ERASMUS UNIVERSITY ROTTERDAM

Master thesis

# Optimal Features
# of Micro Pension Schemes

Econometrics and Management Science

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# Abstract

Micro pension is a pension scheme designed for the people with low income. In this thesis I analyze elements of the traditional pension scheme in order to select optimal settings for micro pensions. I use both the Expected Utility framework and Cumulative Prospect Theory to evaluate the theoretical benefits versus the observed behavior. In the first part of the research I compare different contract types, which combine elements like minimum income guarantee, direct investment and trading the upside potential for the lower price of the contract. I use the arbitrage free pricing principle and extend the model used in the literature by incorporating stochastic interest rates. I find that in the micro pension setting the contracts that include minimum guarantee perform the best. In the second part of the research I look at the payout phase, and compare an annuity contract versus lump sum withdrawal. I introduce the prediction error that the retiree makes while planning her consumption and evaluate its consequences. I find that annuity is an optimal choice as an insurance against longevity except for the cases of confirmed terminal illness. Loss aversion in CPT explains the choice against annuitization that people often make in the real life. Taken together, both parts provide a theoretical guidance for constructing the micro pension contract.

**Keywords:**
Micro pensions, annuity, lump sum, minimum income guarantee, Expected Utility theory, Cumulative Prospect theory, loss aversion, longevity, prediction error, arbitrage-free pricing.

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Remaining mistakes in this thesis are entirely my responsibility.

# Contents

# Glossary

**Annuity** – a payment of a predefined size distributed by the pension provider to the retiree on regular basis (monthly or annually) during the years of his/her life

**Bequest motive** – willingness of the person to leave a part of his pension as an inheritance to the family members

**Cocontribution** – additional periodical contribution that the state or the company makes together with the person who will receive the pension in the future

**DHAN foundation** - Development of Humane Action Foundation, a professional development organization in Tamil Nadu, India, which works towards bringing significant changes in the livelihoods of poor people through innovations in themes and institutions[1]

**Grameen bank** – a microfinance organization and community development started in Bangladesh that provides financial services (such as small loans, savings opportunities and pensions) to the impoverished people[2]

**Microfinance** – "banking the unbankables, bringing credit, savings and other essential financial services to the millions of people who are too poor to be served by regular banks, in most cases because they are unable to offer sufficient collateral"[3]

**Micro finance institution (MFI)** – an organization that provides microfinance services, can be broadly defined as any financial organization—credit union, down-scaled commercial bank, financial NGO, or credit cooperative—that provides financial services for the poor

**Self-Help-Groups (SHG)** - a village-based financial intermediary usually composed of between 10-15 local women. Most self-help groups are located in India, though SHGs can also be found in other countries, especially in South Asia and Southeast Asia[4]

**SHEPHERD –** Society for Human Equallity People's Health Education and Rural Development, an organization in Tamil Nadu Madurai District, South India, which is running poverty alleviation programs for rural poor women[5]

**Social pension** – cash-transfer based programs providing non-contributory old age provision for underprivileged citizens, similar to grants or humanitarian aid (as opposed to contributory pensions, where the future retiree is himself responsible for building the future welfare by investing some initial capital)

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[1] http://www.dhan.org/
[2] http://www.grameen-info.org/
[3] Gert van Maanen, 2004. Microcredit: Sound Business or Development Instrument, Oikocredit.
[4] http://en.wikipedia.org/wiki/Self-help_group_%28finance%29
[5] http://shepherdindia.org/index.php

# 1. Introduction

The economy of the modern world is developing in quite a heterogeneous way: the average life level across the globe differs dramatically. There is a huge gap of the life style implied behind the words "developing country" and "developed country". The level of the social and financial security is usually a reflection of the economic and political situation in the state. Old people, belonging to the part of the society which solely depends on the state and family due to objective physical reasons, are always among the first to take the hit of the imperfect social protection system.

Global aging is the issue that all the countries in the world need to face. For developing countries this problem looms even larger, as according to US National Institute of Aging, The current rate of growth of the older population in developing countries is more than double that in developed countries. As of 2008, 62 percent (313 million) of the world's people 65 and older lived in developing countries. By 2040, today's developing countries are likely to be home to more than 1 billion people 65 and over, 76 percent of the projected world total. This striking statistics means that in several decades there will be many more old people, and that their life length will get higher as well. All these people will need financial support, and if the social security system cannot handle the loading now, the chance is slight that the necessary reforms will catch up fast enough. Therefore the people need to participate in ensuring their own old age financial provision themselves. In this thesis I analyze several elements of the traditional pension schemes in order to select optimal settings for micro pensions, which are one of the ways for the people with low income to build their old age provision.

All developed countries have safety nets implied in social security systems to prevent poverty among older people. They are called first-tier redistributive schemes provided by the public sector and are mandatory. The second pillar consists of the industry- or company-based mandatory pension plans for employees and the third pillar includes voluntary contributory individual pension and savings plans arranged by individual investors themselves (Pensions Panorama (2007)).

In developing countries, where the first pillar pension plans are not developed or not properly organized, second pillar pension plans often cover a very small percentage of population. In India, for example, depending on estimates, 87% to 89% of the working population are self-employed or belong to the unorganised sector. These people, when growing old, are uncovered by any type of formal old age financial provision (Gianadda (2007)). Relying on the help of the family or community is a usual practice among old poor as it is the only source to expect the support from. However, recent developments towards society modernization have arrived even in the countries like India, where the life pattern is strongly influenced by historical habits and traditions. This causes gradual erosion of the joint family system, entailing the growing distance between generations and weakening the support seen in the past.

The government reforms, such as the New Pension Scheme in India, strive to reconstruct the first pillar pension plans, which in case of successful implementation will have an important impact on the life of people in the long run. However, corruption, poor infrastructure and cultural context can make these positive changes sensible only in the far away future.

Individual pension schemes based on personal contributions gain crucial importance in this situation. However, it is hard for individuals to get the opportunities to invest the money saved during the active period of life on their own – a more efficient and reliable option would be to purchase the pension products available in the financial market. The problem is that most of the available schemes are designed for investors with regular income and stable financial and social situation. This leaves poor people out of the scope.

A recent development in the financial area is the rise of the microfinance. The term itself refers to the provision of financial services to low-income clients [Ledgerwood, 2000]. It includes a wide range of financial services, distinguishing from the regular banking industry by several characteristic features. Among them a much smaller size of the amounts of money involved in the transactions and individual approach to the clients, which implies particular distribution principles, different from the classical bank-customer approach.

