the interest rate. This observation could explained by the fact that high interest rates are associated with high expected investment gains.

Furthermore, by comparing Tables E1 to E3 we see that the fair value of the insurer's account C improves when the profit-sharing β increases irrespective of the development of interest rate r and the profit-sharing rate of insured α. In contracting, the fair balance of the terminal bonus R- is decreasing with β. Not surprisingly, higher β means that lower profit is credited to the reserve account R, and hence leading to a larger deficit on the reserve account. From a risk management perspective, it is thus crucial for insurance companies to find reasonable value for α and β in order to fulfill the fair pricing principle and at the same time to be profitable.

4. Stochastic Modeling of Financial Risks

This chapter is concerned with the long-term financial risk sources which investors have to cope with: investment portfolio risk, term structure and inflation risk. All considered risk factors are captured by separate stochastic diffusion processes, each of which can be potentially correlated. In this chapter, we consider the classical approach based on the real-world measure ($\mathbb{P}$) and the risk-neutral measure ($\mathbb{Q}$). In turn, each risk type is elaborated upon in a separate section.

4.1. Assets dynamics and interest rate modeling

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a complete probability space supporting all sources of financial randomness. The set Ω is the set of all possible outcomes for market movements. The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ represents the flow of information available to the insurer and the policyholder at time t. Consistent with no-arbitrage paradigm, we assume the existence of an equivalent martingale measure $\mathbb{Q}$ for this economy.

With respect to the investment portfolio, we consider an economy with two traded assets − a bond with price process B and a stock with price process S. The modeling of the interest rate crediting mechanism takes the following simplified balance sheet at time t:

Assets	Liabilities
$B(t,T_B) = (1-w)\cdot P_t$	$A_t + R_t^+$
$S_t = w \cdot P_t$	$C_t - R_t^-$
P_t	P_t

Fig. 7. Simplified illustration of the insurer's financial situation at time t for a single premium life insurance contract. The parameters are: bond price $B(t,T_B)$ with maturity T_B, stock price S_t, investment account P_t, insured's account A_t, reserve account R_t, insurer's account C_t and stock weight w,

We let P_t denotes the market value of the insurer's assets at time t. Furthermore, we assume that the insurer invests a fraction w in of the received premium in stocks, S_t, and the rest in risky bonds, $B(t,T_B)$, of maturity $T_B \geq T$. The return gained from segregated fund for each integer t is defined by

$$\delta_t = w\ln\left(\frac{S_t - S_{t-1}}{S_{t-1}}\right) + (1-w)\ln\left(\frac{B_t - B_{t-1}}{B_{t-1}}\right). \tag{4.1}$$

Typical values of the parameter w (i.e. fraction invested in stocks) for Dutch outstanding policies are 10%, 20% and 30%. The liability side comprises three entries: A_t is book value of the policyholder's account; R_t^+ and R_t^- represent the positive and the negative reserve account at time t respectively, and C_t denotes the insurer account.

As aforementioned, the insurer invests in two assets: stock and bond. Under the real-world probability measure $\mathbb{P}$, the stock market uncertainty is considered by modeling the stock index S_t as a Black and Scholes model

$$dS_t = (r_t + \sigma_S \lambda_s) S_t dt + \sigma_S S_t d\overline{W}_t^S, \tag{4.2}$$

where r_t is the instantaneous spot rate, σ_S is the constant stock price volatility, $\lambda_s \geq 0$ represents the market price of equity risk, and $\overline{W}_t^S$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ in the interval $[0, T]$. Then after risk adjustment [see Baxter and Rennie (1996) for an introduction], (4.2) can be written in the form

$$dS_t = r_t S_t dt + \sigma_S S_t dW_t^S, \tag{4.3}$$

where W_t^S is a standard Brownian motion under risk-neutral measure $\mathbb{Q}$. The transition from martingale $\mathbb{P}$ to $\mathbb{Q}$ is facilitated through the market price of equity risk:

$$\lambda_s = \frac{\mu_t - r_t}{\sigma_S}, \tag{4.4}$$

where $\mu_t = r_t + \sigma_S \lambda_s$. We see from Equations (4.2) and (4.3) that the market price of equity risk (λ_s) is zero under risk-neutral martingale measure. As expected, if the investors were risk-neutral, no excess return is required for taking additional risk. The investor would discount all cash flows − irrespective of their risk − at the risk-free rate (or short rate). In this context, computations are done in the real historical world and valuations take place under a risk-neutral probability measure.

Under martingale $\mathbb{Q}$, Ito's lemma with $f(S_t) = \ln(S_t)$ gives

$$df(S_t) = \frac{\partial f}{dt} dt + \frac{\partial f}{dS} dS_t + \frac{1}{2} \frac{\partial^2 f}{dS^2} [dS_t dS_t]$$

$$\Longrightarrow \qquad d\ln(S_t) = \frac{1}{S_t} (r_t S_t dt + \sigma_S S_t dW_t^S) - \frac{1}{2} \sigma_s^2 dt$$

$$\Longrightarrow \qquad d\ln(S_t) = \left(r_t - \frac{\sigma_S^2}{2}\right) dt + \sigma_S dW_t^S. \tag{4.5}$$

It follows that

$$S_T = S_t \exp\left\{\int_t^T \left(r_u - \frac{\sigma_S^2}{2}\right) du + \int_t^T \sigma_S dW_u^S\right\}.$$

The continuous nominal short rate process r_t can be modeled by many different interest models. The most well-known term structure models are: Vasicek (1977), Cox, Ingersoll and Ross (1985), Ho and Lee (1986), and Hull and White (1990). For a complete overview of these stochastic interest models we refer to the textbook of Hull (2006). Here, we limit our attention to the Vasicek (1977) model. The Vasicek model enjoys popularity among academics and practitioners. One of the reasons that this model is appealing is because it has analytical solution for many interest rate derivatives, meaning that an explicit solution can be found. However, for the modeling of short interest rates, the Vasicek model has a disadvantage because they allow the rates to become negative. In practice, the probability of r going below zero is almost negligible for typical parameter values.

The dynamics of the one-factor Vasicek (1977) under the risk-neutral probability measure $\mathbb{Q}$, is given by the following stochastic process

$$dr_t = a[\theta_r - r_t]dt + \sigma_r dW_t^r, \tag{4.6}$$

where θ_r describes the long-run mean of the interest rate, a is a constant representing the reversion speed, σ_r is the instantaneous volatility of the interest rate and W_t^r is a standard Brownian motion under $\mathbb{Q}$. The mean reversion characteristic implies that the interest rates have the tendency to move back towards the long-run mean θ_r, where the parameter a indicates how strong the force of mean reversion is. The correlation between the Wiener process of the risky asset dW_t^S and the Wiener process of the risk-free rate dW_t^r is given by the $\rho_{S,r}$. In this thesis, the interest rate risk can be defined as the risk of a change in fair value caused by the transformation from the stochastic interest rates to the deterministic interest rates.

The value of a zero-coupon bond in the Vasicek model with time to maturity T is given by

$$P(t,T) = e^{-A(t,T)-B(t,T)r_t}, \tag{4.7}$$

where

$$A(t,T) = R(\infty)\big(T - B(t,T)\big) + \frac{\sigma_r^2}{4a}\big(B(t,T)\big)^2,$$
$$B(t,T) = \frac{1}{a}(1 - e^{-aT}),$$

and where $R(\infty) = \theta + \frac{\sigma_r \lambda_r}{a} - \frac{1}{2}\frac{\sigma_r^2}{a^2}$ describes the nominal interest rate of a bond with maturity approaches infinity [see Sørensen (1999)].

Under the real-world probability measure $\mathbb{P}$, the dynamics of the value of the risky bond portfolio B with a given modified duration $D = -\frac{\partial B}{\partial r}\frac{1}{B}$, by Ito's lemma, is assumed to follow the stochastic differential equation [see Munk, Sørensen and Vintner (2004)]:

$$dB_t = (r_t + \sigma_B \lambda_r) B_t dt + \sigma_B B_t dW_t^r, \tag{4.8}$$

where λ_r is the market price of interest rate risk, $\sigma_B = \sigma_r D$ is the instantaneous standard deviation of the bond portfolio and W_t^r presents the standard Brownian motion of the interest rate process. The duration D can been associated with the elasticity of the bond price with respect to the short interest rate. We mention that the short interest rate and the return on the bond are perfectly negatively correlated and with covariance rate $\sigma_{B,r} = -D\sigma_r^2 = -(1/D)\sigma_B^2$. For the correlation between stock and bonds it holds that $\rho_{S,B} = -\rho_{S,r}$. The variance-covariance matrix of stocks and bonds can be summarized in matrix form by

$$\Sigma = \begin{pmatrix} \sigma_S^2 & \sigma_S \sigma_B \rho_{S,B} \\ \sigma_S \sigma_B \rho_{S,B} & \sigma_B^2 \end{pmatrix}.$$

Under the equivalent martingale measure $\mathbb{Q}$, the stochastic differential equation of the bond price process is given by

$$dB_t = r_t B_t dt + \sigma_B B_t dW_t^r, \tag{4.9}$$

where the term $\sigma_B \lambda_r$ is omitted due to the well-known fact that derivative pricing takes place under a risk-neutral probability measure.

4.2. Inflation risk

In Chapter 3 we have assumed that the annual inflation rate is deterministic and equals the expected inflation $\bar{\pi}$. From now onwards, we model the inflation rate risk stochastically because the effect of uncertain inflation becomes especially important when we consider long-maturity options. We build upon the article of Maurer, Schlag and Stamos (2008) when we describe the impact of stochastic inflation rates.

We assume that the nominal price level of the consumption good denoted by Ψ_t evolves according to

$$d\Psi_t = \pi_t \Psi_t dt + \sigma_\Psi \Psi_t dW_t^\Psi \tag{4.10}$$

where π_t is the expected rate of inflation, σ_Ψ is the volatility of the price index and σ_Ψ thus presents the degree of the unexpected short-run inflation movements in the economy. W_t^Ψ is a

Wiener process with incremental variance dt under martingale $\mathbb{Q}$. The expected inflation rate itself is given by the next stochastic process

$$d\pi_t = \gamma(\bar{\pi} - \pi_t)dt + \sigma_\pi dW_t^\pi, \tag{4.11}$$

where $\bar{\pi}$ describes the long-run mean of the rate of inflation, γ represents the degree of mean reversion, σ_π is the volatility of the inflation rate and W_t^π is a standard Brownian motion under measure $\mathbb{Q}$. Equation (4.11) is known as Ornstein–Uhlenbeck process with mean reversion, which is similar to the approach of interest rate modeling. The Ornstein–Uhlenbeck process is linked with the Vasicek (1977) model and has thus a disadvantage because they allow the rates to become negative. This is not a problem for the modeling of inflation rates, since deflation (negative inflation rate) is a common economic issue. Furthermore, changes in the price of the consumption good and the inflation rate are correlated with the stock index and the short rates. The corresponding correlation between inflation and two stochastic processes, i.e. interest rate and stock returns is given by $\rho_{r,\pi}$ and $\rho_{S,\pi}$ respectively. In addition, the correlation between the price index and the expected inflation is defined as $\rho_{\Psi,\pi}$.

Note that the estimates of the inflation process and the covariance matrix are obtained from the article proposed by Maurer, Schlag and Stamos (2008). They present a similar valuation approach to model the various stochastic processes and their dependence structure. The specific framework is based on expressing the model in state space form and then using the Kalman filter to obtain the relevant log-likelihood function to be maximized.[14] Historical data about the development of the consumer price index in Germany are taken from the German Central Bank (Bundesbank). Our choice here is justified because the German data can be seen as the benchmark for most West-European countries, in particular the Netherlands.

4.3. Multivariate risk framework

Modeling and measurement of multivariate risk in insurance and finance is an extremely challenging and important. The values of our stochastic processes are all correlated. In order to assess the risk profile of the insurance contract in an appropriate manner, we employ the Cholesky decomposition. The Cholesky decomposition is commonly used in the Monte Carlo method for simulating systems with multiple correlated variables.

Applying Cholesky decomposition we obtain the following correlation matrix

[14] We refer to Babbs and Nowman (1999) and Harvey (1989) for the application of the Kalman filter to the calibration of the Vasicek model.

$$U = \begin{array}{c} \overbrace{\begin{array}{cccc} r_t & S_t & \Psi_t & \pi_t \end{array}} \\ \begin{pmatrix} 1 & \rho_{r,S} & \rho_{r,\Psi} & \rho_{r,\pi} \\ \rho_{r,S} & 1 & \rho_{S,\Psi} & \rho_{S,\pi} \\ \rho_{r,\Psi} & \rho_{S,\Psi} & 1 & \rho_{\Psi,\pi} \\ \rho_{r,\pi} & \rho_{S,\pi} & \rho_{\Psi,\pi} & 1 \end{pmatrix} \end{array}, \tag{4.12}$$

where U is a symmetric positive definite matrix. Then U can be factored into an upper triangular matrix L such that $U = LL'$ where L refers to its conjugate transpose. A similar procedure can be found in Brigo and Mercurio (2006), where only interest rate and stock return dynamics are considered. Then L can be expressed as

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho_{r,S} & \sqrt{1-\rho_{r,S}^2} & 0 & 0 \\ \rho_{r,\Psi} & \frac{\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}}{\sqrt{1-\rho_{r,S}^2}} & \sqrt{1-\rho_{r,\Psi}^2-\frac{\left(\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}\right)^2}{1-\rho_{r,S}^2}} & 0 \\ \rho_{r,\pi} & \frac{\rho_{S,\pi}-\rho_{r,S}\rho_{r,\pi}}{\sqrt{1-\rho_{r,S}^2}} & Y & Z \end{pmatrix}, \tag{4.13}$$

where

$$Y = \frac{\rho_{\Psi,\pi}-\rho_{r,\Psi}\rho_{r,\pi}-\frac{\left(\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}\right)\left(\rho_{S,\pi}-\rho_{r,S}\rho_{r,\pi}\right)}{1-\rho_{r,S}^2}}{\sqrt{1-\rho_{r,\Psi}^2-\frac{\left(\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}\right)^2}{1-\rho_{r,S}^2}}} \tag{4.14}$$

and

$$Z = \sqrt{1-\rho_{r,\pi}^2-\frac{\left(\rho_{S,\pi}-\rho_{r,S}\rho_{r,\pi}\right)^2}{1-\rho_{r,S}^2}-\frac{\left(\rho_{\Psi,\pi}-\rho_{r,\Psi}\rho_{r,\pi}-\frac{\left(\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}\right)\left(\rho_{S,\pi}-\rho_{r,S}\rho_{r,\pi}\right)}{1-\rho_{r,S}^2}\right)^2}{1-\rho_{r,\Psi}^2-\frac{\left(\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}\right)^2}{1-\rho_{r,S}^2}}}. \tag{4.15}$$

Thus

$$\begin{cases} dW_t^r = d\widetilde{W}_t^r \\ dW_t^S = \rho_{r,S} d\widetilde{W}_t^r + \sqrt{1-\rho_{r,S}^2}\, d\widetilde{W}_t^S \\ dW_t^\Psi = \rho_{r,\Psi} d\widetilde{W}_t^r + \frac{\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}}{\sqrt{1-\rho_{r,S}^2}} d\widetilde{W}_t^S + \sqrt{1-\rho_{r,\Psi}^2 - \frac{\left(\rho_{S,\Psi}-\rho_{r,S}\rho_{r,\Psi}\right)^2}{1-\rho_{r,S}^2}}\, d\widetilde{W}_t^\Psi + Zd\widetilde{W}_t^\pi \\ dW_t^\pi = \rho_{r,\pi} d\widetilde{W}_t^r + \frac{\rho_{S,\pi}-\rho_{r,S}\rho_{r,\pi}}{\sqrt{1-\rho_{r,S}^2}} d\widetilde{W}_t^S + Yd\widetilde{W}_t^\Psi + Zd\widetilde{W}_t^\pi \end{cases} \cdot \tag{4.16}$$

4.4. Numerical results

We show several numerical illustrations to convey the impact of stochastic interest rates and inflation rates on the insurance policy. Unless otherwise stated, the basic set of parameters used in the simulation is reported in Table 3. We analyze the case in which the insurance client contributes the initial investment value $P_0 = 100$ and enters a single premium participating contract offering a 3% minimum guarantee r_G; 20% profit-sharing rate α in the excess returns generated by the segregated fund is attributed to the policyholder and 30% of the excess return β is assigned to the insurer.[15] We also assume that the insurance contract has a life span of 40 years given that the insured is aged 25 at the inception of the contract and retires at the age of 65.

Table 3
Parameters used in the simulation

Contract				Asset		Interest rates		Inflation		Correlations			
										S	r	Ψ	π
r_G	0.03	P_0	100	σ_S	0.15	a	0.10	σ_Ψ	0.0101	1	−0.0531	−0.0675	0.0026
α	0.20	A_0	0	D	5	θ_r	{0.06; 0.08; 0.10}	$\bar{\pi}$	0.0240		1	0.0516	0.5248
β	0.30	C_0	0	σ_B	0.05	σ_r	{0.01; 0.02; 0.03}	γ	0.4740			1	0.1641
T	40	R_0	0	w	∈ [0,1]			σ_π	{0.01; 0.02; 0.03}				1

This table reports the parameters used in the simulation for the basic contract, the interest rates and the inflation process, and the corresponding covariance matrix. The parameters are: minimum guaranteed rate r_G, profit-sharing rate insured α, profit-sharing rate insurer β, contract maturity T, initial investment P_0, initial insured's account A_0, initial reserve account R_0, initial insurer's account C_0, stock price volatility σ_S, bond duration D, bond price volatility σ_B, stock weight w, mean reversion rate of the interest rate a, long-term mean of the interest rate θ_r, volatility of the interest rate σ_r, volatility of the price index σ_Ψ, long-term inflation rate $\bar{\pi}$, mean reversion rate of the inflation rate γ and volatility of the inflation rate σ_π.

The covariance matrix in Table 3 is obtained from the article proposed by Maurer, Schlag and Stamos (2008). For instance, the correlation coefficient between the stock return and the short rate is given by -0.0531 ($\rho_{r,S}$). This means that when interest rates go up, stock prices will go

[15] In Subsection 3.6.3 we found that $\{r^* = 8\%,\ \alpha^* = 0.20,\ \beta^* = 0.30\}$ is an appropriate fair parameter combination. As an example, we use these specific parameter values as our starting point for analyzing the fair guarantee contract.

down. This relationship is interpreted as follows. First, higher interest rates make borrowing more costly. This tends to put downward pressure on corporate investments and consumer spending. As a result, the economy will slow down and stock prices will decrease. A second reason for stock prices to decrease after a rise in central bank policy rate is that many investors who had been buying stocks are seeking safe investment opportunities such as corporate or government bond. Money leaving the stock market and entering the bond market can be associated with a bearish stock market. Moreover, companies that sell long-term debt will pay more now that rates are higher and in turn it reduces their earnings power. Another essential observation is the correlation between the interest rate and the inflation rate. The core monetary policy objective of the European Central Bank (ECB) is to maintain price stability with achieving an inflation target of 2%, as measured by the annual change in the consumer prices index (CPI). The correlation coefficient between the interest rate and the expected inflation rate ($\rho_{r,\pi}$) is 0.5248. This is contrary to our beliefs, since an increase in interest rate lowers the economy growth, and thus reduces the price inflation. However, prices tend to have a direct but lagging relationship to interest rates. This means that falling prices have followed falls in interest rates, and rising prices have followed rises in interest rates. In addition, a rise in interest rates pushes up the cost of lending as mentioned earlier. To some extent, business may decide to pass on this higher marginal cost of capital to the consumer. Therefore, it could take some time for a rate change to work its way through into prices.

4.4.1. Asset allocation

For purposes of illustrating the financial effect of asset allocation on the fair contract and its associated risks, in Fig. 8 we show the results of asset allocation for several combinations of stock and bond − the proportion invested in stock is denoted by weight w. In Fig. 8 we also illustrate the mark-to-market values of the options embedded in the insurance contract for several different levels of the long-term mean of the interest rate, i.e. $\theta_r = \{10\%, 8\%, 6\%\}$.

A closer look at the fair values in Fig. 8 reveals that, when we invest more in bonds (or w becomes lower), the fair value of the insurance contract decreases. This is consistent with our intuition, since bonds are considered as less risky asset ($\sigma_B = 5\%$) with regards to stocks ($\sigma_S = 15\%$). However, Fig. 8 also shows that the change in fair value stabilizes around stock weight $w = 20\%$, and even slightly increases when the stock weight w declines towards zero. Therefore, it is not very plausible to construct a portfolio with bonds only. However, due to either higher profit-sharing rate to the insured α or a higher minimum guaranteed rate r_G, the insurer is forced to adopt a more aggressive investment strategy in order to meet the target demand. Moreover, the patterns of variability in the fair values for different levels of the long-term mean of the interest rate θ_r are quite similar for different stock weights. Again, we have illustrated that high interest rates have negative impact on the risk-neutral embedded value. This is in line with our findings from the previous chapter (e.g. see Chapter 3 – Table 2).

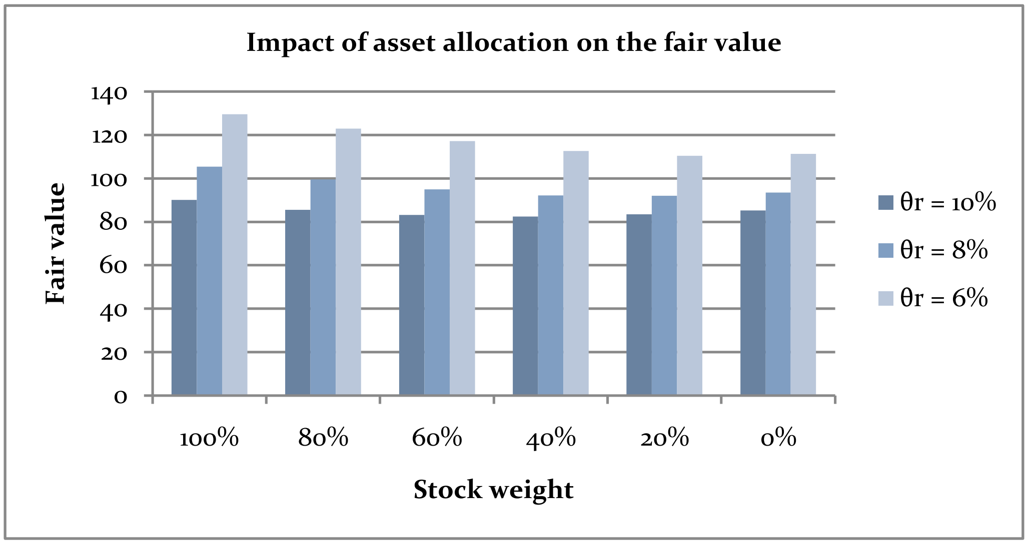

Fig. 8. The figure illustrates the financial impact of asset allocation (bond or stock) on the fair value of the insurance contract for different weights in stock $\{1; 0.8; 0.6; 0.4; 0.2; 0\}$ and different levels of the long-run mean of the interest rate $\{10\%; 8\%; 6\%\}$. Other parameters used are: minimum guaranteed rate $r_G = 0.03$, profit-sharing rate insured $\alpha = 0.20$, profit-sharing rate insurer $\beta = 0.30$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_S = 0.15$, $\sigma_B = 0.05$, $\sigma_r = 0.01$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$ and scenario $= 50000$.

In this respect, insurance companies should implement an asset allocation strategy for the reference portfolio to minimize the financial risk induced by the insurance policy [see, e.g., Brinson et al. (1991)]. However, the objective of this research project is not to find an optimum investment strategy (e.g. optimum bond-stock mix), but to focus on the pricing of insurance contracts. Hence in the remaining parts of the thesis, we consider two cases: 1) we invest in stocks only (i.e. $w = 100\%$); 2) we invest 30% in stocks and 70% bonds (i.e. $w = 30\%$). For more information regarding the topic of asset allocation on participating life insurance policies, see for example, Ballotta and Haberman (2009).

4.4.2. *Impact of stochastic interest rates*

The short rate is now modeled as a stochastic process using the Vasicek (1977) term structure model such as defined in Section 4.1. We examine the financial impact of stochastic interest rates on the fair price of the financial guarantee with profit-sharing rate of insured $\alpha = 0.20$ and profit-sharing rate of insured $\beta = 0.30$. The volatility of the risk-free rate σ_r is assumed to be deterministic and constant over time and is chosen from the set $\sigma_r = 1\%, 2\%, 3\%$. The present value of the single premium life insurance policy is presented by

$$\Pi(0, X_T) = E_Q\left[e^{-\sum_{t=1}^{T}(r_t - \bar{\pi})} X_T\right], \tag{4.23}$$

where $\Pi(0, X_T)$ presents the fair value of X_T at time 0, $E^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, $\mathbb{Q}$, given the information at date zero, X_T presents the market value of the insurance account at expiry date, r_t is the short rate of interest at time t and $\bar{\pi}$ describes the long-term mean of the inflation rate.

In Table 4, we list the numerical results of the fair values of the European life insurance contract. The computed fair values are based on three different levels of interest rate volatility ($\sigma_r = 1\%, 2\%, 3\%$) and two different asset allocations ($w = 100\%, 30\%$). Furthermore, the percentage increase of the change in the risk-neutral option value as regards to the standard case (i.e. $\sigma_r = 0\%$ and $\sigma_\pi = 0\%$) is given in parenthesis.

Table 4
Fair values of a single premium European life insurance contract for different levels of long-run mean of the interest rates (θ_r) with profit-sharing rate insured $\alpha = 0.20$ and profit-sharing rate insurer $\beta = 0.30$. Assumptions used are: 1) stochastic equity return; 2) deterministic or stochastic interest rate; 3) deterministic inflation rate; 4) no mortality risk; 5) no surrender risk.

Interest rate scenario	Chapter 3	Chapter 4		
	$\sigma_r = 0\%$ and $\sigma_\pi = 0\%$	$\sigma_r = 1\%$ and $\sigma_\pi = 0\%$	$\sigma_r = 2\%$ and $\sigma_\pi = 0\%$	$\sigma_r = 3\%$ and $\sigma_\pi = 0\%$
$w = 1$				
$\theta_r = 10\%$	81.21	85.57 (+5%)	104.05 (+28%)	166.73 (+105%)
$\theta_r = 8\%$	96.06	109.51 (+14%)	163.08 (+70%)	304.87 (+217%)
$\theta_r = 6\%$	182.07	213.93 (+18%)	331.64 (+82%)	622.49 (+242%)
$w = 0.30$				
$\theta_r = 10\%$	77.99	77.38 (−1%)	94.30 (+21%)	154.67 (+98%)
$\theta_r = 8\%$	73.02	84.51 (+16%)	140.18 (+92%)	281.13 (+285%)
$\theta_r = 6\%$	101.03	148.25 (+47%)	271.46 (+169%)	569.60 (+464%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates, in column 3 to 5 we report the fair values under the assumption of stochastic interest rates and deterministic inflation rates. In addition, we report the fair values for two different portfolios where stock weight $w = \{1, 0.30\}$. Between parentheses are the percentage increases of the fair values as regards to the basic contract value from Chapter 3. The present value of the insurance policy is defined as $\Pi(0, X_T) = E_Q\left[e^{-\sum_{t=1}^{T}(r_t - \bar{\pi})} X_T\right]$, where X_T presents the market value of the insurance account at expiry date, r_t is the interest rates at time t and $\bar{\pi}$ describes the long-term inflation rate. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$ and scenario = 50000.

From Table 4 we observe that the interest rate risk effect on the contract value is extremely large. For example, if we look at the case where the long-term interest rate $\theta_r = 10\%$, the fraction of investment in stocks $w = 100\%$ and the interest rate volatility $\sigma_r = 0\%$, the corresponding value of the basic contract is given by 81.21. If we increase the interest rate volatility σ_r from 0% to 1%, 2% or 3%, the fair value becomes 85.57, 104.05 and 166.73 with a respective upwards change of 5%, 28% and 105%. It can be seen that the contract value increases exponentially. Similar developments are found for $\theta_r = 8\%$ and $\theta_r = 6\%$, where the influence of stochastic interest rates is even more impacting. When we turn our attention to the case where the stock weight $w = 30\%$, we notice that the fair values are much lower as compared to the case of $w = 100\%$. Indeed, the main purpose of asset allocation is to enhance the likelihood of achieving desired investment returns and to reduce risk and volatility in the investment portfolio. With respect to the initial example, the risk-neutral contract value now becomes 77.99 which is an improvement of approximately 4% $(= 81.21/77.99)$. However, the fair value of the diversified portfolio is more sensitive to the interest rate risk, especially when the market interest rate drops and its volatility increases. This result could be explained by the dynamics of the instantaneous standard deviation of the bond portfolio, i.e., $\sigma_B = \sigma_r D$. The bond volatility σ_B is thus an increasing function of the parameter σ_r and D, where σ_r denotes the interest rate volatility and D represents the bond duration. Based on $D = 5$, an increase in interest rate volatility of 1% corresponds with an increase in bond volatility of 5%. This change is relatively high as compared to the stock volatility $\sigma_s = 15\%$ which is assumed to be constant over time.

Regarding the fair pricing principle, given in (3.15), we mention that the fair contract from Chapter 3 $(r^* = 8\%,\ \alpha^* = 0.20,\ \beta^* = 0.30, w = 100\%,\ 96.06)$ is not valid anymore under the current assumptions. When more variability in interest rates is added, e.g. $\sigma_r = 1\%$, the respective contract value is given by 109.51 (+14%) which exceeds the initial premium paid at the inception of the contract (i.e. $P_0 = 100$). In order to fulfill the fair pricing principle, the insurance provider may raise the profit-sharing rate β, or alternatively, reduce the rate of excess return shared with the insurance client α.

4.4.3. Impact of stochastic inflation rates

In this subsection, we determine the impact of inflation rate risk on the fair pricing of embedded options. More specifically, we assume that the price inflation process is stochastic, whereas the interest rate dynamics remains deterministic and constant during the lifetime of the contract. The present value of the insurance policy is given by

$$\Pi(0, X_T) = E_Q\left[e^{-\sum_{t=1}^{T}(r-\pi_t)} X_T\right], \tag{4.24}$$

where $\Pi(0, X_T)$ presents the fair value of X_T at time 0, $E^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, $\mathbb{Q}$, given the information at date zero, X_T presents the market value of the insurance account at expiry date, r is the constant short rate of interest and π_t describes the inflation rate at time t. To properly assess the inflation rate risk, we compare the results from Chapter 3 – i.e. deterministic and constant interest rate and inflation rate (see Table 2) – with the new case wherein the inflation rates are assumed to be stochastic. These comparisons are highlighted in Table 5.

Table 5
Fair values of a single premium European life insurance contract for different levels of long-run mean of the interest rates (θ_r) with profit-sharing rate insured $\alpha = 0.20$ and profit-sharing rate insurer $\beta = 0.30$. Assumptions used are: 1) stochastic equity return; 2) deterministic interest rate; 3) deterministic or stochastic inflation rate; 4) no mortality risk; 5) no surrender risk.

Interest rate scenario	Chapter 3	Chapter 4		
	$\sigma_r = 0\%$ and $\sigma_\pi = 0\%$	$\sigma_r = 0\%$ and $\sigma_\pi = 1\%$	$\sigma_r = 0\%$ and $\sigma_\pi = 2\%$	$\sigma_r = 0\%$ and $\sigma_\pi = 3\%$
$w = 1$				
$\theta_r = 10\%$	81.21	81.17 (−0%)	82.65 (+2%)	84.27 (+4%)
$\theta_r = 8\%$	96.06	97.98 (+2%)	101.78 (+6%)	107.71 (+12%)
$\theta_r = 6\%$	182.07	184.63 (+1%)	194.76 (+7%)	208.55 (+15%)
$w = 0.30$				
$\theta_r = 10\%$	77.99	77.83 (−0%)	77.24 (−1%)	76.57 (−2%)
$\theta_r = 8\%$	73.02	73.05 (+0%)	73.34 (+0%)	75.53 (+3%)
$\theta_r = 6\%$	101.03	104.88 (+4%)	115.06 (+14%)	131.71 (+30%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates, in column 3 to 5 we report the fair values under the assumption of deterministic interest rates and stochastic inflation rates. In addition, we report the fair values for two different portfolios where stock weight $w = \{1, 0.30\}$. Between parentheses are the percentage increases of the fair values as regards to the basic contract value from Chapter 3. The present value of the insurance policy is defined as $\Pi(0, X_T) = E_Q\left[e^{-\sum_{t=1}^{T}(r-\pi_t)}X_T\right]$, where X_T presents the market value of the insurance account at expiry date, r is the constant short rate of interest and π_t describes the inflation rate at time t. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_\pi = 0.0101$, mean reversion rate of the inflation rate $\gamma = 0.4740$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{\Psi,\pi} = 0.1641$, and scenario = 50000.

Based on Table 5, several important statements can be made. First, it can be observed that the market-consistent contract value is increasing with the respective level of the inflation rate volatility. For instance, for $\theta_r = 8\%$ and $w = 100\%$, the risk-neutral contract value increases by 2% when the inflation rate volatility σ_π is set to 1%. When the volatility level σ_π rises from 0% to 2% and 3%, the European contract value increases by respectively 6% and 12%. As expected, the more volatile the expected inflation rate process, the greater the uncertainty regarding the actual future value of our investment. Furthermore, the impact of inflation rate risk is most remarkable for low levels of market interest rates. In fact, low market interest rates are typically associated with adverse market conditions and thus poor investment performances. In such situation, any reduction in the fund value due to inflation risk will adversely impact the contract value. This finding is similar to the case of stochastic interest rates as described in Subsection 4.4.2. We have also conducted the same analysis for a portfolio consisting of 30% stock and 70% bonds. The respective lower fair values are exactly in line with our expectations and our previous findings.

Based on the results obtained from Table 4 and Table 5, we argue that the impact of interest rate risk on the contract values is substantially larger than the impact of inflation rate risk. This can be explained as follows. There is a huge difference between the mean-reversion parameter of the two models. More specifically, the constant mean-reversion speed of the interest rate model a in (4.6) is assumed to be 0.10, while the estimated mean-reversion rate of the expected inflation process γ in (4.11) equals 0.4742 (see Table 3). Hence, the tendency to revert to its long-term level is much stronger for the latter case. This will certainly weaken the effect of inflation rate variability on the contract value. From an economic point of view, the main task of the European Central Bank (ECB) is to keep inflation rates low and stable over time. Therefore, a high rate of mean-reversion is consistent with our intuition.

4.4.4. *Impact of stochastic interest rates and inflation rates*

Up to now we have only considered the impact of interest rate risk and inflation rate risk separately. In this section, we wish to study these risks together and also include the correlations between each risk processes. The fair value of the single premium participating insurance contract at the inception is expressed by

$$\Pi(0, X_T) = E_Q\left[e^{-\sum_{t=1}^{T}(r_t-\pi_t)}X_T\right], \tag{4.25}$$

where $\Pi(0, X_T)$ presents the fair value of X_T at time 0, $E^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, $\mathbb{Q}$, given the information at date zero, X_T presents the market value of the insurance account at expiry date, r_t is the interest rates at time t and π_t describes the inflation rate at time t. The results are reported in Table 6.

As aforementioned, the impact of stochastic inflation rates is relatively weak (column 4) when compared to the impact of stochastic short rates (column 3). This is a well-known fact since insurance companies are predominantly exposed to interest rate risk. In the last 2 columns of Tables 6 we report the fair values where both financial risk effects are analyzed. In order to reveal the influence of dependence structures among stochastic financial processes, as an example, in column 5 we assume that interest rate and price inflation are uncorrelated (other correlations are incorporated), whereas in column 6 the respective correlation coefficient $\rho_{r,\pi} = 0.5248$ is investigated. It is shown that the contract values given in column 3 (i.e. $\sigma_r = 1\%$ and $\sigma_\pi = 0\%$) and column 5 (i.e. $\sigma_r = 1\%$, $\sigma_\pi = 1\%$ and $\rho_{r,\pi} = 0$) exceed the contract values presented in column 6. Based on Equation (4.16), it can be clearly seen that the Brownian motion of the expected inflation process (W_t^{π}) is affected by the correlation value $\rho_{r,\pi}$ and presents a significant downward change in the contract value. The above example shows once again how important and relevant multivariate risk modeling is as regards to the pricing of life insurance contracts and embedded options.

Table 6

Fair values of a single premium European life insurance contract for different levels of long-run mean of the interest rates (θ_r) with the profit-sharing rate insured $\alpha = 0.20$ and the profit-sharing rate insurer $\beta = 0.30$. Other assumptions used are: 1) stochastic equity return; 2) no mortality risk; 3) no surrender risk.

Interest rate scenario	Chapter 3	Chapter 4			
	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$	$\sigma_r = 1\%$ $\sigma_\pi = 0\%$	$\sigma_r = 0\%$ $\sigma_\pi = 1\%$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ $\rho_{r,\pi} = 0$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ $\rho_{r,\pi} = 0.5248$
$w = 1$					
$\theta_r = 10\%$	81.21	85.57 (+5%)	81.17 (−0%)	84.47 (+4%)	83.59 (+3%)
$\theta_r = 8\%$	96.06	109.51 (+14%)	97.98 (+2%)	110.94 (+15%)	107.32 (+12%)
$\theta_r = 6\%$	182.07	213.93 (+18%)	184.63 (+1%)	215.87 (+19%)	207.62 (+14%)
$w = 0.30$					
$\theta_r = 10\%$	77.99	77.38 (−1%)	77.83 (−0%)	77.50 (−1%)	76.63 (−2%)
$\theta_r = 8\%$	73.02	84.51 (+16%)	73.05 (+0%)	85.04 (+16%)	81.71 (+12%)
$\theta_r = 6\%$	101.03	148.25 (+47%)	104.88 (+4%)	150.20 (+49%)	142.35 (+41%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates (see Chapter 3); in column 3 to 4 we report the fair values under the assumption of either stochastic interest rates or stochastic inflation rates; and in column 5 and 6 we report the fair values under the assumption of stochastic interest rates and stochastic inflation rates, where in column 5 $\rho_{r,\pi} = 0$ and in column 6 $\rho_{r,\pi} = 0.5248$. In addition, we report the fair values for two different portfolios where stock weight $w = \{1, 0.30\}$. Between parentheses are the percentage increases of the fair values as regards to the contract value given in column 2. The present value of the insurance policy is defined as $\Pi(0, X_T) = E_Q\left[e^{-\sum_{t=1}^{T}(r_t - \pi_t)} X_T\right]$, where X_T presents the market value of the insurance account at expiry date, r_t is the interest rates at time t and π_t describes the inflation rate at time t. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_S = 0.15$, $\sigma_\pi = 0.0101$, mean reversion rate of the inflation rate $\gamma = 0.4740$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, and scenario = 50000.