The ultimate goal of delivering these financial services is poverty reduction and building a sustainable system allowing an access to financial resources for the people with lower income. Until recent years, the focus was mainly made on providing microcredit to individuals or communities as an alternative for the informal moneylenders. However, after a certain period of time several studies researching the impact of these financial services on the life of poor people showed that microcredit is not a panacea as it often helps to generate an extra income, but doesn't help people to smooth the consumption over time. Savings and insurance schemes start to gain some popularity among microfinance institutions. The reason is twofold: growing demand for savings products on the one side and the possibility to use collected funds to finance own credit services and in this way reduce dependence on sponsors on the other. Logically, pension schemes ought to be the next step in delivering financial services to the poor people.

The role of micro pension can be valuable in many aspects of life. Its impact extends to the family of the retiree and besides the poverty reduction may contribute to protect the dignity of the old person and prevent her from suffering while not being treated respectfully and as a "financial burden". Besides that, it often happens that old people are forced to be responsible for raising their grandchildren, missing one link in the generation sequence due to health related or other problems that make their children disabled.

The goal of this research is to investigate the elements of the regular pension scheme and select or modify those that are the most beneficial for the underprivileged people as a client group. The topic arose in the context of the project for Indian rural poor, that the development organization DHAN is currently planning to implement. DHAN works with over 600,000 poor and underprivileged households in India (http://www.dhan.org). The goal of the project is to create a sustainable pension scheme involving people with the lowest income with the coverage 25.000 people after the period of five years. Given the original source of this thesis idea, most of the time I narrow the examples to the case of India.

The relevant existing research can be divided into two parts: the literature investigating pensions and retirement and the research in the area of microfinance.

Existing literature in the area of microfinance is more focused on the aspects of micro lending and micro insurance (as e.g. Basu (2006), Tazul (2007), Case Studies by Centre for Microfinance (2006)), while savings and pension schemes are more out of scope. There also exists scientific research regarding regular pension schemes for developed countries and their different elements like annuity distribution, guarantee, investing of the collected funds etc. Most of this literature, however, is focused on the side of pension provider and dealing with risks associated with such a long-term liability project. Some studies, like Verma (2003) even cover particularly Indian pension market. However, these studies mainly focus on regulatory issues and the schemes provided for the organized sector in India - the first and second pillar pension schemes. Thus it is especially interesting to investigate the microfinance framework, as the target group has some specific features that make regular schemes either not applicable or requiring some significant modifications in order to be implemented for this kind of investors. The existing research papers on micro pensions in India, similar to Gianadda (2007) or Uthira and Manohar (2009), mainly give an overview of the savings schemes most similar to pension principles as at the moment in India there are no schemes providing compulsory annuitization of the accumulated funds.

A lot of relevant research exists also in the area of pension design (e.g. Whitehouse (2007)), but the influence of different schemes on client's welfare remains a topic that researchers usually leave out of the scope. One of the few works which uses the approach form the side of the investor is Doskeland and Nordahl (2007). The authors introduce several contract types and investigate their influence on customers' welfare by combining the pricing principles with Expected Utility theory and Cumulative Prospect Theory. I build up the first part of my thesis based on this article. I follow the approach of the authors – Expected Utility and behavioral finance framework in the second part of the research, while comparing annuity contract and the lump sum withdrawal.

Doskeland and Nordahl (2007) introduce different contract types, but exclude an annuity from the scope of the research. The reason is that the actual validity of annuity scheme starts at the

retirement date, while Doskeland and Nordahl (2007) focus on the investment opportunities before retirement. In the micro pension setting it is very interesting to combine different investment options with annuitization at retirement date. Besides that, all the introduced pension contracts have two features in common: their payoff has an upside potential and in some cases a minimum guarantee upon maturity. In the framework of micro pensions minimum guaranteed income at retirement is definitely an important part of the scheme, but the benefit of the upside potential is more questionable. In order to test the value of this feature for low income investors I introduce an additional contract type, which mimics the so called Bull Spread investment strategy. In this contract the upside potential is traded for the lower price for the participant. However, in the evaluation phase this contract type doesn't prove to be the best as the lower price that the investor need to pay is not overweighting the disadvantage of giving away a prospective of extra income.

In my research I do not consider the case of social pensions, although it is a wide-spread form of providing financial help to aging people in the developing countries. The reason is that the social pensions are based on a donor help, and then the questions to answer are only related to the definition of the eligible recipients. The goal of my research is to investigate the opportunities to create a sustainable pension scheme for underprivileged people which can work in the long run without any external financial support. Thus, in the current model the investor himself is sharing the responsibility for his old age welfare together with the pension scheme provider, similar to the pension arrangements in developed countries.

The main value of this paper from the point of view of microfinance is providing a technical base to select the optimal elements of the micro pension scheme for the future implementation. It provides objective argumentation for the choice of annuity or minimum income guarantee as the target features to implement. In the pensions framework, it extends the research of Doskeland and Nordahl (2007) for the case of stochastic interest rates and provides a new behavioral angle to the question of optimal annuitization, as well as new considerations about the consumption planning of the people who take their pension as a lump sum.

The structure of this report is the following. In section 2 I provide a general description of micro pension and its place in the financial system of developing countries. Section 3 introduces the financial setup and investment contracts. Overall the scheme consists of two parts: the first one is the investment period before retirement described in section 3, when future retiree places the initial capital according to a certain investment scheme. Second part, described in section 5, is the period after retirement, when the retiree receives accumulated funds in a form of a pension. Chapter 4 introduces Expected Utility theory and Cumulative Prospect Theory framework. In the first part I select the best investment contract type, while in the second - the most beneficial type of the

pension distribution. I investigate two options for the latter: the annuity contract and receiving a capital as a lump sum. Every step of the research, I build a model originally for the regular pension scheme (not particularly micro finance) and after that review the parameters and provide some considerations based on the facts of real life in order to interpret the necessary modifications.

As a conclusion, I make several suggestions regarding the practical implementation of the pension scheme for the underprivileged people in developing countries. The main finding is that the annuity is a very desirable feature for the micro pension scheme as it is the only way to deliver the pension *during the whole period of life* of the retiree. Another useful feature should be the minimum income guarantee, which is, however, difficult to implement in reality. To my knowledge, there are currently no pension schemes designed for people with lower income that would include at least one of the two features.