5. Insurance Contracts and Mortality Risk

The prior chapter demonstrated the importance of stochastic modeling of financial risks. This chapter examines how uncertainties regarding future mortality and life expectancy outcomes affect the fair value of life insurance participating policies. In this regard, we estimate and forecast the stochastic mortality rates by means of the Lee-Carter model (1992). The Lee-Carter model is applied to historical population data from the Netherlands in the long-term perspective. The estimation of the model's parameter is estimated using the Singular Value Decomposition (SVD) technique. In the end of the chapter, we present numerical results to reveal the impact of mortality risk on the fair valuation of embedded options.

5.1. Introduction of mortality risk

For life insurance and annuity products whose payoffs depend on future mortality rates, it is essential to study the so-called mortality risk. We distinguish two types of mortality risk: micro-longevity risk and macro-longevity risk. Micro-longevity risk defines the risk related to uncertainty in time of death if survival probabilities are known with certainty, while macro-longevity risk refers to the risk of unexpected improvements in future life expectancies [see Hari et al. (2007a)]. In practice, micro-longevity risk can be reduced by means of portfolio diversification or by pooling arguments [see, e.g., Olivieri (2000), Coppola et al. (2000, 2002, 2003), and Di Lorenzo and Sibillo (2002)]. In contrast, the fact that macro-longevity risk is a systematic risk weakens the diversification principle and increasing the portfolio size is therefore no longer applicable. Our objective is to quantify the second type of mortality risk, namely macro-longevity risk, for the solvency of minimum rate of return guarantees. For convenience, we use the general term *mortality risk* to refer exclusively to the uncertainty in future mortality and life expectancy outcomes, regardless of whether it leads to longer or shorter than expected lifetime.

Owing to the pace of ongoing improvements in medical support, nutrition, safety precautions, and a more health-oriented focus of lifestyle, human mortality rates have decreased considerably over the past centuries. These declines stem from substantial reductions in mortality rates at younger ages and, to some extent, improvements at old-ages. Although this is a positive development it brought considerable stress in pension plans for the elderly. As a consequence, adverse financial impacts caused by improper pricing and risk managing of mortality risk has been blamed as one of the main reasons for the collapse of the British *Equitable Life Assurance Society (ELAS)* in 2000, the world's oldest life insurance office. The unexpected level of population ageing, together with the introduction of market-consistent accounting and risk-based solvency requirements, has called for an integration of mortality risk analysis into stochastic valuation models.

Actuaries (or insurance statisticians) have traditionally valued premium and reserves using deterministic mortality intensity, with the implication that the past represents the future. The deterministic approach fits curves to mortality as a function of age and time to approximate mortality rates. [see, e.g., Gompertz (1825) and Makeham (1860)]. A main disadvantage of deterministic mortality approach is that the model uncertainty is not taken into account. As a result, the traditional deterministic actuarial approach is now seen to be inadequate for the assessment of fair pricing of pension liabilities. The newest direction in the study of human survival is to model and/or forecast mortality as a random process in which the variability of mortality rates is incorporated. Over the past ten years, a number of new approaches have been developed for forecasting mortality using stochastic models, such Lee-Carter (1992), Biffs (2005) and Hari et al. (2007a,b).

In this chapter, we analyze the impact of financial risks and mortality risk on participating life insurance contracts, where the mortality risk is modeled according to the Lee-Carter (1992) method. Moreover, our model framework relies on the assumption of independence between financial and actuarial risk.

5.2. The Lee-Carter method

The Lee-Carter (1992) model is one of the most popular methodologies for mortality trend fitting and projection. This model is computationally simple to apply and it has given satisfactory results in fitting mortality rates for many countries, for instance, U.S. and Canada [(Li and Chan (2007)], Chile [Lee and Rofman (1994)], Japan [Wilmoth (1996)], and the Netherlands [Hari et al. (2007a,b)].

During the last decade, various extensions of the Lee-Carter model have been developed for forecasting mortality using stochastic models, see for instance, Lee and Miller (2001), Brouhns et al (2002), Renshaw and Haberman (2003), Girosi and King (2005) and Hari et al. (2007a,b). Our attention focuses on the method proposed by Lee and Carter (1992) and used for projections of the age-specific death and survival probability rates for men and women in the Netherlands.[16] To fit the Lee-Carter model we require both the central rates of death and the exposures to risk. We use 157 yearly observations of age-specific death numbers and exposures for men and women in the Netherlands, from 1850 till 2006, provided by The Human Mortality Databases.[17]

The Lee-Carter model, specified for the logarithm of $m_{x,t}$ the central death rate for age x at time t, has the form:

[16] Variables such as number of policy years, smoking/non-smoking and some medical issues are not considered in this thesis.

[17] Human Mortality Database, University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 01/09/2009).

$$\ln(m_{x,t}) = a_x + b_x\kappa_t + \varepsilon_{x,t}, \quad x = x_1, \dots, x_N \quad \text{and} \quad t = 1, \dots T. \tag{5.1}$$

Here a_x coefficients describe the average shape of $\ln(m_{x,t})$ across age, the b_x coefficients describe the way mortality varies at the age x as a reaction to the change of the general level of mortality κ_t and x denotes the specific age of the insured.[18] If κ_t falls, mortality improves, and if κ_t rises, mortality worsens. The error term $\varepsilon_{x,t}$ illustrates the deviation of the model from the observed log central death rates and is expected to be Gaussian $\varepsilon_{x,t} \sim N(0, \sigma_\varepsilon^2)$. The Lee-Carter model presented in (5.1) is solved under the constraints $\sum_t \kappa_t = 0$ and $\sum_x b_x = 1$ to ensure identifiability of the model. We note that the central death rate is obtained by $m_{x,t} = \frac{D_{x,t}}{E_{x,t}}$, where $D_{x,t}$ denotes the number of people with age x that died in year t, and $E_{x,t}$ being the number of person years with age x in year t, the so-called exposure.

As all parameters at the right-hand side of Equation (5.1) are unobservable, the model cannot be fit by the ordinary least squares method. Nonetheless, we overcome this problem by employing the Lee and Carter's (1992) two-stage estimation procedure, which gives exact solutions. In the first stage, Singular Value Decomposition (SVD) is applied to retrieve the underlying latent process [Lawson and Hanson (1974)].[19] In the second stage, the time series of κ_t is re-estimated by solving for κ_t such that

$$D_t = \sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\}, \tag{5.2}$$

where D_t denotes total number of deaths in year t. The second stage guarantees that the life tables fitted over the sample years will reconcile the total number of deaths and the population age distributions.

The most distinguishable aspect of the Lee-Carter model is that the model allows uncertainties for forecasts. In other words, the mortality index κ_t is intrinsically viewed as a random process. Thus the resulting estimate of the time-varying parameter κ_t is then modeled as a stochastic time series using standard Box-Jenkins (1976) methods. An autoregressive integrated moving average (ARIMA) model is then used to model the dynamics of the latent factor κ_t. When estimate the Lee-Carter model, one usually models the mortality index κ_t as a random walk with drift:

$$\kappa_t = c + \kappa_{t-1} + \eta_t, \tag{5.3}$$

[18] In Lee and Carter (1992), the authors define x as the set of age groups, i.e., 1-4, 5-9, 10-14, ..., 80-84, and 85+.

[19] Two alternative approaches to SVD are: a Weighted Least Square [Wilmoth (1993)] and a Maximum Likelihood Estimation [Wilmoth (1993) and Brouhns et al. (2002)].

where permanent shocks $u_t \sim N(0, \sigma_u^2)$ are white noises and thus corresponds to ARIMA(0,1,0). In this case, the forecast of the mortality index κ_t changes linearly and each forecasted death rate $m_{x,t}$ changes at a constant exponential rate. In addition, in order to deal with potential outliers, the estimated latent process may be extended by including dummy variables to capture the effect of temporary shocks, e.g. pandemics or wars. This is exemplified by Lee and Carter (1992), who viewed subjectively the known influenza epidemic in 1918 as an anomaly and dealt with it by means of an intervention model with a dummy variable. Further note that $\varepsilon_{x,t}$ and η_t are independent, satisfying the distributional assumptions

$$\begin{pmatrix} \varepsilon_{x,t} \\ u_t \end{pmatrix} \bigg| \mathcal{F}_{t-1} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_\varepsilon & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix} \right). \tag{5.4}$$

5.3. Modeling of the old-age

Data quality issues have made the modeling of the old-age mortality difficult. In more detail, the number of deaths and exposures-to-risk at advanced ages are relatively low at the middle of 19th century and at the first half of 20th century. The occurrence of the few centenarians may lead to large sampling errors and highly crude death rates. Therefore, our approach of model fitting is simply based on the age range 0-85. Since we are also interested in the death and survival probabilities of individuals aged 85+, we need a method that can extrapolate survival distribution for this segment of the population. In order to tackle this issue, we employ mathematical extrapolation technique as presented by Coale and Guo (1989) and Coale and Kisker (1990) to provide a consistent model framework that also accounts for the emergence of older ages. The Coale and Kisker method is appealing due to its simplicity and ease of calculation.

The Coale-Kisker method has been widely applied to the mortality data of various developed countries. This method is based on the assumption that old-age mortality rates increase at a varying rate instead of a constant rate as the Gompertz law of mortality (1982) assumes.[20] The Coale-Kisker extrapolation age starts at age of 86 and defines the following relation

$$ck_{x,t} = ck_{x-1,t} - R_t, \qquad \text{for } x \geq 86, \tag{5.5}$$

where $ck_{x,t} = \ln(m_{x,t}/m_{x-1,t})$, i.e. the change in log central death rate at age x, and R_t is a constant number. We then extend the Lee-Carter model with the highest attained age, which is commonly referred to as ω in the actuarial literature. The maximum attainable age is assumed to be 110. Extending the formula (5.5) up to $\omega = 110$ and summing, the following condition is sattisfied

[20] The Gompertz (1928) model proposed that mortality rate increased exponentially with age.

$$ck_{86,t} + \cdots + ck_{110,t} = 15ck_{85,t} - (1 + 2 + \cdots + 25)R_t. \tag{5.6}$$

Solving for R_t, we obtain

$$R_t = \frac{25ck_{85,t} + \ln(m_{85,t}) - \ln(m_{110,t})}{(1 + 2 + \cdots + 25)}. \tag{5.7}$$

As proposed by Coale and Kisker (1990), the level of mortality at age 110 years (m_{110}) is fixed at 1.0 for males, since there are almost no male survivors left at ages greater than 110. With respect to the level of mortality at age 110 years for females, the authors assumed that m_{110} is fixed at 0.8. This is because women tend to have longer life expectancies than their male counterparts.

It is worth noting that various alternatives are available to model the old-age mortality. For example, Panjer and Russon (1992) and Panjer and Tan (1995) use cubic polynomial to extend survival distribution beyond age 100; Heligman and Pollard (1980) proposed a discrete version of the Gompertz law of mortality to describe the mortality rates; Himes, Preston and Coudran (1994) suggest a method using logit regression to extrapolate any life table by relating it to the extended 'standard' mortality schedule; more recently, Li, Hardy and Tan (2008) developed a threshold life table using extreme value theory. This model allows us to extrapolate a survival distribution without the need for accurate mortality data.

5.4. Estimation procedure of Lee-Carter model

Before we reach the stage of forecasting the survival probability, we first provide a more detailed treatment of the Lee-Carter model as described above. In order to estimate the parameters a_x (intercept), b_x (slope) and the mortality index κ_t of the Lee-Carter model for a given matrix of log central death rates $\ln(m_{x,t})$, we employ the following estimation procedures:

1. We assume that for any integer x, and any time t, the force of mortality is constant during the year: $m_{x+u,t+u} = m_{x,t}$, for all $u \in [0,1)$. Then, one can verify that $p_{x,t} = e^{-\int_0^1 m_x(s)ds} = \exp(-m_{x,t})$.
2. The estimator of a_x are given by the means: $\hat{a}_x = \frac{\sum_t \ln(m_{x,t})}{T}$.
3. Let Z denote the demeaned matrix of the log central death rate $\ln(m_{x,t})$, $Z = \ln(m_{x,t}) - \hat{a}_x \iota$, where ι is a row vector of ones.
4. We estimate $\hat{b}_x$ and $\hat{\kappa}_t^{(1)}$ using the Singular Value Decomposition, i.e. $Z = USV^T$.

5. We re-estimate $\hat{\kappa}_t^{(2)}$ with values $\hat{a}_x$ and $\hat{b}_x$ obtained from the previous steps to satisfy the condition $D_t = \sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\}$.

The most challenging part of estimating the Lee-Carter model is Step 4, namely the implementation of Singular Value Decomposition (SVD). The SVD is a widely used technique to decompose a matrix into several component matrices, exposing many of the useful and interesting properties of the original matrix. The intuition behind this approach is that the difference to the mean is decomposed into a time trend κ_t and an age-specific factor b_x that determines the strength with which the time trend affects a certain age. A SVD can be constructed by $Z = USV^T$(see Appendix D1).

As it turns out, the vectors in the expansion of Z are the eigenvectors of the square matrices ZZ^T and Z^TZ. The column of U are called left singular vector of Z, and the columns of V are called right singular vector of Z. The singular values are the nonzero square roots of the eigenvalues from ZZ^T and Z^TZ. The SVD can be represented equivalently as

$$Z = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \ldots + \sigma_r u_r v_r^T,$$

where u_j and v_j are the jth columns of U and V respectively, and r is the minimum of m and n, denoted $r = \min(m, n)$.

If most singular values are smaller than the others, then it might be reasonable to approximate Z by a few of the terms with the largest singular values. As proposed by Lee and Carter (1992), the first singular value is significantly larger that other singular values, one can use one factor (rank 1) to approximate the log death rates. We have

$$Z \approx \sigma_1 u_1 v_1^T = \hat{b}_x \hat{\kappa}^{(1)},$$

where

$$\hat{b}_x = \frac{\sigma_1}{\bar{u}_1 N} \qquad \text{and} \qquad \hat{\kappa}^{(1)} = \sigma_1 v_1 \bar{u}_1 N. \tag{5.8}$$

To deal with the unity condition of the slope coefficient b_x, we divide $\hat{b}_x$ by $\bar{u}_1 N$ such that $\sum_x b_x = 1$ holds. Here the term $\bar{u}_1$ denotes the average of the first column of U and N is the total number of age groups.

During the estimation procedure the mortality index κ_t is first estimated to minimize errors in the log of central death rates rather than the death rates themselves. At this point, we proceed to Step 5 to re-estimate the mortality index κ_t for a given population age distribution. This is done by taking the previous estimates of a_x and b_x. Next, we aim to find $\hat{\kappa}^{(2)}$ such that

$$D_t = \sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\}. \tag{5.9}$$

Since no analytic solution is available for $\hat{\kappa}_t^{(2)}$, this can be done only by searching over a range of values of the mortality level κ. The searching algorithm can be schematized in the following steps:

1. Define the relation $D_t = \sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\}$, where $\hat{\kappa}_t^{(1)}$ presents the initial value of κ_t
2. Check which of the following three cases is satisfied
   - If $\sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\} - D_t > 0$ and $\begin{cases}\kappa_t > 0, \text{then } \kappa_t = \kappa_t(1-\lambda) \\ \kappa_t < 0, \text{then } \kappa_t = \kappa_t(1+\lambda)\end{cases}$
   - If $\sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\} - D_t < 0$ and $\begin{cases}\kappa_t > 0, \text{then } \kappa_t = \kappa_t(1+\lambda) \\ \kappa_t < 0, \text{then } \kappa_t = \kappa_t(1-\lambda)\end{cases}$
   - If $\sum\{\exp(a_x + b_x\kappa_t)E_{x,t}\} - D_t = 0$, then $\hat{\kappa}_t^{(2)} = \kappa_t$ and stop iteration (+ repeat all the steps for $t = t+1$)
3. Return to Step 1

   (Remark: λ is a small number, e.g. $\lambda = 0.001$)

After obtaining the new values $\hat{\kappa}_t^{(2)}$, we attempt to find an appropiate model for the mortality index κ_t. The latent factor κ_t modeled for both male and female according to historical Dutch mortality data is given by

$$\kappa_t^m = -1.3361 + \kappa_{t-1}^m + 20.52 \cdot flu_t + 11.92 \cdot WWII_t + \eta_t^m \tag{5.10}$$

$$\kappa_t^f = -1.6035 + \kappa_{t-1}^f + 22.04 \cdot flu_t + 7.51 \cdot WWII_t + \eta_t^f \tag{5.11}$$

where two dummies variables are included to deal with temporary shocks, i.e. (1) the 'flu' dummy which stands for the 1918 Spanish flu pandemic and (2) the 'WWII' dummy which refers to the event of World War II dated from 1939 till 1945 for European countries.[21] The corresponding standard errors of the mortality index model for male and female are $\sigma_\eta^m = 5.91$ and $\sigma_\eta^f = 5.90$ respectively. Finally, we use the estimated parameter values to derive the log central death rates $\ln(m_{x,t})$ by applying formula (5.1).

Fig. 9 shows the estimated Lee-Carter (1992) model parameters values of the intercept coefficient a_x, the slope coefficient b_x and the mortality level κ_t for men and women in the

[21] The flu dummy takes non-zero values for years {1918 = 1; 1919=-1} and zero elsewhere. The WWII dummy takes non-zero values for years {1940-1945 = 1 and 1946 =-6} and zero elsewhere.

Netherlands. Additionally, this figure presents the corresponding log central death rates, $\ln(m_{x,t})$, for the period 1850 till 2006. The a_x coefficients are the average values of the logs of the death rates. Not surprisingly, the male coefficients lie above those of the female at all ages, reflecting the fact that mortality was higher for men on average from 1850 to 2006. The b_x coefficients describe the relative sensitivity of death rates to variation in the latent factor κ_t. If the b_x is high for some age x, then this means that the mortality rate improves faster at this age than in general. If it were negative at some ages, in particular for old ages, this would mean that the likelihood of dying at those ages are increasing. As can be seen from the graph, the set of coefficient is quite similar for both genders. As the b_x's are controlled to sum to 1, their absolute levels have no particular meaning.

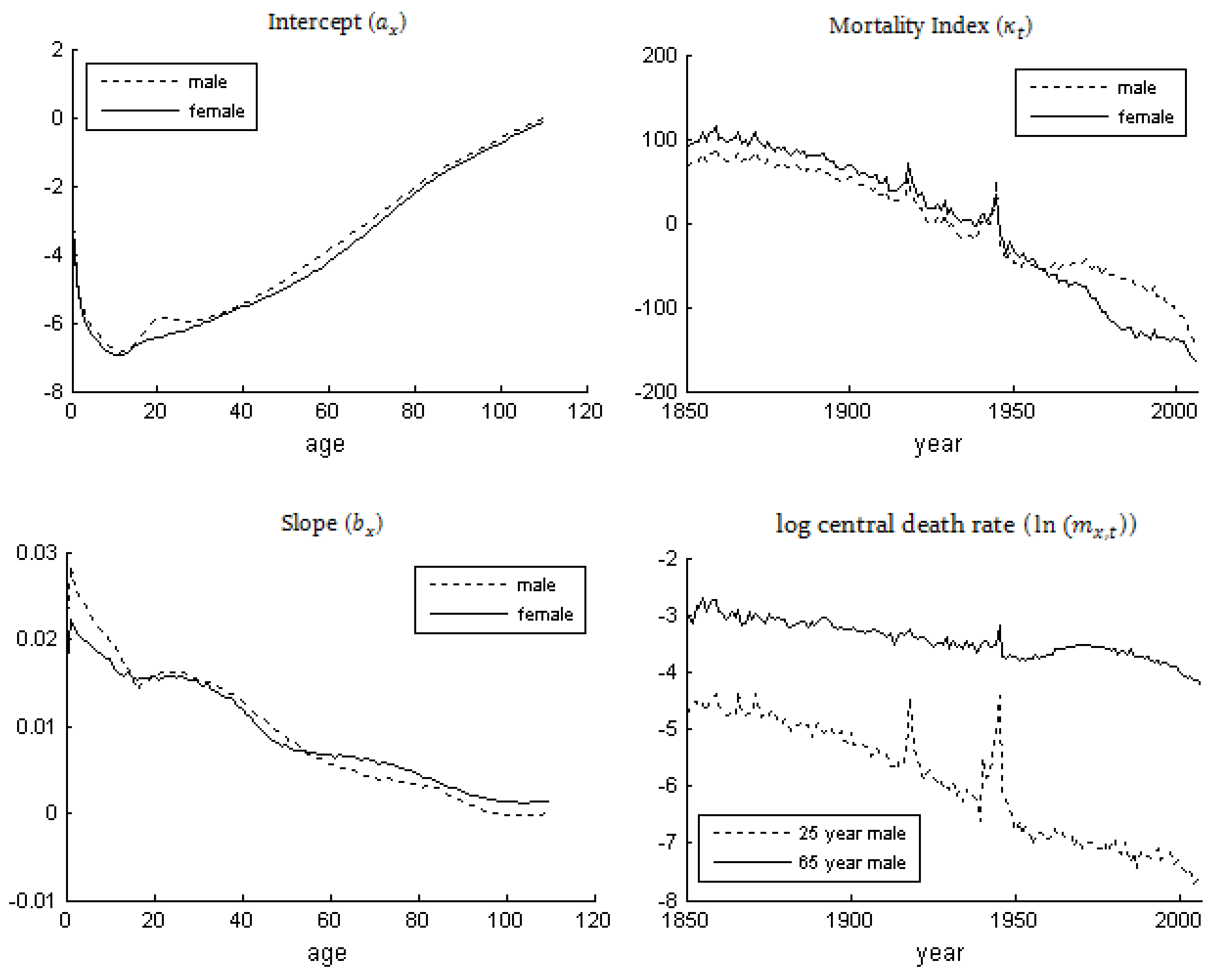

Fig. 9. The estimated Lee-Carter (1992) model: $\ln(m_{x,t}) = a_x + b_x\kappa_t + \varepsilon_{x,t}$, with a_x (left upper panel), κ_t (right upper panel), b_x (left lower panel) for both male and female, and $\ln(m_{x,t})$ for $x = 25$ and $x = 65$ aged male (right lower panel). The model is constructed using Dutch mortality data for the period 1850 till 2006 and the considered age range is 0-110.

Furthermore, in the right upper panel of Fig. 9 we illustrate the development of the mortality index κ_t from the period 1850 to 2006. It is clear that the mortality data is rather volatile, particular around the period of the Spanish flu (1918) and the Second World War (1939-1945). Overall, the mortality level has been decreasing almost linearly during the sample period, for both male and female, which suggests that the human life expectancy is improving over years.

In the following, we compare the log central death rates, $\ln(m_{x,t})$, of a male policyholder aged 25 and a male policyholder aged 65 during the sample period 1985-2006 in the right lower panel of Fig 9. We notice that the fitted values of $\ln(m_{x,t})$ are decreasing for both age categories, reflecting an increase in the life expectancy as expected. It should also be addressed that the improvements in mortality for a 25 aged male are more significant than the case of 65 aged male. Another essential observation from Fig. 9 is that the log central death rates of the 25-year-old policyholder are severely affected by the Spanish flu (1918) and the WWII (1939-1945), whereas the mortality intensity of the 65-year-old policyholder remains almost unchanged. Moreover, it worth noting that the main findings obtained in Fig. 9 is also consistent with the results from Cui (2008).

5.5. Forecasting future mortality

Having developed and fitted the demographic model we are now ready to move to the problem of forecasting. Fig. 10 displays actual fitted base period value of the mortality index κ_t for the Netherlands from 1850 to 2006 and their forecasts from 2007 to 2100. More specifically, we have performed one-step ahead forecasts for the latent process κ_t. Note that the points forecast are essentially linear extrapolations of the base period series. 95% confidence intervals are also shown for the forecast of κ_t.

The next step is to convert the forecasts of κ_t into the forecasts of log death rates using the previously estimated age-specific coefficients a_x and b_x and model (5.1). Once the implied forecast of the central death rates has been obtained, any desired life table function can be constructed.

Using the forecasted results as outlined above, we derive the survival probability and the expected remaining lifetime of an individual with age x at time t. Let us denote ${}_{\tau}p_{x,t}$ for the probability that an x-year-old insurance client at time t will survive at least another τ years with the convention that ${}_{1}p_{x,t} = p_{x,t}$, i.e.

$$ {}_{\tau}p_{x,t} = p_{x,t} \times p_{x+1,t+1} \times \dots \times p_{x+\tau-1,t+\tau-1}. \tag{5.12} $$

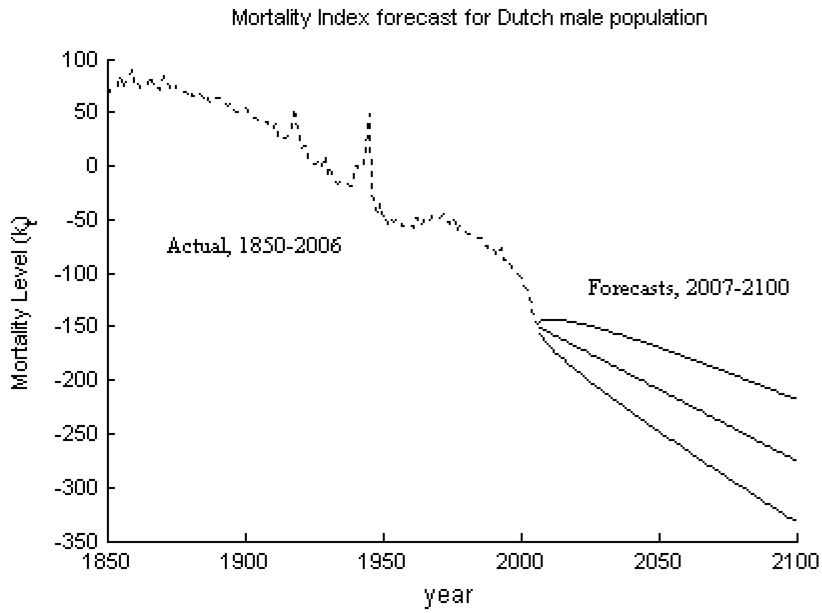

Fig. 10. Mortality forecast from actual data (1850-2006) to forecasts (2007-2100). The actual data are modeled as ARIMA(0,1,0) model from 1850 and has one dummy for the influenza epidemic and one dummy for the WWII.

The remaining lifetime of an x-year-old individual at time t is given by $T_{x,t}$, and $1_{(T_{x,t} \geq \tau)}$ refers to the indicator random variable that indicates whether an x-year-old person at time t will survive at least τ more years [see Hari et al. (2007a)]. Then, conditional on survival rates up to period t, the expected curtate remaining lifetime of a policyholder aged x at time t is given by

$$\begin{aligned} E_t[T_{x,t}] &= \sum_{\tau=1}^{\omega-x} E_t\left[1_{(T_{x,t} \geq \tau)}\right] \\ &= \sum_{\tau=1}^{\omega-x} E_t[\,{}_{\tau}p_{x,t}] \\ &= \sum_{\tau=1}^{\omega-x} E_t\left[\exp\left(-\sum_{s=0}^{\tau-1} m_{x+s,t+s}\right)\right], \end{aligned} \tag{5.13}$$

where the second expression of (5.13) follows from the law of iterated expectation, and where ω denotes the maximum attainable age, i.e. $\omega = 110$.

To reveal the size of improvements in human life expectancy, we define the expected remaining lifetime for selected historical years under the assumption that there are no further improvements in survival rates and compare it to the case when improvements in mortality are included. The expected remaining lifetime without accounting for future mortality development is calculated by

$$E_t\left[T_{x,t}\right] = \sum_{\tau=1}^{\omega-x} E_t\left[\,{}_{\tau}p_{x,t}\right] = \sum_{\tau=1}^{\omega-x} E_t\left[\exp\left(-\sum_{s=0}^{\tau-1} m_{x+s,t}\right)\right]. \tag{5.14}$$

Life tables for which age-specific probabilities of death are calculated under the assumption that further improvement in survival probability are not taken into account are known as period life tables. In contrast, cohort life table are calculated using age-specific mortality rates which allow for known or projected changes in mortality in later years.

Table 7
Expected remaining lifetime based on period and cohort life table

Gender	Year	Period Life Table		Cohort Life Table	
		25	65	25	65
Men	1900	39.7	10.6	44.3	11.3
	1925	45.3	12.3	46.4	12.2
	1950	48.5	13.7	49.4	13.9
	1975	47.7	13.0	52.6	13.4
	2000	51.1	15.0	54.7	15.9
Women	1900	40.7	11.2	44.7	11.8
	1925	45.4	12.9	51.3	12.4
	1950	49.9	14.2	55.4	15.3
	1975	53.5	16.8	57.5	18.5
	2000	56.3	19.3	59.7	19.5

The table shows the expected remaining lifetime at the age of 25 and 65 for men and women according to two different life tables. The period life table is based on the assumption that there is no further improvement in mortality, while the cohort life table includes the future improvement of mortality rates.

Table 7 reports the expected remaining lifetime computed by the period - and the cohort life table for historical time periods from 1900 to 2000. The analysis is implemented on Dutch male and female policyholders at the age of 25 and 65. Table 7 shows that the expected remaining lifetime based on the period life table is somewhat lower as compared to the one obtained by the cohort life table. For instance, according to the period life table the expected remaining lifetime of a 25-year-old male in year 2000 is 51.1, while based on the cohort table we observe an expected remaining lifetime of 54.7, which is an increase of approximately 7%. The results confirm the fact that mortality calculations which are based on period life tables seriously underestimate future life expectancy. For this reason, cohort life tables are regarded as a more appropriate measure of expected life expectancy than period life tables. The analysis of mortality risk in this thesis is carried out by using cohort life tables [see formula (5.14)].

5.6. Effect of mortality on fair value

In this section, we explore the impact of mortality risk on the fair value of the single premium insurance contract as described in Chapter 3. We build upon the approach of Bacinello (2003b). According to Bacinello (2003b), the fair European contract adjusted for death and survival probabilities can be expressed by

$$\Pi(0, X_T) = E_Q\left[\sum_{t=1}^{T-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{t=1}^{T-1}(r_t-\pi_t)} X_T \cdot {}_Tp_x\right], \tag{5.15}$$

where $\Pi(0, X_T)$ presents the fair value of X_T at time 0, $E^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, $\mathbb{Q}$, given the information at date zero, X_T is the liability without consideration for the mortality risk, ${}_{t-1/1}q_x$ presents the probability of an x-year-old policyholder dies at t-th year and ${}_Tp_x$ presents the probability of an x-year-old policyholder survives another T years. The parameter r_t and π_t denote respectively the stochastic interest rate and the short inflation rate at time t. Furthermore, we have assumed that the remaining lifetime $T_{x,t}$ is stochastically independent of the Brownian motions W_t^r, W_t^S, W_t^Ψ and W_t^π. Intuitively, this means that the event death is independent of stochastic financial processes such as the term structure, the price inflation and the performance of the stock returns, and vice versa.

It is worth noting that the reference insured is a Dutch male aged $x = 25$ at the inception of the contract. A similar analysis can be conducted for a female policyholder; although a minor modification of the respective mortality rates would be required. Moreover, in the previous sections we have assumed that the insurance client is aged 25 and is alive at the age of 65 (i.e. the retirement age). The contract maturity is therefore $T = 40$ years $-$ this implicitly also assumes that the death probability of an x-year-old policyholder in t-th year ${}_{t-1/1}q_x = 0$ and the probability of an x-year-old policyholder survives another T years ${}_Tp_x = 1$. In our current setting, where mortality risk is incorporated, feasible values of the death probability ${}_{t-1/1}q_x$ and the survival probability ${}_Tp_x$ are elements of the unit interval [0,1].

Next, Table 8 reports fair values of a single premium European life insurance contract for different levels of long-run mean of the interest rates (θ_r) and stock weight $w = 0.3$. The last column of Table 8 reports the fair value of the European contract adjusted for mortality risk, while the other columns report the fair values in which the mortality risk is neglected. Between parentheses we give the percentage changes in fair values with respect to the basis contract from

Chapter 3 (i.e. column 2). Turning to the last column of Table 8, we observe the fair value to be highest when all the three risk effects are taken into account. This is hardly surprising since each risk factor poses risk premium to the insurer. To be able to gain more insight regarding the mortality effect, we set the volatility of interest rates and inflation rates both to zero (i.e. $\sigma_r = 0$ and $\sigma_\pi = 0$). The corresponding results are provided in column 6 of Table 8.

Table 8
Fair values of a single premium European life insurance contract for different levels of long-run mean of the interest rates (θ_r) and stock weight $w = 0.3$. Other assumptions used are: 1) stochastic equity return; 2) no surrender risk.

$w = 0.30$	Chapter 3	Chapter 4			Chapter 5	
Interest rate scenario	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 1\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 0\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$
Contract type: $\alpha = 0.20$ and $\beta = 0.30$						
$\theta_r = 10\%$	77.99	77.38 (−1%)	77.83 (−0%)	76.63 (−2%)	86.38 (+11%)	85.79 (+10%)
$\theta_r = 8\%$	73.02	84.51 (+16%)	73.05 (+0%)	81.71 (+12%)	91.89 (+26%)	101.94 (+40%)
$\theta_r = 6\%$	101.03	148.25 (+47%)	104.88 (+4%)	142.35 (+41%)	137 (+36%)	179.51 (+78%)
Contract type: $\alpha = 0.20$ and $\beta = 0.40$						
$\theta_r = 10\%$	73.75	72.71 (−1%)	73.27 (−1%)	71.95 (−2%)	80.83 (+10%)	80.31 (+9%)
$\theta_r = 8\%$	67.11	79.34 (+18%)	67.01 (+0%)	76.65 (+14%)	84.58 (+26%)	95.11 (+42%)
$\theta_r = 6\%$	98.86	144.83 (+47%)	102.37 (+4%)	139.42 (+41%)	133.71 (+35%)	174.08 (+76%)
Contract type: $\alpha = 0.20$ and $\beta = 0.50$						
$\theta_r = 10\%$	68.98	68.20 (−1%)	69.18 (+0%)	67.64 (−2%)	75.25 (+9%)	74.42 (+8%)
$\theta_r = 8\%$	61.23	74.58 (+22%)	60.85 (−1%)	71.63 (+17%)	77.48 (+27%)	88.60 (+45%)
$\theta_r = 6\%$	97.05	143.04 (+47%)	100.55 (+4%)	136.46 (+41%)	130.32 (+34%)	169.88 (+75%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates (see Chapter 3); in column 3 to 5 we report the fair values under the assumption of either stochastic interest rates or stochastic inflation rates or both (see Chapter 4); in column 6 we report the fair values under the assumption of stochastic mortality and in column 7 we report the fair values under the presence of interest rate risk, inflation risk and mortality risk. Between parentheses are the percentage increases of the fair values as regards to the contract value given in column 2. The present value of the insurance policy is defined as $\Pi(0, X_T) = E_Q\left[\sum_{t=1}^{T-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{t=1}^{T-1}(r_t-\pi_t)} X_T \cdot {}_Tp_x\right]$, where X_T is the liability without consideration for the mortality risk, ${}_{t-1/1}q_x$ presents the probability of an x-year-old policyholder dies at t-th year, ${}_Tp_x$ presents the probability of an x-year-old policyholder survives another T, r_t is the interest rates at time t and π_t describes the inflation rate at time t. The reference insured is a male aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_\pi = 0.0101$, mean reversion rate of the inflation rate $\gamma = 0.4740$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, and scenario = 50000.

As an example, we will look at the contract type $\alpha = 0.20$ and $\beta = 0.30$. Among the three risk sources, it can be seen that the influence of mortality risk is strongest when the long-term mean of the interest rate $\theta_r = 10\%$. We also observe that the increase in fair values due to change in market interest rates occur almost linearly for the mortality risk, whereas the increase in fair values is somewhat exponentially for the interest rate risk. This implies that the effect of different type of risk is heavily dependent on the level of the market interest rate, or more generally, the state of the economy. Moreover, Table 8 also reflects the fact that higher contribution in profit-sharing rate β is associated with a decrease in the contract value. The present value of the insurer's account $\Pi(0, C_T + R_T^-)$ improves due to an increase in the risk premium caused by an upward change in β. Based on the results, an appropriate value for the profit-sharing rate β would be near 0.4 or 0.5.

Based on the analysis outline above and the results from Chapter 3 and 4, we argue that a substantial amount of the fair value change is dedicated to the risk of mortality. The huge impact of mortality risk on the fair premium can be explained as follows. In normal circumstances, the life insurance benefit would be paid to the policyholders providing that they survive until maturity, T, and there might be a payment on earlier death. In the case that the policyholder dies within the investment period $[0, T]$, say in year $t = 5$, the insurer is obliged to pay a specified amount of benefit to the beneficiary of the insured client. In this thesis, we have assumed that the amount of the death benefit is equal to the accrued policyholder's account at the end of the year of death. If the investment performance in the first 5 years is worse or at least below the guaranteed benefits, the insurer has to cover any deficits on the insurance account. This will negatively impacts the fair contract value. In addition, we would like to remark that the definition of mortality risk used in the thesis is somewhat different from other papers. In practice, the mortality risk is often referred as the systematic deviations of the realized mortality rates from the projected rates. Since we have initially assumed that the policyholder does not die within the period of interest $[0, T]$, our projected mortality rates is rather unrealistic because ${}_{t-1/1}q_x = 0$ and ${}_Tp_x = 1$ are used. This would explain the major impact as well.