## 2. Micro pensions: an overview

### 2.1. Elements of micro pension scheme

A micro pension is an old age financial security scheme for investors with a low income. It combines both the elements of a usual pension scheme and specific features of microfinance. As this pension product is not provided by the government or employer, but requested by the investors themselves, micro pensions can be classified as a second or third pillar pension scheme. However, there are several differences from the setting of the regular pension, as familiar to the citizens of all developed countries. One of these differences is that self-employed people, or those belonging to the unorganized sector, do not have any official fixed salary. This makes it impossible to use the mechanism of collecting money as taxation of the income, as in most second pillar pension schemes. The income of self employed people is often irregular and more vulnerable to the risks like natural disasters health emergency or theft than that of people working in the organized sector. This makes the collected amount and consequently the future stream of cash flows more uncertain. This even leaves aside the simple fact that it is often not possible to have a personal bank account for the money accumulation.

There is no strict definition for the micro pension as such. The reason among others is that at the moment all over the world there are almost no examples of sustainable pension schemes designed specifically for investors with a low income. Still, as in the case of a classic pension scheme, the ultimate goal of micro pension is to provide financial security to the elderly, when they cannot continue to work due to physical reasons. Therefore, the pension should ensure the long-time financial provision in the first place.

The two stages of participating in the scheme for every individual investor include the accumulation phase and the payment phase. In the first stage, a future retiree is makes regular contributions to the pension fund and the fund collects and invests accumulated money according to predefined investment scheme. The retirement date is the beginning of the second stage, when the retiree starts to receive and consume the accumulated wealth. At the retirement date a few options may appear: the investor can withdraw the accumulated funds as a phased withdrawal (obtaining a share of the accumulated funds on regular basis), a lump sum or an annuity. I investigate and compare the benefits of the latter two options. I leave phased withdrawal out of the scope, as it is an option that ensures a stream of income for the retiree only during a certain period of time. It is therefore in principle similar to taking the capital as a lump sum, with the only difference that the pension fund instead of the person defines the consumption planning.

In practice, in most developed countries the retirement date for the first and second pillar pension schemes is usually defined as a certain age, therefore both the investor and the pension fund know in advance the length of the accumulation period before retirement for every individual

investor. The payment phase after retirement contains more uncertainty compared to accumulation phase, as it is not possible to predict in advance for how many years the investor will receive the payments, should she choose the annuitization option.

What makes micro pensions different from the classical setting, is the scale of initial investments and the risks involved. In the case of pure defined contribution scheme, the pension fund plays a simple role of the interagent between the investor and pension provision, collecting and investing the money and delivering back a final payoff. In the case of micro pensions it is a questionable strategy, as poor people cannot take the risks similar to those that regular investors can afford. This means that there might be a need in a downside protection, which is an element of defined benefit pension scheme.

SHEPHERD India, for example, has established some pension schemes based on defined contribution principle. However, DHAN foundation for their prospective pension scheme requires for their clients some protection against the investment risk. The argument is that these people do not possess necessary knowledge to protect themselves from the fluctuations in the financial market or even to choose the right investment strategy due to low financial literacy and cannot afford to lose any share of the accumulated funds[6]. It is important to remember that in case of a defined benefit pension scheme managing the risks involved in the investment scenario becomes a responsibility of the pension provider. The future payments guarantee becomes its liability. A pension fund has to define a reasonable future size of it based on the investments expected from the future retiree during the productive period of his work, and create a setting to ensure the payments at the retirement date.

People with a very low income are much more vulnerable to any kind of instability in their life, such as health problems or natural disasters, which has a direct impact on their ability to pay regular contributions. This means that in case any elements of the defined benefit pension scheme are included in the model, the risk for the financial institution is much higher than when dealing with regular financially stable investors.

## 2.2. Implementing the scheme: the challenges

This section briefly mentions the problems that occur while implementing the micro pension scheme in a real world situation. This list is not complete, and unfortunately not all of them can be addressed in the current research, partly due to the fact that it is hard to formulate them in the numerical or theoretical values. It is still important to describe these issues to get a better feeling of the environment the pension scheme should exist.

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[6] Source: interviews conducted by Van Dullemen, C. with DHAN foundation representatives in India, September 2009.

Most features that distinguish micro pensions from the old age provision for regular investors, entail the challenges for implementation in a real life. The experience of some microfinance institutions in India, Bangladesh and Philippines that tried to create savings or pension products for people with low income (see Appendix 1), showed that there are few particular areas where most difficulties are predetermined.

### 2.2.1. Operational issues: collecting payments, distribution and transaction costs

Collecting the payments from the clients via bank department or any institutional branch like a post office is in not always efficient or even possible while serving the poor. There are two reasons for that – from the side of the client, traveling to the nearest institution can be too costly in terms of time and money, and from the side of the financial institution the transaction costs can be too high when working with small amounts of money. The solution for the fund manager is to have an intermediate agent, like a Self-Help-Group representative or a local Micro Finance Institution (MFI), which can collect the payments until they reach a certain size, and only then send them to the fund manager. The negative consequence of this solution is that the interest rate for individual investments is calculated only at the moment of reaching the fund manager, which may reduce the size of the final payment.

For the clients, the most preferable way is door-to-door collecting of the periodical payments, or bringing them to the locally reachable MFIs or meetings of the community units, like Self-Help-Groups. The first principle is used by for example informal money borrowers, who come to their clients to collect their profit and store it as a part of their savings on a daily basis. In fact, poor people are even willing to pay for this kind of service, introducing in this manner a negative interest rate for their investments (Ledgerwood (2000)). This can serve an illustration of the importance of the savings opportunities for the poor people and the ability to reach them in efficient way.

### 2.2.2. Regulatory issues

Many MFIs are not allowed to collect and mobilize deposits on their own. In India, for example, MFIs are permitted to attract deposits only if they are registered as local area banks or cooperatives (Shankar (2009)). One of possible solutions is so called partner-agent model, which means that financial product is designed and distributed in the collaboration of the larger financial institution, which stores the money and actually performs fund management, and microfinance institution, which plays the role of a distribution channel as it has the biggest access to the poor people, their trust and possibly an established coverage (Gianadda (2008)).

An example of successful attracting of saving deposits from the poor people is the Grameen bank in Bangladesh, which during the first years after introducing the savings products has collected

US$ 826 million as deposits, significantly more than the total sum of the outstanding loans (Dowla and Barua (2006)).