Additionally, we show in Table 9 the fair values of the single premium contract where no asset allocation is applied ($w = 100\%$). The result reveals that the undiversified portfolio leads to a higher fair premium, which is in line with our feeling. However, we also find that the financial impact of mortality risk in Table 9 is significantly lower than the mortality impact observed in Table 8. Interesting, the difference between the two portfolios is biggest when $\theta_r = 6\%$. In this case, it is very difficult for the insurance company to make enough investment return to meet the minimum guaranteed rate r_G. We know that the average inflation is around 2.5%, which implies that the real interest rate should be around 3.5% ($= 6\% - 2.5\%$), that is near the minimum required. Hence, in order to meet the guaranteed benefits, one way is to invest 100% in stocks.

Table 9
Fair values of a single premium European life insurance contract for different levels of long-run mean of the interest rates (θ_r) and stock weight $w = 1$. Other assumptions used are: 1) stochastic equity return; 2) no surrender risk.

$w = 1$	Chapter 3	Chapter 4			Chapter 5	
Interest rate scenario	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 1\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 0\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$
Contract type: $\alpha = 0.20$ and $\beta = 0.30$						
$\theta_r = 10\%$	81.21	85.57 (+5%)	81.17 (−0%)	83.59 (+3%)	89.17 (+10%)	92.95 (+14%)
$\theta_r = 8\%$	96.06	109.51 (+14%)	97.98 (+2%)	107.32 (+12%)	113.66 (+18%)	125.03 (+30%)
$\theta_r = 6\%$	182.07	213.93 (+18%)	184.63 (+1%)	207.62 (+14%)	214.61 (18%)	240.72 (+32%)
Contract type: $\alpha = 0.20$ and $\beta = 0.40$						
$\theta_r = 10\%$	73.99	77.85 (+5%)	75.11 (+2%)	77.18 (+4%)	79.47 (+7%)	82.80 (+12%)
$\theta_r = 8\%$	89.07	102.26 (+15%)	89.93 (+1%)	100.34 (+13%)	102.90 (+16%)	114.43 (+28%)
$\theta_r = 6\%$	177.28	208.28 (+17%)	180.27 (+2%)	203.08 (+15%)	203.17 (+15%)	229.14 (+29%)
Contract type: $\alpha = 0.20$ and $\beta = 0.50$						
$\theta_r = 10\%$	68.10	70.67 (+4%)	67.38 (−1%)	70.31 (+3%)	71.28 (+5%)	74.04 (+9%)
$\theta_r = 8\%$	82.71	97.16 (+17%)	84.32 (+2%)	93.58 (+13%)	92.62 (+12%)	104.70 (+27%)
$\theta_r = 6\%$	173.08	204.00 (+18%)	176.27 (+2%)	199.14 (+15%)	192.89 (+11%)	219.46 (+27%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates (see Chapter 3); in column 3 to 5 we report the fair values under the assumption of either stochastic interest rates or stochastic inflation rates or both (see Chapter 4); in column 6 we report the fair values under the assumption of stochastic mortality and in column 7 we report the fair values under the presence of interest rate risk, inflation risk and mortality risk. Between parentheses are the percentage increases of the fair values as regards to the contract value given in column 2. The present value of the insurance policy is defined as $\Pi(0, X_T) = E_Q\left[\sum_{t=1}^{T-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{t=1}^{T-1}(r_t-\pi_t)} X_T \cdot {}_Tp_x\right]$, where X_T is the liability without consideration for the mortality risk, ${}_{t-1/1}q_x$ presents the probability of an x-year-old policyholder dies at t-th year, ${}_Tp_x$ presents the probability of an x-year-old policyholder survives another T, r_t is the interest rates at time t and π_t describes the inflation rate at time t. The reference insured is a male aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_S = 0.15$, $\sigma_\pi = 0.0101$, mean reversion rate of the inflation rate $\gamma = 0.4740$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, and scenario = 50000.

6. Life Insurance Contracts and Surrender Risk

The final phase of our research aims at studying the valuation of single premium life insurance contracts with a surrender option. In this chapter, we describe the Least Squares Monte Carlo (LSMC) approach as proposed by Longstaff and Schwartz (2001) to price the value of American-like claims. Here, we focus on financial and demographic drivers of surrender decisions including variability of the term structure, price inflation, stock market performance and the uncertainty regarding future mortality and life expectancy. We build upon the insurance contract as described in previous chapters by incorporating the possibility to exit (surrender) the contract prior maturity. We present numerical results relative to the proposed evaluation problem the final section.

6.1. Introduction of surrender option

A profit-sharing policy embeds a surrender option gives the policyholder the chance to terminate the contract before its maturity. If the policyholder decides to early abandon the contract because it is financially convenient, he/she receives a sum that is defined as the *surrender value*, minus any costs charged by the insurance company. If the policyholder decides to hold the contract until the maturity date, he/she receives the stated survival benefits. Since life insurance contracts are long-term contracts, many clients may be reluctant to the idea of locking high amounts of money for such long periods. For this reason, the insurer may offer the possibility of early withdrawals to ensure that the policyholder will not perceive insurance security as an illiquid investment.

The additional flexibility in early termination of the contract might cause some profit reduction for the insurer if optimally exercised. It is thus crucial for insurance providers to understand the surrender behavior and its financial impact. Empirical evidence suggests that surrender decisions are mainly driven by several factors including changes the in competitive environment (e.g. risk premium paid by the insurance client), financial market conditions (e.g. level of interest rate), investment performance and deterioration/improvements of the policyholders' health. High interest rates as well as poor investment results are usually associated with the policyholder leaving the contract and engage in more rewarding investment opportunities. In our analysis, we will assume that the policyholder is perfectly informed and takes rational decisions. The policyholder will take the decision to surrender in order to maximize the value of his wealth.

Surrenders are not welcomed by insurance companies for several reasons. First, the insurance company might incur financial losses from early payments caused by leaving customers due to reduction in the assets base (liquidity) and increase in the costs per-policy. Second, in the long term the insurance company might suffer from adverse selection since policyholders that have health problems will not surrender. This may generate imbalances in the exposure to the mortality risk of remaining insureds. In practice, to minimize surrender risk insurance companies will partially cover the cost of surrender options by applying surrender penalties. The penalties are often predefined as a percentage of the guaranteed benefit. This percentage could be for instance a decreasing function of the in force period of the contract.

From the point view of asset pricing theory, surrender options can be viewed as derivatives with American-style exercise features. The valuation of American-style claims is one of the most important practical problems in financial engineering due to the early exercise feature, particularly when more than one factor affects the value of the option. Indeed, because of the complexity of the underlying dynamics, there is no analytical solution available for the valuation of American options in the standard Black and Scholes (1973) framework.

Two broad approaches are usually distinguished in the academic literature. In the first, finite difference (e.g. binomial/multinomial trees) methods generate a discrete lattice in time and space and iterate backwards in time from expiration.[22] Lattice methods are well-suited for the valuation of American option when the dimension of outcome space is low. However, as the dimension of the problem increases, computing this conditional expectation can become computationally prohibitive. In contrast to traditional finite difference and lattice methods, Monte Carlo simulation methods [see Boyle (1977)] are more widely applicable. Since their convergence rate does not depend on the number of dimensions, Monte Carlo simulations can be effective for high-dimensional pricing problems.

There are a number of studies that have considered the pricing of American options. Some of the first treatments analyzing it are by Tilley (1993), Fu and Hu (1995), Carrière (1996), Carr (1998), Longstaff and Schwartz (2001), Broadie and Glasserman (2004), and Laprise et al. (2006). In particular, Longstaff and Schwartz (2001) employed the least squares regression to determine the optimal exercise time of the problem by backward dynamic programming. The main idea is to estimate the condition expected payoff from continuation at each exercise date (i.e. continuation value), and compare it with the payoff from immediate exercise. Its success is due to its general applicability combined with its ability to deal with both high-dimensional models and multiple exercise times. Applications of the Least Squares Monte Carlo (LSMC) approach on life insurance contracts with surrender option have been tackled by Andreatta and Corradin (2003) and

[22] For instance the approach presented in Bacinello (2003a,b), who analyzes the surrender option in an Italian life insurance contract with single and periodic premiums, and provides numerical illustrations based on a recursive binomial formula implemented in the Cox-Ross-Rubinstein (1979) framework.

Bacinello, Biffis and Millossovich (2008). In this thesis, we apply the LSMC approach to price the participating life insurance contract with surrender option.

6.2. The Longstaff-Schwartz approach

Formally, the Longstaff-Schwartz (2001) approach to price American-style claims assumes an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and finite time horizon $[0, T]$, where the set Ω is called the sample space of all possible outcomes, the σ-algebra $\mathcal{F}$, as the subset of Ω, are called events, and $\mathbb{Q}$ is a probability measure defined on the elements of $\mathcal{F}$. Let m denote a sample path of underlying asset prices generated by Monte Carlo simulation over a discrete N exercise times $0 < t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_N = T$. For the sake of simplicity, we assume that the right to surrender can only be done at the end of the year until the maturity of the contract. Hence this is a Bermudan-type option rather than a truly American one – a Bermuda option can be exercised on predetermined dates during its life, for example on annual basis [Hull (2006)]. Next, we introduce the notation $\mathcal{C}(m, s; t, T)$ to denote the path of cash flow generated by the option, conditional on the option not being exercised at or before time t and on the optionholder following the optimal exercise policy for all subsequent s, $t < s \leq T$. Moreover, we assume that the asset value underlying the option follows a Markov process.

At maturity, the investor exercises the option if it is in the money, or allows it to expire if it is out of money. At time t_n prior to the terminal expiration date $(n = 1,2, \ldots, N-1)$, the optionholder must decide whether to exercise at that point or to hold the option and revisit the decision at the next exercise date. Therefore, the optimal stopping problem changes to comparing the immediate exercise value with the conditional expectation from continuation. Unfortunately, the policyholder has no idea of the true future payoff from continuation at time t_n. As is common when dealing with American options, one can introduce the Snell envelope to evaluate the American-style option features (see Appendix D2). For more information regarding the implementation of the Snell envelope, see for instance, Bensoussan (1984) and Karatzas (1988).

According to no-arbitrage valuation theory, $F(m; t_n)$, the continuation value at time t_n for m-th path with respect to the risk-neutral pricing measure $\mathbb{Q}$, is given by

$$F(m; t_n) = E^Q \left[\sum_{j=n+1}^{N} \exp\left(-\int_{t_n}^{t_j} r(m, s) ds \right) \mathcal{C}\left(m, t_j; t_n, T\right) \middle| \mathcal{F}_{t_n} \right], \tag{6.1}$$

where $r(m, t)$ is the stochastic riskless discount rate, and the expectation of the cash flows is taken conditional on the information set $\mathcal{F}_{t_n}$ at time t_n. The idea underlying the Least Squares Monte Carlo (LSMC) approach methodology is that the conditional expectation, as presented in

(6.1), can be approximated by a least squares regression for each exercise date.[23] Let $\hat{F}(m;t_n)$ denote this estimate and let L_h denote the h-th basis function, then we can represent the estimated continuation value as

$$\hat{F}(m;t_n) = \sum_{h=1}^{H} a_h(t_n)\, L_h\big(X(m;t_n)\big), \tag{6.2}$$

where $a_h(t_n)$ is the coefficient corresponding to the h-th basis function L_h at time t_n and $X(m;t_n)$ presents the asset price at the n-th timestep for the scenario m.

To implement the LSMC method, we have approximated $\hat{F}(m;t_n)$ using the first $H < \infty$ basis functions. Now, $a_h(t_n)$ can be estimated by a least squares regression of $F(m;t_n)$ onto the basis $L_h\big(X(m;t_n)\big)$. The coefficient vector can be expressed as

$$\big(\hat{a}_1(t_n), \dots, \hat{a}_H(t_n)\big)^T = \big(\Psi^T\Psi\big)^{-1}\Psi^T\big(y_1(t_n), \dots, y_M(t_n)\big)^T, \tag{6.3}$$

where $y_m(t_n) = e^{-r_{t_n}} X(m;t_{n+1})$, and $\Psi_\mathrm{m} = \Big(L_0\big(X(m;t_n)\big), \dots L_H\big(X(m;t_n)\big)\Big)$ for $m = 1, \dots, M$ realization paths. Since we work in a Markovian environment, we can use Laguerre polynomials as choice of basic functions as suggested by Longstaff and Schwartz (2001). Mathematically, the Laguerre polynomials are given by

$$\begin{aligned} L_1(X) &= 1 \\ L_2(X) &= -X + 1 \\ L_3(X) &= \frac{1}{2}(X^2 - 4X + 2) \\ &\vdots \\ L_H(X) &= \frac{e^X}{H!}\frac{d^H}{dX^H}(X^H e^{-X}), \end{aligned} \tag{6.4}$$

where X denotes the underlying cash flow. Hence, the continuation function $F(m;t_n)$ is approximated as a linear combination of a set of Laguerre polynomials as described above. Other types of the basic function are for example Hermite, Legendre, Chebyshev, Gegenbauer, Jacobi polynomials or also powers of X. Numerical analyses for several choices of basis functions are reported in Moreno and Navas (2003) and Stentoft (2004).

[23] The motivation for approximating the conditional expectation can be given in terms of projection theory of Hilbert spaces. We restrict our attention to derivatives with payoffs that are elements of the space of square-integrable or finite-variance functions $L^2(\Omega, \mathcal{F}, \mathbb{Q})$. Since L^2 is a Hilbert space, it is assumed that the conditional expectation, $F(m;t_n)$, can be expressed as a linear combination of a countable set of orthonormal basis functions. For a comprehensive view of Hilbert space theory and Hilbert space representations of square-integrable functions, see Royden (1968).

The above pricing methodology relies on the assumption of no mortality risk. When deciding whether to surrender or not, we must take into account the fact that the policyholder might die in the next year. Hence at time t_n, the adjusted continuation value can be defined as

$$\hat{F}^C(m;t_n) = e^{-\left(r_{t_n}-\pi_{t_n}\right)}\left(\hat{F}(m;t_n)_{t_n-1/1}q_x + \hat{F}(m;t_n)_{t_n}p_x\right) \tag{6.5}$$

where ${}_{t-1/1}q_x$ presents the probability of an x-year-old annuitant dies at t-th year and ${}_{\tau}p_x$ presents the probability of an x-year-old annuitant survives another τ years. The term $e^{-\left(r_{t_n}-\pi_{t_n}\right)}$ denotes the discount rate at time t_n.

Once the continuation value, $\hat{F}^C(m;t_{N-1})$, is estimated at time t_{N-1}, we can compare it with the payoff of immediate exercise, after which the optimal exercise strategy along each exercise point can be obtained. The recursion proceeds by rolling back to time t_{N-2} and repeating the process until the exercise decisions at each time along each path have been identified. After having obtained the optimum exercise time t_m^* for each scenario, the value of the surrender contract is then approximated by averaging these values for all paths results in the present value of the option. Given the valuation problem in (5.15), a Bermuda-style contingent claim on X and expiring at T can obtained by choosing

$$\Pi(0,X_\tau) = \sup_{\tau\epsilon\mathcal{T}_{(0,T)}} E_Q\left[\sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)}X_t\cdot {}_{t-1/1}q_x + e^{-\sum_{k=1}^{\tau}(r_k-\pi_k)}X_\tau\cdot {}_{\tau}p_x\right], \tag{6.6}$$

where $\Pi(0,X_\tau)$ presents the fair value of X_τ at time 0, $E^Q[\cdot]$ denotes the conditional expectation under an equivalent martingale measure, $\mathbb{Q}$, given the information at date zero, $\mathcal{T}_{0,T}$ is the class of stopping times taking values in $[0,T]$, τ denotes the stopping time dependent on whether the contract is ended by maturity ($\tau = T$), death ($\tau \in [0,T-1]$), or surrender ($\tau = t_m^*$). As already stated, the contract can only be stopped at the end of each year until the maturity; the stopping time τ is therefore an integer. Accuracy of the estimates of the value of the American contingent claim can be increased by increasing the number of time steps (N), the number of realization paths (M) and the number of basis function (H). Clément, Lamberton and Protter (2002) give a detailed analysis of convergence theorems and proves that the algorithm converges to the actual value of the claim if M and H approaches infinity. In addition, they also determine the rate of convergence of the Monte Carlo procedure.

Subsequently, we replace the term X in relation (6.6) by the insurance accounts of interest, namely insured's account (A), insurer's account (C) and reserve account (R^+ and R^-). The premium for the surrender option is then given by the difference between the fair premium for the whole contract (i.e. including surrendering) and the fair premium for the nonsurrenderable European participating contract (see Chapter 5).

With respect to the surrender penalty, we make the assumption that the policyholder receives the surrender value $F^S(m;t)$ at the scenario m, if he/she surrenders the policy at time t:

$$F^S(m;t) = (1-\xi)A_{t-1}e^{r_G+\alpha(\delta_t-r_G)^+}, \tag{6.7}$$

where $\xi \in [0,1]$ denotes the constant penalty rate charged by the insurance companies in case of surrendering and A_{t-1} presents the insured's account at time $t-1$. A more realistic penalty would be for instance a decreasing rate over years.

6.3. The LSMC algorithm

We describe the LSMC algorithm to the application of profit-sharing insurance contract with minimum guarantee. The method uses M simulated paths and in each of path m $(m = 1, \dots M)$ the optimum surrender time t_m^* has to be determined. Since it is not allowed for the policyholder to surrender at the inception of the contract or at the maturity date, we divide the interval $[1, T-1]$ into discrete integral time steps, i.e. $t = \{T-1, T-2, \dots, 1\}$. Starting from time $T-1$, the optimum policy is then obtained by using backward dynamic programming.

In practice, the policyholder will surrender because of financial reasons if

$$F^S(m;t) > \hat{F}^C(m;t), \tag{6.8}$$

where $F^S(m;t)$ represents the benefits entitled to the policyholder in the case of surrendering, and $\hat{F}^C(m;t)$ denotes the estimated continuation value of the contract at time t. Since the valuation algorithm start at time $T-1$, the decision rule along path m is made by comparing the payoff of immediate exercise (including surrender penalty) at time $T-1$ with the discounted payoff from time T. The value of the contract at expiration date for m-th path is given by $V_T(m) = A_T^m + \max(R_T^m, 0)$. We use the contract value $V_T(m)$ to derive the corresponding continuation value $\hat{F}^C(m;T)$. This value is adjusted for both financial and actuarial risk factors. When $F^S(m;T-1) > \hat{F}^C(m;T-1)$, we set $t_m^* = T-1$ and $V_{T-1}(m) = F^S(m;T-1)$. In the case that $F^S(m;T-1) \le \hat{F}^C(m;T-1)$, then we simply update the contract's value on path m by discounting it for one more step, i.e., we set $V_{T-1}(m) = e^{-(r_{T-1}-\pi_{T-1})}\big(V_{T-1}(m) \cdot {}_{j-1/1}q_x^m + V_{T-1}(m) \cdot {}_{T-1}p_x^m\big)$. Subsequently, this procedure is repeated by going backwards in time till $t = 1$. Once the optimal set of stopping times t_m^* has been identified for each single scenario, implicit option elements can be approximated by Equation (6.6). Further note that the pseudo-code of the algorithm is also provided in Fig. 11.

The LSMC pricing algorithm

for $m \leftarrow 1$ to M

  generate interest rates $(r_1^m, \ldots, r_T^m)$; inflation rates $(\pi_1^m, \ldots, \pi_T^m)$; equity returns $(\delta_1^m, \ldots, \delta_T^m)$

  determine insured's account $(A_1^m, \ldots, A_T^m)$; insurer's account $(C_1^m, \ldots, C_T^m)$; reserve account $(R_1^m, \ldots, R_T^m)$

  generate survival probability ${}_t p_x$ and death probability ${}_{t-1/1} q_x$

  $V(m) \leftarrow A_T^m + \max(R_T^m, 0)$

end

for $j \leftarrow T - 1$ to 1

  $Y \leftarrow V; X \leftarrow A_j; \Psi \leftarrow$ matrix of basis functions (see Eq. 6.2)

  compute $\hat{a}_1, \ldots, \hat{a}_H \leftarrow (\Psi^T \Psi)^{-1} \Psi^T Y$

  $\hat{F}(\Omega, j) \leftarrow (\hat{a}_1, \ldots, \hat{a}_H)\Psi$

  $\hat{F}^C(\Omega; j) \leftarrow e^{-(r_j - \pi_j)} \left(\hat{F}(\Omega, j) \cdot {}_{j-1/1} q_x + \hat{F}(\Omega, j) \cdot {}_j p_x \right)$

  $F^S(\Omega; j) \leftarrow (1 - \xi)X$

  for $m \leftarrow 1$ to M

    if $F^S(m; j) > \hat{F}^C(m; j)$

      $t_m^* \leftarrow j$

      $V_j(m) \leftarrow F^S(m; j)$

    else

      $V_j(m) \leftarrow e^{-(r_j - \pi_j)} \left(V_j(m) \cdot {}_{j-1/1} q_x^m + V_j(m) \cdot {}_j p_x^m \right)$

    end

  end

end

for $m \leftarrow 1$ to M

  $\tau = t_m^*$

  $\hat{V}_0[m] \leftarrow \sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t} (r_k^m - \pi_k^m)} X_\tau^m \cdot {}_{t-1/1} q_x^m + e^{-\sum_{k=1}^{\tau} (r_k^m - \pi_k^m)} X_\tau^m \cdot {}_\tau p_x^m$, for $X_\tau^m \leftarrow \{A, C, R^+, R^-\}$

end

return $\sum_{m=1}^{M} \hat{V}_0[m] / M$

Fig. 11. Pseudocode describing the Least Squares Monte Carlo (LSMC) pricing algorithm for a profit-sharing life insurance contract.

6.4. Effect of surrender risk on fair value

In this section, we present the results for a single premium life insurance contract by focusing on the surrender option when exercised rationally. Furthermore, financial risks and mortality risk are considered when analyzing the fair financial contract. Following the approach presented by Bacinello, Biffis and Millossovich (2008), we employ the Least Squares Monte Carlo valuation algorithm with Laguerre polynomial basis function of order 3. Hence, we regress the discounted realized cash flows on a constant and two nonlinear functions of the underlying asset index (see Equation 6.4). The results of the European type contract (Chapter 3 to 5) and the American type contract (Chapter 6) with stock weight $w = 30\%$ and $w = 100\%$ are reported respectively in Table 10 and Table 11. Furthermore, in order to achieve a better understanding of surrender behavior, from Table E4 to Table E6 in Appendix E we present the fair values of the decomposed American-style contract.

Based on results from numerical experiments in Table 10 and in Table 11, we show that the American type contract (i.e. the last column) may be either more or less valuable to the insurer as opposed to the European counterpart. It is obvious that whether an insured will choose surrender or not and when surrender will occur depends on the parameters used in the model. In particular, the American contract value depends on the share of the positive surplus that is distributed to the insurer (β), the long-term mean of the interest rate (θ_r) and more importantly, the proportion invested in stocks (w).

First, the incentive to early exercise prematurely will gradually disappear as the profit-sharing rate β drops. This can be observed by comparing the percentage increases in the last two columns of the two tables. For example, in Table 10 with a given interest rate level $\theta_r = 10\%$, the percentage changes are $\{\beta = 0.30\colon +10\%; +6\%\}$, $\{\beta = 0.40\colon +9\%; +9\%\}$ and $\{\beta = 0.50\colon + 8\%; +14\%\}$). This suggests that for the two different portfolios, an increase in β must be followed by an increase in the incentive to surrender. The intuition behind this behavior is simple. Due to an increase in the β, more surplus is attributed to the insurance company and lower amount of surplus is assigned to the reserve account R. This is further illustrated in column 5 (terminal bonus R^{+}) and column 7 (insurer's account C) from Table E4 to Table E6 given in Appendix E. According to the approach of the Least Squares Monte Carlo (LSMC), higher βs are often associated with lower terminal bonuses and thus lower continuation values (i.e. rewarding from not exercising). Thus, the numerical results obtained are in good agreement with the theoretical analysis.

Second, the influence of the long-term mean of the interest rate θ_r on the American contract value depends on the risk profile of the investment portfolio. If we limit our attention to the case where the proportion of investment in stocks is 30% (see Table 10), we observe the surrender effect is strongest when $\theta_r = 8\%$ and is significant lower for other market interest levels. It can be also seen that the American contract value is mostly below the European contract value, except for one case, i.e., $\alpha = 0.20$, $\beta = 0.50$ and $\theta_r = 10\%$ with the respective fair value of the American type contract 78.42 > 74.42 (European contract). Before we give any economic interpretations why this could happen, we first motivate the surrender effect at $\theta_r = 8\%$ and why it positively impacts the fair value. The reason why policyholders decide to cancel the insurance policy before time T is because they believe that the discounted future claims will be significantly lower than the present claim value.[24] This additional American-option feature would make the contract more valuable to the policyholder.[25] However, this condition does not necessarily have to hold for the insurer (see previous example). Instead, it is very likely that not sufficient returns will be made by the insurer to meet the required minimum guaranteed rate $r_G = 3\%$. Therefore, leaving customers seems desirable to the insurance company in this situation. In addition, we observe a similar development for $\theta_r = 6\%$. However, the difference between the European contract and the American contract becomes much lower with respect to the initial case $\theta_r = 8\%$. In this thesis, the market interest rate $\theta_r = 6\%$ is associated with extremely adverse economic condition. Hence, most of the policyholders will stick to the insurance contract and a very few will surrender. We partially explain this observation by looking at the surrender penalty value (column 11) and the terminal bonus account (column 5) in Table E4 to Table E6. As expected, the surrender penalty value is significantly lower than the case $\theta_r = 8\%$ and the terminal bonus is far above the values obtained for $\theta_r = 8\%$ for each contract types; this implies that less people are leaving the contract. Similar analysis can be conducted for $\theta_r = 10\%$.

Third, the risk of a change in the value of an insurance policy caused by a deviation of the actual surrenders (premature terminations) affects the two portfolios differently. Up to now, we have evaluated the impact of market interest rates on the diversified portfolio. What about the case when all pension premium is invested into stocks? Does the surrender risk affect the American contract value differently when the investment portfolio becomes more risky? The answer is yes. For example, if we look at the case where the market interest rate level $\theta_r = 10\%$. Table 11 shows that, for each contract type, the American contract values exceed the corresponding European contract values (definitely not favorable for the insurer). Not surprisingly, since this is a good state of economy, it is relative easy for the insurance company to maintain the promised rate of 3%. On top of that, leaving customers (slim chance here) implies substantial reductions in account values of the insurance company (C). It is apparent that when

[24] Although the insured has no idea about the 'real' account value in the near future, he/she might know the 'expected' account value.

[25] In order to compensate for this increase in value, it is common for insurance companies to impose a penalty charge if the policyholder opts to surrender the policy prematurely. In this thesis, the surrender penalty ξ is set to 5% of the initial surrenderable value.

the portfolio consists of stocks only, higher returns can be made as compared to a diversified portfolio. This is reflected in the terminal bonus account R^+ (column 5) and the insurer account C (column 7) of Table E4, Table5 and Table E6 presented in Appendix E. We notice that the respective values of the terminal bonus R^+ and the insurer's account C are much larger for stock weight $w = 100\%$ than stock weight $w = 30\%$. Furthermore, the surrender behavior is also subject to the profit-sharing rate β. The higher the fraction of surplus distributed to the insurer, the less attractive for policyholder to keep the contract. The surrender penalty result from Table E4 to E6 clearly confirms our expectation. Subsequently, we remark that the risk-neutral values of the American contract for different contract types all fall beneath the initial premium level $P_0 = 100$ and the European counterpart. The corresponding fair values are: $\{\alpha = 0.20;\ \beta = 0.30;\ \theta_r = 10\%;\ 98.60\}$, $\{\alpha = 0.20;\ \beta = 0.30;\ \theta_r = 10\%;\ 94.14\}$ and $\{\alpha = 0.20;\ \beta = 0.30;\ \theta_r = 10\%;\ 89.73\}$. For $\theta_r = 8\%$ and $\theta_r = 6\%$, no fair contract can be obtained for $w = 100\%$. As the market interest rate drops from 10% towards 6%, it becomes very risky for the insurance company since stocks are considered as extreme volatile assets. For example, the amount of potential losses for the surrendered policies is reported from Table E4 to Table E6 in column 10. More precisely, column 10 represents the surrender reserve value R^*-. This value indicates the initial market value of the loss when surrendering occurred and is much higher than the case where $w = 30\%$. Moreover, the incentive to surrender increases as the interest rate level falls. This is again confirmed by the development in the surrender penalty values. Based on the provided analysis, the American contract value improves with respect to the European contract due to extreme adverse economic condition presented here, i.e. when $\theta_r = 6\%$.

Based on the numerical results illustrated in this section and the extensive discussions, we are ready to find an appropriate parameter combination that satisfies the fair pricing principle. We know that $\theta_r = 6\%$ is the worst case scenario presented in this thesis. No matter what setting the insurance company chooses, a fair contract cannot be obtained. In the remaining economic scenarios, a suitable value for the profit-sharing rate β would be around 0.30 or 0.40; with respect to the profit-sharing rate α, at least 20% of the positive excess should be credited to the insured.

Table 10

Overview of fair values of the single premium profit-sharing life insurance contract based on a portfolio consisting of 30% stocks and 70% bonds

$w = 0.30$	Chapter 3	Chapter 4			Chapter 5		Chapter 6
Interest rate scenario	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 1\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 0\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$ $T \in [0,40]$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$ $T \in [0,40]$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$ $T \in [0, t_m^*]$
Contract type: $\alpha = 0.20$ and $\beta = 0.30$							
$\theta_r = 10\%$	77.99	77.38 (−1%)	77.83 (−0%)	76.63 (−2%)	86.38 (+11%)	85.79 (+10%)	82.52 (+6%)
$\theta_r = 8\%$	73.02	84.51 (+16%)	73.05 (+0%)	81.71 (+12%)	91.89 (+26%)	101.94 (+40%)	87.50 (+20%)
$\theta_r = 6\%$	101.03	148.25 (+47%)	104.88 (+4%)	142.35 (+41%)	137 (+36%)	179.51 (+78%)	177.09 (+75%)
Contract type: $\alpha = 0.20$ and $\beta = 0.40$							
$\theta_r = 10\%$	77.99	77.38 (−1%)	77.83 (−0%)	76.63 (−2%)	80.83 (+10%)	80.31 (+9%)	80.26 (+9%)
$\theta_r = 8\%$	73.02	84.51 (+16%)	73.05 (+0%)	81.71 (+12%)	84.58 (+26%)	95.11 (+42%)	85.49 (+27%)
$\theta_r = 6\%$	101.03	148.25 (+47%)	104.88 (+4%)	142.35 (+41%)	133.71 (+35%)	174.08 (+76%)	171.32 (+73%)
Contract type: $\alpha = 0.20$ and $\beta = 0.50$							
$\theta_r = 10\%$	68.98	68.20 (−1%)	69.18 (+0%)	67.64 (−2%)	75.25 (+9%)	74.42 (+8%)	78.42 (+14%)
$\theta_r = 8\%$	61.23	74.58 (+22%)	60.85 (−1%)	71.63 (+17%)	77.48 (+27%)	88.60 (+45%)	84.54 (+38%)
$\theta_r = 6\%$	97.05	143.04 (+47%)	100.55 (+4%)	136.46 (+41%)	130.32 (+34%)	169.88 (+75%)	164.55 (+70%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates (see Chapter 3); in column 3 to 5 we report the fair values under the assumption of either stochastic interest rates or stochastic inflation rates or both (see Chapter 4); in column 6 we report the fair values under the assumption of stochastic mortality and in column 7 we report the fair values under the presence of interest rate risk, inflation risk and mortality risk (Chapter 5); finally in column 8 we report the fair values under the presence of both financial and actuarial risks. Between parentheses are the percentage increases of the fair values as regards to the contract value given in column 2. The present value of the insurance policy is defined as $\Pi(0,\tau) = \sup_{\tau \in \mathcal{T}_{(0,T)}} E_Q\left[\sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{k=1}^{\tau}(r_k-\pi_k)} X_\tau \cdot {}_\tau p_x\right]$, where $\mathcal{T}_{0,T}$ is the class of stopping times taking values in $[0,T]$, τ denotes the stopping time dependent on whether the contract is ended by maturity ($\tau = T$), death ($\tau \in [0, T-1]$), or surrender ($\tau = t_m^*$), X_T is the liability without consideration for the mortality risk, ${}_{t-1/1}q_x$ presents the probability of an x-year-old policyholder dies at t-th year, ${}_Tp_x$ presents the probability of an x-year-old policyholder survives another T, r_t is the interest rates at time t and π_t describes the inflation rate at time t. The reference insured is a male aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_\pi = 0.0101$, mean reversion rate of the inflation rate $\gamma = 0.4740$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, surrender penalty $\xi = 0.05$ and scenario = 50000.

Table 11
Overview of fair values of the single premium profit-sharing life insurance contract based on a portfolio consisting of 100% stocks and 0% bonds

$w = 1$	Chapter 3	Chapter 4			Chapter 5		Chapter 6
Interest rate scenario	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 1\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 0\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x = 0$ ${}_Tp_x = 1$ $T = 40$	$\sigma_r = 0\%$ $\sigma_\pi = 0\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$ $T \in [0,40]$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$ $T \in [0,40]$	$\sigma_r = 1\%$ $\sigma_\pi = 1\%$ ${}_{t-1/1}q_x \in [0,1]$ ${}_Tp_x \in [0,1]$ $T \in [0, t_m^*]$
Contract type: $\alpha = 0.20$ and $\beta = 0.30$							
$\theta_r = 10\%$	81.21	85.57 (+5%)	81.17 (−0%)	83.59 (+3%)	89.17 (+10%)	92.95 (+14%)	98.60 (+21%)
$\theta_r = 8\%$	96.06	109.51 (+14%)	97.98 (+2%)	107.32 (+12%)	113.66 (+18%)	125.03 (+30%)	114.01 (+19%)
$\theta_r = 6\%$	182.07	213.93 (+18%)	184.63 (+1%)	207.62 (+14%)	214.61 (18%)	240.72 (+32%)	210.37 (+16%)
Contract type: $\alpha = 0.20$ and $\beta = 0.40$							
$\theta_r = 10\%$	73.99	77.85 (+5%)	75.11 (+2%)	77.18 (+4%)	79.47 (+7%)	82.80 (+12%)	94.14 (+27%)
$\theta_r = 8\%$	89.07	102.26 (+15%)	89.93 (+1%)	100.34 (+13%)	102.90 (+16%)	114.43 (+28%)	107.47 (+21%)
$\theta_r = 6\%$	177.28	208.28 (+17%)	180.27 (+2%)	203.08 (+15%)	203.17 (+15%)	229.14 (+29%)	190.53 (+7%)
Contract type: $\alpha = 0.20$ and $\beta = 0.50$							
$\theta_r = 10\%$	68.10	70.67 (+4%)	67.38 (−1%)	70.31 (+3%)	71.28 (+5%)	74.04 (+9%)	89.73 (+32%)
$\theta_r = 8\%$	82.71	97.16 (+17%)	84.32 (+2%)	93.58 (+13%)	92.62 (+12%)	104.70 (+27%)	103.34 (+25%)
$\theta_r = 6\%$	173.08	204.00 (+18%)	176.27 (+2%)	199.14 (+15%)	192.89 (+11%)	219.46 (+27%)	198.30 (+15%)

The table shows the fair values of a single premium European life insurance contract under the risk-neutral measure $\mathbb{Q}$. More specifically, in column 2 we report the fair values under the assumption of deterministic interest rates and inflation rates (see Chapter 3); in column 3 to 5 we report the fair values under the assumption of either stochastic interest rates or stochastic inflation rates or both (see Chapter 4); in column 6 we report the fair values under the assumption of stochastic mortality and in column 7 we report the fair values under the presence of interest rate risk, inflation risk and mortality risk (Chapter 5); finally in column 8 we report the fair values under the presence of both financial and actuarial risks. Between parentheses are the percentage increases of the fair values as regards to the contract value given in column 2. The present value of the insurance policy is defined as $\Pi(0,\tau) = \sup_{\tau\epsilon\mathcal{T}_{(0,T)}} E_Q\left[\sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{k=1}^{\tau}(r_k-\pi_k)} X_\tau \cdot {}_\tau p_x\right]$, where $\mathcal{T}_{0,T}$ is the class of stopping times taking values in $[0,T]$, τ denotes the stopping time dependent on whether the contract is ended by maturity ($\tau = T$), death ($\tau \in [0, T-1]$), or surrender ($\tau = t_m^*$), X_T is the liability without consideration for the mortality risk, ${}_{t-1/1}q_x$ presents the probability of an x-year-old policyholder dies at t-th year, ${}_Tp_x$ presents the probability of an x-year-old policyholder survives another T, r_t is the interest rates at time t and π_t describes the inflation rate at time t. The reference insured is a male aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_\pi = 0.0101$, mean reversion rate of the inflation rate $\gamma = 0.4740$, long-term inflation rate $\bar{\pi} = 0.0240$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, surrender penalty $\xi = 0.05$ and scenario = 50000.

7. Concluding Remarks

7.1. Summary and conclusions

This thesis provides valuable insights into the key factors of Dutch single premium participating life insurance policies. The aim of participating life insurance policies is to provide sustainable and stable medium- to long-term returns through the combination of guaranteed benefits and non-guaranteed bonuses to policyholders. These policies are endowments that contain implicit options-like features such as minimum interest rate guarantees, stochastic annual surplus participation, terminal bonus and a surrender option. It is essential to get a thorough understanding of these items so that fluctuation in the embedded option value can be anticipated and explained.