### 2.2.3. Cultural context

Although being mentioned last among the potential challenges, cultural context can certainly appear one of the most serious obstacles on the way of implementing the pension scheme. Poverty often comes with a low educational level and that means that traditions and superstitions might play a big role in the behavior of people.

One example is a distribution of the finances in the family: female clients of the Grameen bank in Bangladesh (see Dowla and Barua (2006)) were often expected to bring all the money in the family and not to set any part of income aside for themselves as the pension product supposes. The bank has solved this problem by making the participation in the savings or pension scheme mandatory for obtaining the loan, in this way eliminating the pressure on the female clients.

The client's gender also makes a substantial difference. For example, most of the clients receiving micro credit from the Grameen bank and members of Self-Help-Groups are female. Statistics shows that women are more likely than men to spend money on the predefined purpose of the credit and are more accurate in repaying the loans. From the micro pensions perspective, the implication is that women might be more systematic in making regular payments and can therefore create less uncertainty and risk for the pension fund. However, this can also come automatically if the future subscribers for micro pensions scheme are the existing clients of the microfinance institutions, because most of them are currently female. For the pension scheme, the clients' gender will also have an impact on the longevity risk, as women and men have a different life expectancy.

For this research I do not set a framework of any particular country, trying to obtain a result applicable for different cultural contexts. I do, however, bring up the examples of particular countries (like India or Bangladesh) in order to somewhat link the theory to the real world situation.

## 2.3. Two sides of one coin

Recently a lot of discussion appeared in the literature and media regarding commercialization of the microfinance area. One of the most repeated claims is that because poor people are often in a desperate need of funding, they are willing to pay higher rates than they should actually be charged fairly, even given the additional risk that serving them incurs for the financial institution. Another issue is that as these people do not have any extra money to spend, they should not be charged the full price for the services, and that the government or development organization should consider providing a cocontrition. However, this suggestion has a serious drawback – external cash flows make the MFI rely and depend on them and eventually entail the danger for the sustainability of the financial system. The cases of dramatic consequences of the external financial support are not that rare. One example is the case of community-based micro insurance products distribution in India, Karnataka, where the United Nations Development Programme paid the first full three-year premiums for the participants who belonged to the low casts or tribes and half of the premiums for

the families living below the poverty line. When the participants were asked to continue paying the contributions themselves, enrolment fell from 82.000 participants to 25.000[7]. Another example includes the funded microcredit programs, when the participants rely on the external sponsor to ensure their debts and therefore lose the feeling of responsibility toward repayment need. The outcome of such a situation is obviously disastrous for the micro finance institution providing the services and thus for the clients themselves.

There is a principal difference between the non-profit MFIs and those who have a final goal to receive the profit which is above the level necessary for the sustainable work and development. "Profit for the profit" is absolutely unjustified when serving the poor clients, but that in order to establish a sustainable system, the regular business cycle should be in place. The area where the initial sponsorship is appropriate and often necessary is the initial phase – establishing the project and marketing means. On the later stage, when the system works successfully, the potential clients will naturally get informed and involved, as the news in the communities spreads fast enough.

Another problem with commercialization is the fact that MFIs tend to attract more people who are not able to pay back in case of getting the credit (simply because of objective lack of investment opportunities) or pay regular contributions in case of micro pension. This phenomenon can be compared to the credit crunch on the micro level[8]. Similar to the commercial banks which distributed housing loans to the people not able to repay them in time, Micro Finance Institutions do not always verify the financial stability of their clients in a responsible way. The solution to this problem is to subscribe the members of the scheme on the first place to other services, like savings, in order to make sure that they have an opportunity and will to set a part of their money aside from everyday spending.

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[7] Source: http://www.prb.org/Articles/2006/CommunityBasedHealthInsuranceShowsPromiseinIndia.aspx
[8] Source: http://www.economist.com/node/14298996

## 3. Financial model

In this chapter I specify the model framework and define the space in which the pension scheme participants act. I introduce several types of investment contracts which reflect the characteristic features of pension schemes in the real world. The goal of this section is to describe investment schemes potentially available for the future retiree in order to choose an optimal scheme later on.

In this and the next chapter I follow the approach of Doskeland and Nordahl (2007) by describing several contract types and later on evaluating their benefit for the investor with the help of Expected Utility theory and Cumulative Prospect Theory. To the three contracts introduced by the authors I add the fourth one, which has a particular feature of making the price lower for the investor by the costs of trading the upside potential to the pension fund. As I am interested in the features of the pension scheme from the point of view of an investor, the choice the investor makes in case he is given an opportunity to make this decision on his own, the approach of the authors provides a path to answer these questions. The additional element of behavioral finance framework helps to evaluate the product features taking into account the way people think in the real world.

In the financial model, I add the stochastic interest rate, versus fixed one in Doskeland and Nordahl (2007) and use numerical procedures to price the described pension contracts in the extended setting.

### 3.1. General setup

The following parties form the model framework: the pension provider and the future retiree (investor). In developed countries, the pension provider for regular retirees is a pension fund, and the contributions for the future pension are usually collected in a form of a tax or percentage of the salary during the years when the person is still actively working. In case of investors with lower income most of the time specialized microfinance institutions or a fund manager (for example a large bank or insurance company) to which the money is transferred, plays the role of the pension fund. People subscribe for the micro pension scheme unrelated to their working environment, as the target group belongs to the unorganized sector. For simplification, I assume that the pension fund collects the money not in installments, but in one initial lump sum payment denoted as $L_0$. The funds are then invested according to the predefined investment scheme so that at the retirement date *T* the

I build the financial framework based on two assets representing a riskless and a risky investment: a risk free bank account *D* and an equity index *S*. In a real life this is interpreted as investing in stocks and putting money on a bank deposit or investing it in less risky securities, like bonds.

Following the common approach in the literature, I use stochastic differential equations to model the assets dynamics and the solutions obtained in this chapter are derived with the help of Ito's lemma. The risk is modeled with the help of Brownian Motion, which is widely used in finance to

represent the uncertainty present in the dynamics of the assets and financial instruments over time. A formal definition of the Brownian Motion is provided in Appendix 2 (see also Baxter and Rennie (1997)). The solutions of the stochastic differential equations in this chapter are formally verified in Appendix 3.

The dynamics of the assets can be described by two equations:

$$dD_t = rD_t dt, D_0 = 1 \tag{1}$$

$$dS_t = \mu S_t dt + \sigma S_t dZ_t^P, S_0 = 1 \tag{2}$$

where $r$ is the risk-free rate, $\sigma$ is volatility of the stock, $\mu$ represents expected return for the equity index and $Z_t^P$ is a Brownian Motion under the real world measure $\mathbb{P}$.

The process for the stock has the following form:

$$S_t = S_0 \exp((\mu - 0.5\sigma^2)dt + \sigma dZ_t^P) \tag{3}$$

where $S_0$ is the initial stock price.

The pension fund invests future retiree's money in the portfolio $A_t$ which consists of the two assets described above. Proportion θ of initial sum is invested in the stock and (1-θ) in the riskless asset. Portfolio value will then have the following form:

$$A_t = \frac{A_0 \theta}{S_0} S_t + \frac{A_0(1-\theta)}{D_0} D_t \tag{4}$$

where $A_0$ is the initial investment and $D_0$ - initial bond price. The total dynamics of the portfolio under the real-world measure evolves according to the equation (see Appendix 3 )

$$dA_t = A_t \left((r + \theta(\mu - r))dt + \theta \sigma dZ_t^P\right) \tag{5}$$

Similar to Doskeland and Nordahl (2007), I assume that the proportion θ of initial capital invested in the risky asset, once defined, stays constant and thus there is no dynamic rebalancing of the portfolio. In practice, this simplification doesn't crucially restrict the model, as micro finance institutions usually try to keep the frame of the financial products they deliver to poor people as simple as possible. Portfolio structure which is predefined when the person gets the subscription is therefore quite a realistic assumption.

### 3.1.1. Contract types

Now, when the underlying assets are defined, I formulate different types of pension contracts, representing the most characteristic features of pension schemes existing in reality and calculate their final payoffs at the retirement date.

The pension scheme in the research is optionally based on the following types of the contracts.

- *Merton's problem*:  the customer has a possibility to choose asset allocation directly (no limitations or guarantees) ;

- *Implicit put :* the customer gets a minimum periodical return guarantee on the initial investment ;

-  *Simple insurance* - the customer has a minimum guarantee on the return, but the company can default. In the latter case the losses for the investor are not limited;

- *Implicit bull spread* – the type of the contract where investor gets a minimum income guarantee and his gains are limited to a certain amount. This means that the downside and upside potential are limited simultaneously and investor has a certain boundaries in which he expects his gains and losses to vary.

### 3.1.2. Pricing the contracts

Fair pricing implies the arbitrage-free principle, which means that the price of the contract at time 0 satisfies the following condition:

$$L_0 = E^{\mathbb{Q}}[e^{-rT}L_T] \tag{6}$$

where *T* is the end of period or retirement date, $L_0$ and $L_T$ - initial investment and accumulated funds respectively. Expectation in the fair pricing principle is taken under the risk-neutral measure $\mathbb{Q}$. This measure provides a possibility to mathematically reflect the fact that in arbitrage-free economy it is not possible to systematically outperform the market (Pelsser (2000)), as the discounted price processes of the assets transform and under this measure have no drift (become $\mathbb{Q}$-martingales). I provide a more detailed description of this issue in Appendix 4.

### 3.2. Contracts payoffs: case of fixed interest rate

Equation (5) for the assets dynamics after changing the measure from the real world $\mathbb{P}$ to risk-neutral $\mathbb{Q}$ looks the following (see Appendix 4):

$$dA_t = A_t \left( rdt + \theta\sigma dZ_t^{\mathbb{Q}} \right) \tag{7}$$

and the solution to it for the retirement date *T* has the following form –

$$A_T = A_0 \exp\left( (r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}z \right) \tag{8}$$

where $z \sim N(0,1)$. The above holds because of the following property of the Brownian Motion: $Z_t^Q \sim N(0,\sigma T)$ .

$$\mathrm{E}^{\mathbb{Q}}[A_T] = A_0 \mathrm{E}^{\mathbb{Q}}\left[\exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}z\right)\right] = A_0 \exp\left((r - \frac{\sigma^2}{2})T\right)\mathrm{E}^{\mathbb{Q}}[\exp(\sigma\sqrt{T}z)] =$$

$$= A_0 \exp\left((r - \frac{\sigma^2}{2})T + \frac{\sigma^2}{2}T\right) = A_0 \exp(rT)$$

Thus in case of constant interest rate the fair pricing equation takes form

$$\mathrm{E}^{\mathbb{Q}}[A_T] = A_0 \exp(rT) \tag{9}$$

The last expression can be interpreted in the following way: in the risk neutral world with the contracts being fairly priced the investor is indifferent between investing in risk-free asset and a portfolio containing a stock. The right-hand side of the equation (9) shows the average discounted payoff of the contract which equals the initial investment on the left-hand side.

The following example demonstrates the principle for conducting numerical procedures in the thesis research. For the specified process of the portfolio dynamics and chosen parameters $r$, $T$ and $A_0$, the right-hand side of the equation (9) is a known value, the value on the left-hand side can be estimated using numerical procedures.

I conduct a simulation procedure in Excel based on the feature of the Brownian Motion participating in the modeling of the stock and portfolio movement: the increments of $Z_t^{\mathbb{Q}}$ are independent and normally distributed. Equation (8) provides intuition for the simulation algorithm, which is a basic Monte-Carlo procedure. Figure 1 shows two selected simulation paths for the assets value converging to the straight line, which is the theoretical expected value of the assets. I run 10.000 to 1.000.000 simulations and the graph shows the convergence of the obtained numerical assets value to the analytical expected value.
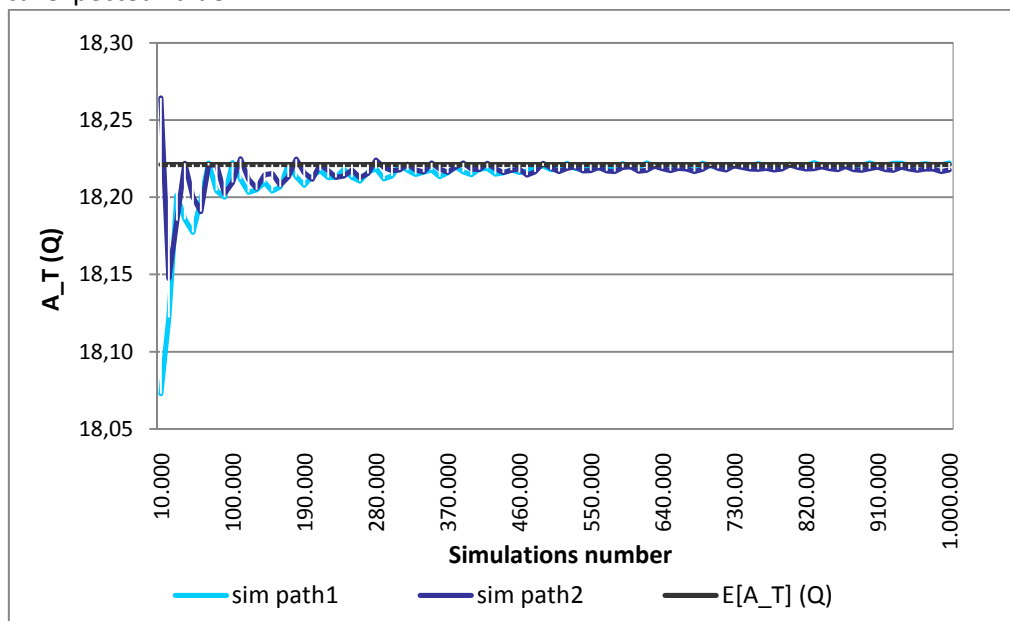
Figure 1. Assets dynamics convergence to the analytical expected value.