Adapting the methodology from Miltersen and Persson (2003), we present a framework in which the different kinds of guarantees or options are considered and priced separately in a market-consistent manner. In particular, we analyze the influence of the term structure of interest, the investment performance, the price inflation, the mortality development, the surrender behavior and the multivariate risk on the risk exposure of an insurer. The fair values of the embedded options are measured using arbitrage-free option pricing techniques and assuming complete markets. Since life insurance products are usually long-term contract, the instantaneous short rate is modeled as a mean-reversion process using the one-factor Vasicek model (1977). We propose to describe the consumer price index dynamics by means of an Ornstein–Ulhenbeck process which takes into account stochastic inflation rates. A standard geometric Brownian motion is used to generate the evolution of the equity price. Furthermore, the mortality risk is modeled according to the Lee-Carter mortality model (1992) with an extension in old-age modeling. The surrender option is modeled in the evaluation framework as a Bermudan option that is exercised by the policyholder only if it financially convenient. We have also shown how to incorporate the multivariate risk into these pricing methods. In addition, we have shown that the typical contract can be decomposed into various basic components: a risk-free bond, an option to receive bonus (for both insured and insurer), and a surrender option.

The valuation of the participating insurance contracts with surplus distribution is conducted in a risk-neutral framework. The insurance contract itself and the embedded options are relatively complex derivatives. It is not possible to obtain closed-form expressions for their risk-neutral value while taken into account all the aforementioned risk factors. Therefore, Monte Carlo simulations are used to derive the risk-neutral value for the European type contract as well as the American (Bermudan) type contract.

The analysis of numerical results led to several conclusions. Our numerical studies showed that the insurance company is mainly exposed to the equity risk, interest rate risk and the mortality risk. In particular, it turns out that the risk-neutral value of an insurance contract with stochastic short rates mostly exceeds the value of a contract with a constant or deterministic short rate for a comparable parameter choice. With increasing volatility of the interest rate process, the fair contract value also increases. The reason why interest rate risk modeling is essential to insurance companies is because they reflect the expected future returns and at the same time interest rates are also used to discount future cash flows (pension benefits). We also found that the effect of the inflation rate risk on the contract value is relatively small due to the strong mean-reversion characteristic of the inflation process. In other words, the tendency to revert to its long-term inflation rate weakens the impact of inflation rate variability. This is also consistent with the primary objective of the ECB to maintain price stability within the Eurozone. The impact of inflation rate risk becomes only noticeable when the rate of volatility increases to extreme high levels or when the market interest rate level drops towards zero. Moreover, we argue that the multivariate risk modeling is important regarding the pricing of life insurance contracts and embedded options. The way how it affects the contract value crucially depends on the value of the correlation parameter. Furthermore, we show that the mortality risk influences the contract value considerably. This can be attributed to the uncertainty surrounding future mortality rates and life expectancy outcomes. With respect to the surrender option, it is unclear whether the possibility of early withdrawals of the contract positively or negatively impacts the risk-neutral value. It is shown that whether an insured will choose surrender or not and when surrender will occur depends on the parameters used in the model. Specifically, the value of the American type contract depends on the share of the positive surplus that is distributed to the insurer (β), the long-term mean of the interest rate (θ_r) and more importantly, the proportion invested in stocks (w) and in bonds $(1-w)$.

Furthermore, the implication of asset allocation is investigated for two simple bond-stock mixes, namely a portfolio consisting of 100% stocks and a portfolio consisting of 30% stocks and 70% bonds. We found that the fair values derived from the diversified portfolio are significantly lower when compared to the undiversified portfolio. This reveals once again how relevant asset allocation can be and how careful the insurers should be in their investment decisions. Moreover, we have also analyzed the impact of the market interest rate level θ_r on the evolution of market-consistent contract values. We found that the contract value is a decreasing function of the market interest rates. Furthermore, the influence of the profit-sharing rate of the insured α, effects the insurance contract negatively. The share of the positive surplus that is distributed to the insured β, however, reduces the fair contract value. The above findings confirm the fact that the contract value highly dependent on the parameterization used in the model. Finally, empirical studies further show that an appropriate fair parameter combination for the profit-sharing rate of the insured $\alpha \geq 0.20$ and the share of the positive surplus that is distributed to the insurer $\beta = \{0.30 - 0.40\}$ for a given minimum guaranteed rate $r_G = 3\%$ and a market interest rate level $\theta_r = \{8\%; 10\%\}$. The respective fair values are below the initial premium paid (P_0) in all cases. In the special case $\theta_r = 6\%$, we observe that the whole contract exceeds the initial payment which implies that the insurance company will make losses on the issued contracts. In this situation, it could be worthwhile for the insurance company to consider hedging strategies.

7.2. Suggestions for further research

Although this thesis made a good attempt at providing a realistic valuation framework for Dutch single premium participating life insurance policies, more work is needed to deal with more complex product and more realistic market parameters. With respect to future studies, there are several interesting directions worth exploring.

First, a natural extension of the current setting is to consider the multi-period version of the insurance contract analyzed in this thesis. In this case instead of requiring a net single premium policy at the beginning of the contract, we allow the policyholder to pay his/her premium at a constant or varying rate for each single year until the pensionable age is reached [see, e.g., Bacinello (2003b) and Bakken, Lindset and Olson (2006)]. In our setup, the total amount of the initial premium P_0 is credited to the participating account. Other possibilities are to deposit parts of the initial deposit P_0 into account the insurer's account C and/or the reserve account R when the contract is initiated. This is especially useful when the first year's investment performance is below the minimum guaranteed rate r_G. It is also straightforward to include varying rate minimum guaranteed rate r_G in our framework. Such extensions would illustrate the use of our methodology in a more realistic setting, appealing to researchers and practitioners in the actuarial and insurance fields.

Furthermore, the accrued pension contribution is considered as a lump sum (single payment) at the maturity date T in this thesis. Another possibility is to study the influence of financial- and actuarial risk on the market value of an annuity with regular payments. The (life) annuity provides the annuitant a predetermined periodical payment, staring at the end of the year in which the annuitant reaches the retirement age (e.g. 65), until he/she is passed away. For the latter case, one should realize that the mortality effects generally play a more important role over longer time horizon and therefore induce a stronger influence on the fair price than the initial case.

In light of the above discussions, the complexity of the evaluation problem becomes considerable when we consider the case of multi-period contracts. The main sources of the difficulty are due to the choice of the risk measure but also the choice of the Monte Carlo simulation approach. In Chapter 3, we have mentioned the drawback of the Monte Carlo simulation method as opposed to closed-form solutions. Recall that simulation is used to solve the valuation problem that is too hard to solve either explicitly or numerically. It is therefore very demanding from a computational point of view. One way to overcome this obstacle is to apply variance reduction techniques to further improve the accuracy and performance of the Monte Carlo approach. With respect to the risk-neutral valuation measure, one may also consider the *T*-forward risk-neutral measure as an alternate method of pricing derivatives securities [see, e.g., Geman et al. (1995) and Brigo and Mercurio (2006) for an introduction]. This method is continuous with respect to a risk-neutral measure $\mathbb{Q}$ but rather than using the money market as numeraire, it uses a zero-coupon bond (ZCB) with maturity T. One can use ZCB observed on the market to construct term structure models and use it to discount future cash flows. Under this approach, the valuation of financial derivatives becomes more efficiency and easier to implement. See Appendix C for complete overview of the application of the *T*-forward risk-neutral measure.

The main emphasis of this thesis has been on pricing of the *value* of options embedded in the life insurance contract. With the downfall of the stock markets, it has become apparent that not only the valuation is crucial, but also the *hedging* of embedded options. Even if the options embedded in insurance products are priced correctly, they still represent uncontrollable liabilities if unhedged. Future research should be conducted to identify tractable and realistic hedging strategies to protect the insurance company against the risk associated with the given guarantees. Effective hedging strategies for life insurance contracts with minimum guarantee have been explored by for instance Coleman et al. (2007) and Fleten and Lindset (2008). Fleten and Lindset (2008) applied the delta-hedging method (i.e. rebalancing techniques) to hedge multi-period guarantees in the presence of transaction costs.[26] Coleman et al. (2007) evaluated performances of delta-hedging, quadratic, piecewise linear local risk minimization methods for discrete hedging of American-type options in an incomplete market setting.

Finally, future research could attempt to investigate the issue of the optimal structure of the insurance company's asset portfolio when guaranteed are offered. In this thesis, we have only considered a very simple bond-stock mix. Given that the investment performance of the participating account is an important component, more flexibility and investigation should be implemented. For example, we could further extend the model by considering an asset portfolio consisting of several different asset classes such as equity shares, publicity traded government and corporate bonds, commercial mortgages, corporate lending, real estates and exchanges.

[26] The typically risk management strategies in this case consist of holding positions in stocks and bonds and dynamically rebalance these positions in order to cover guarantees. The number of shares of the underlying held in a delta-hedging strategy is given by the sensitivity (delta) of the option value to the underlying.

Appendix A – FTK Stress Tests

The testing framework contains three key elements:

1. *Minimum test*: The minimum test ensures that the accrued benefits are funded by sufficient assets in the case of immediate discontinuance. The funding ratio is computed by

$$\text{Funding ratio} = \frac{\text{Pension Asset}}{\text{Pension Liability}}.$$

   According to the FTK regulations, the funding ratio of at least 105% should be maintained by each pension fund. Under the FTK rules, pension funds that fall below the minimum coverage ratio of 105% of their indexed liabilities are required to produce a recovery plan to correct their situation within a predetermined period.

2. *Solvency test*: The second test reveals whether the current value of the assets is at least equal to the current value of the liabilities and whether there is sufficient solvency to absorb a mismatch over a period of one year. Pension funds are therefore obligated to establish sufficient safety capital or any similar buffer against market volatility. Both standard models and own firm based models are allowed to be implemented.

3. *Continuity test*: The continuity test assesses the fund's long-term prospects. The continuity test is intended to show whether the risks in terms of capital solvency will remain within the risk exposure norms applicable at the time. The long-term stress test is focusing on the investment management of a pension fund (i.e. ALM-studies). Based on this test, appropriate ALM strategies are chosen in order to reduce financial risk on the long run.

Appendix B – Profit-sharing Policies

B1. Profit-sharing based on TL-discount or UL-discount

Profit-sharing mechanism provided in contract with TL-discount (t-yield) or UL-discount (u-yield) is determined at the start of the issued contract and is mainly offered to small-middle size companies. The interest rate discount is derived from fictive investment returns on several treasury rates (i.e. t-yield or u-yield) instead of actual returns achieved by insurance company. A single premium discount is given on beforehand, so no more profit-sharing is credited to the policyholder during the contract period, except the guaranteed rate of interest. Furthermore, it should be mentioned that the specific discount system represents the fictive excess return made in the first 10 à 12 years of the contract period. Since the former t-yield is replaced by u-yield in 1 January of 1995, we only focus on the computational aspect of insurance product with UL-discount.

Table B1
Computation 4% interest rates discount

u-yield (U) in %	UL-discount in %
4 - 6	8 x (t -/- 4)
6 - 8	16 + 5.5 x (t -/- 6)
8 - 10	27 + 4.5 x (t-/- 8)

Source: Pension guide 2008 (Pensioengids 2008)

Table B2
Computation 3% interest rates discount

u-yield (U) in %	UL-discount in %
$U \leq 3\%$	0
$3\% < U \leq 5\%$	8.0 x (U - 3)
$5\% < U \leq 7\%$	5.5 x (U - 5) + 16
$7\% < U \leq 9\%$	4.5 x (U - 7) + 27
$9\% < U \leq 11\%$	4.0 x (U - 9) + 36
$11\% < U \leq 14\%$	3.0 x (U - 11) + 44
$U > 14\%$	53

Source: Pension guide 2008 (Pensioengids 2008)

Table B1 summarizes how UL-discount could be measured using t-yield under technical interest rate of 4%. From August of 1999 the insurance tariff is not based on the interest rate of 4%, but 3%. The computation of UL-discount for the current situation is shown in Table B2. For example, suppose that a guarantee contract is issued in the month of January and the corresponding u-yield is 4.5%. The UL-discount on premium tariff is computed as 8.0 x (4.5 – 3) =

12.0%. The contract period is typically five years and the premium tariffs are fixed over this period. By contract extensions, the UL-discount is recalculated according to the new u-yield.

B2. Profit-sharing based on TL-discount or UL-discount with continuation discount

This variant of profit-sharing is quite similar to the contract discussed in the previous section. A key difference is that a continuation discount is added. An extra discount is assigned to the insured's account in case that the contract with TL-discount or UL-discount is extended after a period of 10 years of insurance. The continuation discount consists of partially excess returns obtained after 10 à 12 years of investment. In case the returns remain unchanged during the investment period, the continuation discount could be valued as 70-75% of the total realized excess return by the insurance company.

B3. Profit-sharing based on excess return u-yield

Contracting the first two forms of profit-sharing schemes, this provision provided here is however not determined in a constant manner. The profit-sharing occurs periodically according to the performance of an external reference index, i.e. u-yield. Profit is shared with beneficiaries when the u-yield exceeds the guaranteed rate of interest. Similarly, the returns are simply not based on the actual index performance, but on some fictive returns of mixed government bonds. Due to absence of implied volatilities for government bonds, valuation of this type of product becomes more complex. To able to value this product, it is common practice in the Netherlands to approximate the u-yield by a swap-rates. Evidence on historical data show that u-yield have behaved similarly as swap rate, and is therefore a robust estimator. More precisely, the 7-years swap rates curve has been recognized as a good proxy for the u-yield [see Plat and Gregorkiewicz (2007)]. The third form of profit-sharing policy sold in the Netherlands is applicable to middle-large enterprises.

B4. Profit-sharing based on segregated funds

The most important and well-known profit-sharing policy available in the Netherlands is the profit-sharing based on segregated funds. Segregated funds consist of professionally managed investment portfolios that guarantee to return a predetermined percentage of the investment at maturity or upon the death of the investor. These funds derive their name from the fact that their assets are set aside to pay policyholders and are not part of the issuing insurance company's general assets.

Like the profit-sharing based on excess return u-yield, the excess return credited to the policy owner occur in each policy year. However, the excess return is not determined by some external reference index, but is subject to the performance of the underlying segregated investment funds. The segregated fund is typically composed of stocks, bonds or other securities, and risk sources related to them have to be taken into account. The segregated funds often have complex embedded option features. These contracts typically also offer features such as mortality benefits, where the guarantee is paid off immediately upon death of the investor. Furthermore, because the payment for the guarantee is usually amortized over the life of the contract, there are additional complications due to investor lapsing. Usually, these types of guarantee products are suitable for large enterprises where more flexibility and upfront profit-sharing are implemented in the contract.

Appendix C – The T-forward Risk-neutral Measure

Following the technique proposed by Geman et al. (1995) and Brigo and Mercurio (2006)], we will now derive the dynamics of our model under the T-forward measure for a general maturity T. Let us denote by Q^T the T-forward (risk-adjusted) measure, i.e., the probability measure that is defined by the Radon-Nikodym derivative. We choose the ZCB maturing at T given by $P(t,T)$ as a numeraire

$$\frac{dQ^T}{dQ} = \frac{\exp\left\{-\int_0^T r_u\,du\right\}}{P(0,T)}. \tag{C1}$$

We know that we can rewrite the process r_t as

$$r_t = x_t + \varphi_t, \tag{C2}$$

where the process x satisfies

$$dx_t = -ax_t\,dt + \sigma_r\,dW_t^r. \tag{C3}$$

It can be shown that, denoting by $f^M(0,T)$, the market instantaneous forward rate at time zero for maturity T is given by

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T},$$

with $P^M(0,T)$ the market discount factor for maturity T. In order to exactly fit the observed term structure, we must have

$$\varphi_T = f^M(0,T) + \frac{\sigma_r^2}{2a^2}(1-e^{-2aT})^2.$$

The Radon-Nikodym derivatives from Equation (C1) becomes

$$\begin{aligned}\frac{dQ^T}{dQ} &= \frac{\exp\left\{-\int_0^T x_u\,du - \int_0^T \varphi_u\,du\right\}}{P(0,T)}\\ &= \frac{\exp\left\{-\frac{\sigma_r}{a}\int_0^T\left[1-e^{-a(T-u)}\right]d\widetilde{W}(u) - \int_0^T f^M(0,u)du - \int_0^T \frac{\sigma_r^2}{2a^2}(1-e^{-au})^2du\right\}}{P(0,T)} \qquad \text{(C4)}\\ &= \exp\left\{-\frac{\sigma_r}{a}\int_0^T\left[1-e^{-a(T-u)}\right]d\widetilde{W}(u) - \int_0^T \frac{\sigma_r^2}{2a^2}\left[1-e^{-a(T-u)}\right]^2du\right\}.\end{aligned}$$

The Girsanov theorem implies that the four processes $\widetilde{W}_t^{T,S}$, $\widetilde{W}_t^{T,r}$, $\widetilde{W}_t^{T,\Psi}$ and $\widetilde{W}_t^{T,\pi}$ are independent Brownian motions under measure $\mathbb{Q}^T$, where

$$\begin{aligned} d\widetilde{W}_t^{T,r} &= d\widetilde{W}_t^{r} + \frac{\sigma_r}{a}\left[1 - e^{-a(T-t)}\right]dt \\ d\widetilde{W}_t^{T,S} &= d\widetilde{W}_t^{S} \\ d\widetilde{W}_t^{T,\Psi} &= d\widetilde{W}_t^{\Psi} \\ d\widetilde{W}_t^{T,\pi} &= d\widetilde{W}_t^{\pi} \end{aligned} \tag{C5}$$

Only W_t^r is affected by the change of measure. The three other Brownian motions are not affected by this change of measure. One can find similar development of the change of measure in Brigo and Mercurio (2006). Therefore the (joint) dynamics of r, S, Ψ and π under Q^T with independent Brownian motion are defined by

$$\begin{aligned} dr_t &= a\left[\theta_r - \frac{\sigma_r^2}{a}\left(1 - e^{-a(T-t)}\right) - r_t\right]dt + \sigma_r d\widetilde{W}_t^{T,r}, \\ dS_t &= S_t\left[r_t - \rho_{r,S}\frac{\sigma_S\sigma_r}{a}\left(1 - e^{-a(T-t)}\right)\right]dt + S_t\left[\sigma_S\rho_{r,S}d\widetilde{W}_t^{T,r} + \sigma_S\sqrt{1-\rho_{r,S}^2}\,d\widetilde{W}_t^{T,S}\right], \\ d\Psi_t &= \Psi_t\left[\pi_t - \rho_{r,\Psi}\frac{\sigma_\Psi\sigma_r}{a}\left(1 - e^{-a(T-t)}\right)\right]dt \ldots \\ &\quad + \Psi_t\left[\sigma_\Psi\rho_{r,\Psi}d\widetilde{W}_t^{T,r} + \sigma_\Psi\left(\frac{\rho_{S,\Psi} - \rho_{r,S}\rho_{r,\Psi}}{\sqrt{1-\rho_{r,S}^2}}\right)d\widetilde{W}_t^{T,S} + \sigma_\Psi\sqrt{1-\rho_{r,\Psi}^2 - \frac{\left(\rho_{S,\Psi} - \rho_{r,S}\rho_{r,\Psi}\right)^2}{1-\rho_{r,S}^2}}\,d\widetilde{W}_t^{T,\Psi} + \right. \\ &\quad \left. \sigma_\Psi Z d\widetilde{W}_t^{T,\pi}\right], \\ d\pi_t &= \left[\gamma(\bar{\pi} - \pi_t) - \rho_{r,\pi}\frac{\sigma_\pi\sigma_r}{a}\left(1 - e^{-a(T-t)}\right)\right]dt + \sigma_\pi\rho_{r,\pi}d\widetilde{W}_t^{T,r} + \sigma_\pi\left(\frac{\rho_{S,\pi} - \rho_{r,S}\rho_{r,\pi}}{\sqrt{1-\rho_{r,S}^2}}\right)d\widetilde{W}_t^{T,S} \\ &\quad + \sigma_\pi Y d\widetilde{W}_t^{T,\Psi} + \sigma_\pi Z d\widetilde{W}_t^{T,\pi}, \end{aligned} \tag{C6}$$

where Y and Z are given by Equations (4.14) and (4.15) provided in Chapter 4. Thus, in the absence of actuarial risk, the fair value of liability X_T at time zero under T-forward risk-neutral measure can be expressed as

$$\Pi(0, X_T) = e^{-\int_0^T (r_t - \pi_t)} E^{Q^T}\left[X_T(\Theta)\right], \tag{C7}$$

with $\Theta = \{P_0, \delta_t, r_G, \alpha, \beta\}$; P_0 denotes the initial investment at the inception of the contract, δ_t presents the equity return at time t, r_G is the minimum guaranteed rate, α corresponds to the profit-sharing rate attributed to the insured and β is the profit-sharing rate attributed to the insurer. Hence, the transformation from the risk-neutral measure $\mathbb{Q}$ to the T-forward measure moves the discounting outside of the expectation term. This would accelerate the valuation process in the case that the size of problem that needs to be analyzed is large.

Appendix D – Theorems

Theorem D1 (Singular Value Decomposition)

Let Z be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ 'diagonal' matrix S such that

$$Z = USV^T$$

Then matrix S will have the block form

$$S = \begin{bmatrix} D & O \\ O & O \end{bmatrix}, \quad \text{where } D = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{pmatrix}$$

and each matrix O is a zero matrix of the appropriate size.

Theorem D2 (The Snell envelope)

Let $(Z_n)_{0 \leq n \leq N}$ be an adapted sequence, and define $(U_n)_{0 \leq n \leq N}$ as follows:

$$\begin{cases} U_N = Z_N \\ U_n = max\big(Z_n, E(U_{n+1}|\mathcal{F}_n)\big) \quad \forall n \leq N-1 \end{cases}$$

We call U_n the Snell envelope of Z_n. It is the smallest super-martingale that dominates Z_n.

Appendix E – Tables

Table E1

Fair values of a single premium European life insurance contract for different levels of interest rates (r) and profit-sharing rate insured (α) with a fixed profit-sharing rate insurer $\beta = 0.30$. Assumptions used are: 1) stochastic equity return; 2) deterministic interest rate; 3) deterministic inflation rate; 4) no mortality risk; 5) no surrender risk.

Ps-rate insurer (β)= 0.30		Terminal insured's account ($A_T + R_T^+$)					Terminal insurer's account ($C_T + R_T^-$)			
Interest rate scenario	Ps-rate insured (α)	Bond element (MG) [1]	Bonus insured (A-G) [2]	Insured's account (A) [1]+[2]	Terminal bonus (R+) [3]	Total [1]+[2]+[3]	Insurer's account (C) [4]	Terminal bonus (R-) [5]	Total [4]+[5]	European contract [1]+…+[5]
$r = 10\%$	0.10	15.88	5.93	21.82	69.26	91.07	10.75	−1.93	8.82	82.25
	0.20	15.88	14.19	30.07	60.47	90.54	13.10	−3.77	9.33	81.21
	0.30	15.88	25.76	41.64	49.21	90.85	16.17	−7.45	8.72	82.13
$r = 8\%$	0.10	35.35	11.03	46.37	47.33	93.70	19.87	−13.46	6.41	87.29
	0.20	35.35	25.73	61.07	36.88	97.95	23.55	−21.66	1.89	96.06
	0.30	35.35	45.30	80.64	26.43	107.07	28.09	−35.71	−7.61	114.69
$r = 6\%$	0.10	78.66	20.36	99.03	23.04	122.07	36.52	−59.12	−22.59	144.67
	0.20	78.66	46.39	125.06	15.70	140.76	42.10	−83.42	−41.31	182.07
	0.30	78.66	79.85	158.51	9.94	168.45	48.96	−116.53	−67.58	236.03

The table shows the fair values of the decomposed profit-sharing insurance contract under the risk-neutral measure $\mathbb{Q}$. The present value of the insurance policy is defined as $\Pi(0, X_T) = e^{-(r-\bar{\pi})T} E_Q[X_T]$, where X_T presents the market value of the insurance account at expiry date, r is the constant short rate of interest and $\bar{\pi}$ describes the long-term inflation rate. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, long-term inflation rate $\bar{\pi} = 0.0240$ and scenario = 50000.

Table E2
Fair values of a single premium European life insurance contract for different levels of interest rates (r) and profit-sharing rate insured (α) with a fixed profit-sharing rate insurer $\beta = 0.40$. Assumptions used are: 1) stochastic equity return; 2) deterministic interest rate; 3) deterministic inflation rate; 4) no mortality risk; 5) no surrender risk.

Ps-rate insurer (β)= 0.40		Terminal insured's account ($A_T + R_T^+$)					Terminal insurer's account ($C_T + R_T^-$)			
Interest rate scenario	Ps-rate insured (α)	Bond element (MG) [1]	Bonus insured (A-G) [2]	Insured's account (A) [1]+[2]	Terminal bonus (R+) [3]	Total [1]+[2]+[3]	Insurer's account (C) [4]	Terminal bonus (R-) [5]	Total [4]+[5]	European contract [1]+…+[5]
r = 10%	0.10	15.88	5.93	21.81	66.15	87.96	14.46	−2.55	11.91	76.05
	0.20	15.88	14.20	30.08	56.70	86.78	17.67	−4.88	12.79	73.99
	0.30	15.88	25.73	41.61	46.27	87.88	21.76	−9.53	12.23	75.66
r = 8%	0.10	35.35	11.04	46.38	43.03	89.41	26.77	−16.34	10.43	78.98
	0.20	35.35	25.70	61.04	33.22	94.26	31.69	−26.49	5.20	89.07
	0.30	35.35	45.29	80.64	23.26	103.90	37.83	−42.25	−4.42	108.32
r = 6%	0.10	78.66	20.37	99.04	20.00	119.04	49.14	−68.74	−19.60	138.64
	0.20	78.66	46.45	125.12	13.96	139.08	56.79	−95.00	−38.20	177.28
	0.30	78.66	79.89	158.55	7.90	166.45	65.91	−132.33	−66.42	232.87

The table shows the fair values of the decomposed profit-sharing insurance contract under the risk-neutral measure $\mathbb{Q}$. The present value of the insurance policy is defined as $\Pi(0, X_T) = e^{-(r-\bar{\pi})T} E_Q[X_T]$, where X_T presents the market value of the insurance account at expiry date, r is the constant short rate of interest and $\bar{\pi}$ describes the long-term inflation rate. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, long-term inflation rate $\bar{\pi} = 0.0240$ and scenario = 50000.

Table E3
Fair values of a single premium European life insurance contract for different levels of interest rates (r) and profit-sharing rate insured (α) with a fixed profit-sharing rate insurer $\beta = 0.50$. Assumptions used are: 1) stochastic equity return; 2) deterministic interest rate; 3) deterministic inflation rate; 4) no mortality risk; 5) no surrender risk.

Ps-rate insurer (β)= 0.50		Terminal insured's account ($A_T + R_T^+$)					Terminal insurer's account ($C_T + R_T^-$)			
Interest rate scenario	Ps-rate insured (α)	Bond element (MG) [1]	Bonus insured (A-G) [2]	Insured's account (A) [1]+[2]	Terminal bonus (R+) [3]	Total [1]+[2]+[3]	Insurer's account (C) [4]	Terminal bonus (R-) [5]	Total [4]+[5]	European contract [1]+...+[5]
r = 10%	0.10	15.88	5.93	21.82	63.20	85.01	18.29	−3.27	15.02	70.00
	0.20	15.88	14.19	30.07	54.14	84.21	22.31	−6.20	16.11	68.10
	0.30	15.88	25.80	41.69	43.16	84.85	27.58	−11.74	15.84	69.01
r = 8%	0.10	35.35	11.04	46.38	39.71	86.09	33.80	−20.05	13.75	72.34
	0.20	35.35	25.75	61.10	30.49	91.59	40.12	−31.25	8.87	82.71
	0.30	35.35	45.52	80.86	21.44	102.30	47.98	−49.18	−1.20	103.50
r = 6%	0.10	78.66	20.42	99.08	17.74	116.83	62.23	−78.79	−16.57	133.39
	0.20	78.66	46.54	125.21	11.68	136.89	71.79	−107.98	−36.19	173.08
	0.30	78.66	79.76	158.43	6.50	164.93	83.07	−148.19	−65.12	230.04

The table shows the fair values of the decomposed profit-sharing insurance contract under the risk-neutral measure $\mathbb{Q}$. The present value of the insurance policy is defined as $\Pi(0, X_T) = e^{-(r-\bar{\pi})T} E_Q[X_T]$, where X_T presents the market value of the insurance account at expiry date, r is the constant short rate of interest and $\bar{\pi}$ describes the long-term inflation rate. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, stock weight $w = 1$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, $T = 40$, $\sigma_s = 0.15$, long-term inflation rate $\bar{\pi} = 0.0240$ and scenario = 50000.

Table E4
Decomposed fair values of a single premium American life insurance contract for different levels of the long-run mean of interest rates (θ_r) and profit-sharing rate insured $\alpha = 0.20$ and profit-sharing rate insurer $\beta = 0.30$ with the assumptions: 1) stochastic equity return; 2) stochastic interest rate; 3) stochastic inflation rate; 4) stochastic mortality; 5) surrendering allowed.

Ps-rate insurer (β)= 0.30	Terminal insured's account ($A_T + R_T^+$)					Terminal insurer's account ($C_T + R_T^-$)						
Interest rate scenario	Bond element (MG) [1]	Bonus insured (A-G) [2]	Insured's account (A) [1]+[2]	Terminal bonus (R+) [3]	Total [1]+[2]+[3]	Insurer's account (C) [4]	Terminal bonus (R-) [5]	Surrender reserve (R*+) [6]	Surrender reserve (R*-) [7]	Surrender penalty [8]	Total [4]+...+[8]	American contract [9]
$w = 1$												
θ_r = 10%	59.26	11.03	70.29	33.02	103.32	11.02	−0.22	1.32	−10.21	2.81	4.72	98.60
θ_r= 8%	69.52	19.86	89.38	23.87	113.26	20.04	−1.61	1.51	−23.97	3.28	−0.75	114.01
θ_r = 6%	109.58	48.98	158.56	14.26	172.82	48.04	−25.03	0.68	−65.26	4.03	−37.54	210.37
$w = 0.30$												
θ_r = 10%	74.39	3.45	77.84	13.59	91.43	3.75	−0.03	2.32	−0.79	3.67	8.91	82.52
θ_r= 8%	86.15	2.97	89.12	5.12	94.24	3.25	−0.21	1.83	−2.41	4.28	6.74	87.50
θ_r = 6%	122.86	20.41	143.27	11.44	154.71	18.73	−24.28	0.05	−18.50	1.63	−22.38	177.09

The table shows the fair value of the decomposed profit-sharing insurance contract under the risk-neutral measure $\mathbb{Q}$ for asset allocation. Furthermore, we report the fair values for two different portfolios $w = \{1, 0.30\}$. The present value of the insurance policy is defined as $\Pi(0,\tau) = \sup_{\tau\epsilon\mathcal{T}_{(0,T)}} E_Q\left[\sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{k=1}^{\tau}(r_k-\pi_k)} X_\tau \cdot {}_{\tau}p_x\right]$, where X_T presents the market value of the insurance account at expiry date, r_t and π_t denote respectively the interest rates and the inflation rate at time t. In addition, ${}_{t-1/1}q_x$ presents the probability of an x-year-old Dutch male policyholder dies at t-th year and ${}_{\tau}p_x$ presents the probability of an x-year-old male Dutch policyholder survives another τ years. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, initial surrender reserve $R_0^* = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_r = 0.01$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, long-term inflation rate $\bar{\pi} = 0.024$, surrender penalty $\xi = 0.05$ and scenario = 50000.

Table E5

Decomposed fair values of a single premium American life insurance contract for different levels of the long-run mean of interest rates (θ_r) and profit-sharing rate insured $\alpha = 0.20$ and profit-sharing rate insurer $\beta = 0.40$ with the assumptions: 1) stochastic equity return; 2) stochastic interest rate; 3) stochastic inflation rate; 4) stochastic mortality; 5) surrendering allowed.

Ps-rate insurer (β)= 0.40	Terminal insured's account ($A_T + R_T^+$)					Terminal insurer's account ($C_T + R_T^-$)						
Interest rate scenario	Bond element (MG) [1]	Bonus insured (A-G) [2]	Insured's account (A) [1]+[2]	Terminal bonus (R+) [3]	Total [1]+[2]+[3]	Insurer's account (C) [4]	Terminal bonus (R-) [5]	Surrender reserve (R*+) [6]	Surrender reserve (R*-) [7]	Surrender penalty [8]	Total [4]+...+[8]	American contract [9]
$w = 1$												
θ_r = 10%	61.75	10.21	71.96	29.06	101.02	13.87	−0.25	1.26	−11.01	3.00	6.88	94.14
θ_r = 8%	72.41	17.31	89.72	19.76	109.48	23.82	−1.65	1.20	−24.90	3.54	2.01	107.47
θ_r = 6%	105.92	42.92	148.85	12.07	160.92	58.16	−22.53	0.67	−70.32	4.41	−29.61	190.53
$w = 0.30$												
θ_r = 10%	79.03	2.76	81.78	8.39	90.17	4.35	−0.01	2.41	−0.87	4.04	9.91	80.26
θ_r = 8%	88.15	2.17	90.32	2.55	92.87	3.55	−0.10	1.85	−2.47	4.55	7.38	85.49
θ_r = 6%	121.48	19.91	141.39	10.09	151.48	24.57	−23.96	0.06	−22.46	1.95	−19.84	171.32

The table shows the fair value of the decomposed profit-sharing insurance contract under the risk-neutral measure $\mathbb{Q}$. Furthermore, we report the fair values for two different portfolios $w = \{1, 0.30\}$. The present value of the insurance policy is defined as $\Pi(0,\tau) = \sup_{\tau\epsilon\mathcal{T}_{(0,T)}} E_Q\left[\sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)}X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{k=1}^{\tau}(r_k-\pi_k)}X_\tau \cdot {}_{\tau}p_x\right]$, where X_T presents the market value of the insurance account at expiry date, r_t and π_t denote respectively the interest rates and the inflation rate at time t. In addition, ${}_{t-1/1}q_x$ presents the probability of an x-year-old Dutch male policyholder dies at t-th year and ${}_{\tau}p_x$ presents the probability of an x-year-old male Dutch policyholder survives another τ years. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, initial surrender reserve $R_0^* = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_r = 0.01$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, long-term inflation rate $\bar{\pi} = 0.024$, surrender penalty $\xi = 0.05$ and scenario = 50000.

Table E6

Decomposed fair values of a single premium American life insurance contract for different levels of the long-run mean of interest rates (θ_r) and profit-sharing rate insured $\alpha = 0.20$ and profit-sharing rate insurer $\beta = 0.50$ with the assumptions: 1) stochastic equity return; 2) stochastic interest rate; 3) stochastic inflation rate; 4) stochastic mortality; 5) surrendering allowed.

Ps-rate insurer (β)= 0.50	Terminal insured's account ($A_T + R_T^+$)					Terminal insurer's account ($C_T + R_T^-$)						
Interest rate scenario	Bond element (MG) [1]	Bonus insured (A-G) [2]	Insured's account (A) [1]+[2]	Terminal bonus (R+) [3]	Total [1]+[2]+[3]	Insurer's account (C) [4]	Terminal bonus (R-) [5]	Surrender reserve (R*+) [6]	Surrender reserve (R*-) [7]	Surrender penalty [8]	Total [4]+...+[8]	American contract [9]
$w = 1$												
θ_r = 10%	68.32	8.12	76.44	21.21	97.65	14.37	−0.22	1.19	−10.89	3.47	7.92	89.73
θ_r = 8%	74.20	15.89	90.09	17.43	107.52	27.85	−1.72	0.99	−26.66	3.72	4.18	103.34
θ_r = 6%	109.41	48.77	158.18	11.19	169.37	81.51	−31.49	0.39	−83.47	4.13	−28.93	198.30
$w = 0.30$												
θ_r = 10%	83.41	2.14	85.55	3.49	89.04	4.77	−0.01	2.39	−0.93	4.39	10.62	78.42
θ_r = 8%	89.40	1.76	91.16	1.01	92.17	4.02	−0.07	1.64	−2.68	4.71	7.62	84.54
θ_r = 6%	119.41	19.28	138.70	8.86	147.56	30.03	−23.67	0.06	−25.64	2.22	−16.99	164.55

The table shows the fair value of the decomposed profit-sharing insurance contract under the risk-neutral measure $\mathbb{Q}$. Furthermore, we report the fair values for two different portfolios $w = \{1, 0.30\}$. The present value of the insurance policy is defined as $\Pi(0,\tau) = \sup_{\tau\epsilon\mathcal{T}_{(0,T)}} E_Q\left[\sum_{t=1}^{\tau-1} e^{-\sum_{k=1}^{t}(r_k-\pi_k)} X_t \cdot {}_{t-1/1}q_x + e^{-\sum_{k=1}^{\tau}(r_k-\pi_k)} X_\tau \cdot {}_{\tau}p_x\right]$, where X_T presents the market value of the insurance account at expiry date, r_t and π_t denote respectively the interest rates and the inflation rate at time t. In addition, ${}_{t-1/1}q_x$ presents the probability of an x-year-old Dutch male policyholder dies at t-th year and ${}_{\tau}p_x$ presents the probability of an x-year-old male Dutch policyholder survives another τ years. The reference insured is aged $x = 25$ at time 0. Other parameters included in the model are: minimum guaranteed rate $r_G = 0.03$, initial investment $P_0 = 100$, initial insured's account $A_0 = 100$, initial reserve account $R_0 = 0$, initial insurer's account $C_0 = 0$, initial surrender reserve $R_0^* = 0$, $T = 40$, $\sigma_s = 0.15$, $\sigma_r = 0.01$, $\rho_{r,S} = -0.0531$, $\rho_{S,\Psi} = -0.0675$, $\rho_{S,\pi} = 0.0026$, $\rho_{r,\Psi} = 0.0516$, $\rho_{r,\pi} = 0.5248$, $\rho_{\Psi,\pi} = 0.1641$, long-term inflation rate $\bar{\pi} = 0.024$, surrender penalty $\xi = 0.05$ and scenario = 50000.

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