Parameters values: *r*=0.03, T=20, $A_0 = 10, S_0 = 1, D_0 = 1, \sigma = 0.3, \theta = 0.4$ . Expected value equals 18.22.

For 100.000 simulations the relative error becomes less than one decimal point of a percent, which is low enough for plausible results. Therefore for the research I conduct this number of simulations or more.

Now, after the general principles of pricing the contracts are set, I build the payoff structure for every contract and using the fair pricing principle derive the initial price for the contract depending on its payoff at the terminal date. I first derive the payoffs under the assumption of fixed interest rate, later on incorporating stochastic interest rate and then compare the results.

### 3.2.1. Merton's contract

The original Merton's problem stated in Merton (1969) describes an investor with a finite lifetime who is directly choosing how much to consume and how to allocate his wealth between stocks and a risk-free asset in order to maximize expected lifetime utility. His objective is to maximize

$$\max \mathrm{E}_0[\int^{t}_{\hat{}} \exp(-rt)U\big(c(t)\big)dt + B\big(W(T)\big)] \qquad (10)$$

Where $E_0$ stands for the conditional expectation at time 0, *c(t)* is a consumption function, *U(x)* is utility function, *W(T)* is investor's wealth at the end of period and *B(W(t))* is so called bequest function, which can be interpreted as an indirect utility function (Fwu-Ranq Chang (2003)). As I work in the pensions context, consumption is only relevant for the end of period and therefore the function $c(t) \equiv 0$. Bequest function in this setup is basically utility function over wealth.

The payoff of the contract can then be describes as simply
$$L_T = A_T$$
which means that the investor obtains the price of his invested portfolio at the terminal point. This case is a pure defined contribution setting, when investor is exposed to all the risks involved in his investing scheme and may gain or lose depending on the market conditions.

Naturally, all the possible contracts in Merton's case satisfy the fair restriction (6) - the full initial investment grows to the final payoff, no fee or implicit fee charged. This can also be directly seen from the result obtained in equation (9).

### 3.2.2. Implicit Put

In the implicit put contract the pension fund promises to grant the account with a minimum interest rate g for every period. Here the difference with the regular minimum guarantee arises in the multiple periods setting (in case of Implicit Put the guaranteed amount is calculated cumulatively as interest rate over the growing investments).
The payoff of this contract has a form:

$$L_T = L_0 e^{gT} + \alpha \delta_{ImplPut} \left( A_T - \frac{1}{\alpha} L_0 e^{gT} \right)^+ \qquad (11)$$

where "+" sign stands for the maximum between the value in brackets and 0; $\delta$ is the residual parameter that makes the contract satisfy the fair restriction (6).

The payoff of the contract is effectively a predefined sum plus an upside potential. From the side of the future retiree, his investments are turned into portfolio the value of which at the maturity date is the maximum between the fixed value $L_0 e^{gT}$ and the reduced value of the portfolio. This is an obvious analogy with a put option. From the point of view of the pension fund, it has written a put option with the strike $L_0 e^{gT}$ to the retiree.[9] The maturity date is the retirement date and at that time the retiree can either take the payoff of his portfolio, less the fee, if it is high enough, or "exercise" his option and get the minimum guarantee if the charged portfolio price is lower than that. This payoff construction means that in order to price this contract we can use an approach similar to that one for pricing a regular option. Of course, in reality the maturity date of this embedded option will be longer than any actually available one in the market, but in practice pension funds can use roll-over hedging strategies to replicate the payoff of such a scheme.

Parameter $\delta$ in the equation (11) can be calculated for every $\theta$ with the help of the formula

$$\delta_{ImplPut} = \frac{1 - e^{gT - rT}}{N(d_1) - e^{gT - rT} N(d_2)} \qquad (12)$$

$$d_1 = \frac{(r - g + 0.5 \sigma_A^2)T}{\sigma_A \sqrt{T}}, d_2 = d_1 - \sigma_A \sqrt{T}, \sigma_A = \theta \sigma$$

where derivation steps are done in a way similar to regular Black-Scholes option price. Appendix 4 provides more details on the calculation.

### 3.2.3. Simple insurance

In this type of contract the investor has no efficient minimum guarantee in case of company's default, which means that the initial guarantee is not completely backed by the replicating portfolio. Doskeland and Nordahl (2007) name this contract Simple Life Insurance, as it reminds the payment delivered by life insurance corporations, which in case of default bares no financial responsibility for the subscribers. However, in my framework it also has an interesting though different representation in real life. In the contest of micro pensions this type of payoff can be interpreted as, for example, an implementation where the pension fund invests money collected from the investors to finance the

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[9] The payoff of the appropriate derivative would not be completely equal, as the payoff structure of real put option would be $L_T = L_0 e^{gT} + \left( A_T - L_0 e^{gT} \right)^+$, where $L_0 e^{gT}$ is a strike, but this analogy is the most important for the pricing principle, which in fact is the same, so I adhere to this terminology.

micro credit schemes for the other members of community. In this case the pension fund and thus investor himself face the credit risk: low repayment rates, and in case of liabilities at the maturity date exceeding the assets value, the pension fund cannot provide the promised payments and defaults.

The payoff structure is then described by the following formula:

$$L_T = A_T - \left(A_T - L_0 e^{gT}\right)^+ + \alpha \delta_{ins}\left(A_T - \frac{1}{\alpha}L_0 e^{gT}\right)^+ \qquad (13)$$

where

$$\delta_{Ins} = \frac{\alpha - \alpha e^{gT-rT}N(d_2\,') - 1 + N(d_1\,')}{\alpha\left(N(d_1) - e^{gT-rT}N(d_2)\right)} \qquad (14)$$

and $d_1\,'$ and $d_2\,'$ have a form similar to $d_1$ and $d_2$ in equation (12) and the derivations are done by analogy with the previous contract (see Appendix 4).

A comparison of the payoffs for the first three described contracts is given in the Figure 2.
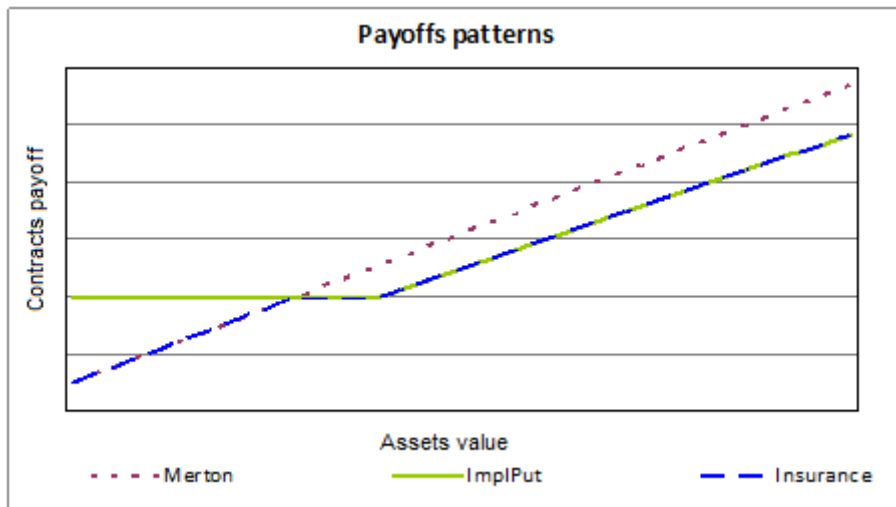


**Figure 2. Payoffs pattern comparison for three contracts: Merton's, Implicit Put and Simple Insurance**

Due to the form of the payoff the shape of implicit put and simple insurance contracts is bended at the values of $A_T$ equal to $L_0 e^{gT}$ and $\frac{1}{\alpha}L_0 e^{gT}$, $L_0 e^{gT} < \frac{1}{\alpha}L_0 e^{gT}$ as to $\alpha \in (0,1)$. The payoffs value for these two contracts differs depending on which of the three intervals the value of $A_T$ belongs, while for the Merton's problem it obviously always equals $A_T$.

   1) $A_T < L_0 e^{gT}$

      Impl. Put:  $L_T = L_0 e^{gT}$

      Insurance:  $L_T = A_T$

2) $L_0 e^{gT} < A_T < \frac{1}{\alpha} L_0 e^{gT}$

Impl. Put: $L_T = L_0 e^{gT}$

Insurance: $L_T = A_T - A_T + L_0 e^{gT} = L_0 e^{gT}$

3) $A_T > \frac{1}{\alpha} L_0 e^{gT}$

Impl. Put: $L_T = L_0 e^{gT} + \alpha \delta_{\text{Im } plPut}(A_T - \frac{1}{\alpha} L_0 e^{gT}) = L_0 e^{gT}(1 - \delta_{\text{Im } plPut}) + \alpha \delta_{\text{Im } plPut} A_T$

Insurance: $L_T = A_T - A_T + L_0 e^{gT} + \alpha \delta_{Ins}(A_T - \frac{1}{\alpha} L_0 e^{gT}) = L_0 e^{gT}(1 - \delta_{Ins}) + \alpha \delta_{Ins} A_T$

We can now see that the payoff patterns in the first interval coincide for Merton's and simple insurance contracts, for the second and third – for implicit put and insurance (in case $\delta$ is equivalent). In reality implicit put and simple insurance contracts will not provide exactly the same payoff as at certain point parameters $\delta$ which ensure the fairness of the price of the contract will differ.

Also parameters $\alpha$ and $\delta$ can now be interpreted more intuitively. The parameter $\alpha$ defines the fee that the pension fund charges the investor for providing the service (it should cover transaction costs, functioning of the organization etc.) The parameter $\delta$ arises from the fair pricing equation and for Implicit Put, as it can be seen from equations and (12), it doesn't depend on the value of $\alpha$. For the Simple Insurance, however (equation (14)), it does depend on it. This means that the slope of the payoffs on the third interval is defined by both parameters, and the magnitude of the down shift of this part of graph is defined by $\alpha$ parameter alone. This can be demonstrated by setting $\alpha=1$ (the pension fund doesn't charge any fee for the service). In this case the second interval vanishes as

$$L_0 e^{gT} = \frac{1}{\alpha} L_0 e^{gT}$$

and the slope of the payoff at the third interval is defined exclusively by value of $\delta$. For different values of $\theta$ – asset allocation proportion I obtain different values of $\delta$ for each contract, ranging from 1 to 0.8 for $\theta$ restricted to the interval *(0;1)*. A shape of the contracts payoff for selected $\delta$ is shown on Figure 3.

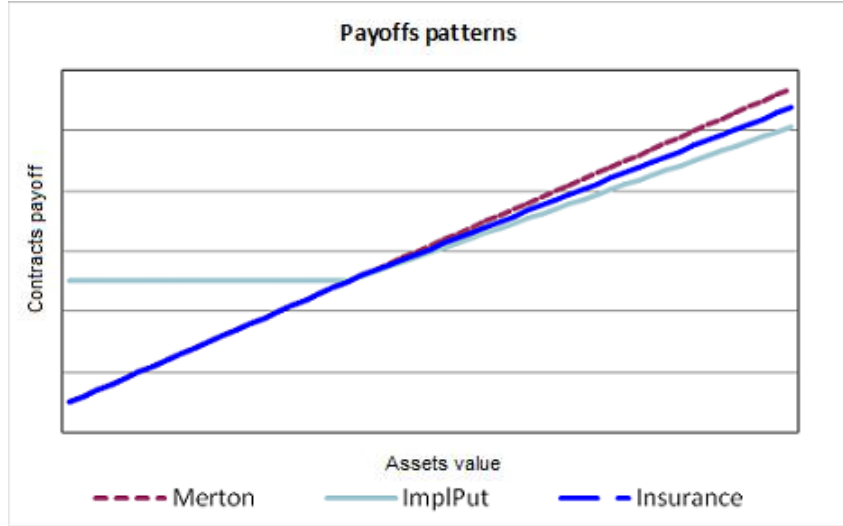**Payoffs patterns**

Contracts payoff

Assets value

- - - - Merton ——— ImplPut —— - Insurance

**Figure 3. Payoffs pattern comparison for three contracts: Merton's, Implicit Put and Simple Insurance in case of α=1 (no fee charged by pension fund).**
Parameters: Implicit Put contract - δ=0.8, Simple Insurance - δ=0.9

Now there are only two intervals where the form of the payoffs differs and the first one corresponds with that described before. In the second interval, the payoffs for Implicit Put contract and simple Insurance are respectively:

Impl. Put: $L_T = L_0 e^{gT} + \delta_{\mathrm{Im}\, plPut}(A_T - L_0 e^{gT}) = \delta_{\mathrm{Im}\, plPut} A_T + L_0 e^{gT}(1 - \delta_{\mathrm{Im}\, plPut})$

Insurance: $L_T = A_T - A_T + L_0 e^{gT} + \delta_{Ins}(A_T - L_0 e^{gT}) = \delta_{Ins} A_T + L_0 e^{gT}(1 - \delta_{Ins})$

Clearly, as on the third interval the guaranteed value $L_0 e^{gT}$ is lower than the value of portfolio $A_T$, the line of the payoffs is laying below the portfolio value, which can be directly seen from the expressions for both contract payoffs. Because $\delta$ is the proportion of the portfolio value, there is no parallel shift but a change in a slope.

### 3.2.4. Implicit Bull Spread

The payoff of the fourth type of contract – Implicit bull spread, has a conceptual difference with the first three, namely that it limits investor's gains in order to make the initial buy off cheaper. In this manner it trades the upside potential of the initial investment for the lower price paid by the customer.

The contract has the following form of the payoff:

$$L_T = L_0 e^{g_1 T} + \alpha\delta_{bull}\left(A_T - \frac{1}{\alpha}L_0 e^{g_1 T}\right)^+ - \alpha\delta_{bull}\left(A_T - \frac{1}{\alpha}L_0 e^{g_2 T}\right)^+ \tag{15}$$
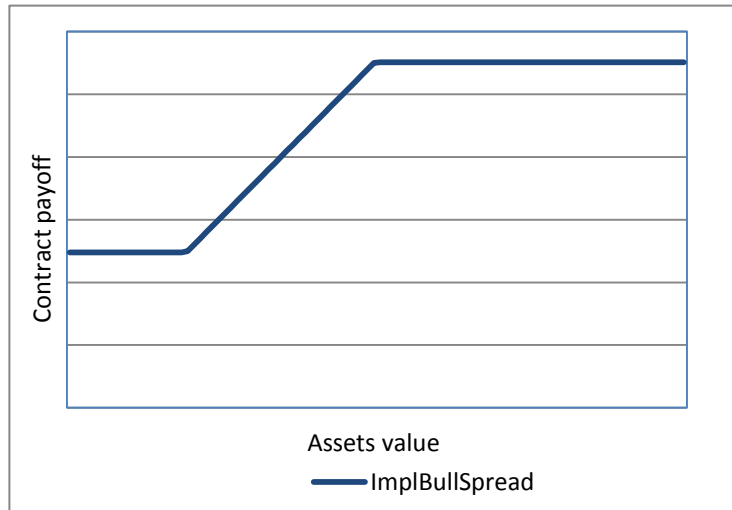
where $g_1 < r < g_2$

$$\delta_{Bull} = \frac{1 - e^{g_1 T - rT}}{N(d_1) - e^{g_1 T - rT} N(d_2) - N(d_3) - e^{g_2 T - rT} N(d_4)} \tag{16}$$

and expression for the parameter δ is derived formally in Appendix 5.

The shape of the payoff has a following form.



**Figure 4.** Payoffs pattern for Implicit Bull Spread contract

It replicates the payoff of the so called bull spread trading strategy created by selling to a customer a put option (a minimum guarantee part of the contract) with a strike $A_0 e^{g_1 T}$ and buying from him another put option with a higher strike $A_0 e^{g_2 T}$ .

Within derivatives analogy framework the restriction $g_1 < r < g_2$ effectively means that the put option with a lower strike is out-of-the-money and the one with higher strike is in-the-money. While implementing the regular put bull strategy the trader bets on the fact that the portfolio value will go up high enough, so that the written put option matures without being executed. The same principle holds for the pension fund, "writing" an implicit put for the future retiree – the outcome when the minimum guarantee is executed is highly undesirable and essentially means that the investor paid part of the initial capital as a fee and didn't get any return.

In this case maximum loss that the retiree is exposed to is limited to the difference between two strike prices minus the premium for selling the put, and maximum gain is limited to the difference between the premiums for the long and short put options.

There are more combinations of the (implicit) derivatives replicating the type of the contract with limited both downside and upside potentials. Due to the nature of the parties involved a bull spread with two put options is the most obvious candidate for the contract replication. Generally, a spread is a type of investment strategy which involves taking a position in two or more options of the

same type (Hull (2006), p.225). I choose this particular type of replicating portfolio based on the consideration that it makes the comparison with the Implicit put contract more straightforward. Moreover, limitation of the upside potential contributes to the main characteristic feature of the payoff and the arbitrage free pricing restriction makes different types of contracts replicating spread payoffs similar in their nature, so picking up just one is enough to capture main investment principle.

### 3.3. Contracts payoffs: stochastic interest rate

In equations (1) and (5) the interest rate $r$ is constant, which allows for the suitable representation of the residual parameters $\delta$. Deterministic interest rates is a quite wide-spread assumption in the literature (see e.g. Hardy (2003) or Brennan and Schwartz (1976)). However, as the investment horizon of the pension fund is very long, one of the most influential risks for the terminal value of the invested money is interest rate risk. The second biggest risk the retiree faces when taking care of his financial well-being after a very long period of time is inflation. Purchasing real estate or life stock might seem for the poor people a more safe investment when evaluating its real value after the years passing. This is also one of the general reasons people decide against long-term savings or purchasing the annuity in the pension scheme. However, as, for example, an investigation by MicroSave in Uganda has shown (see Wright et al (2001)), the inflation-caused losses in real value of the money can still be less severe than losses due to other saving vehicles such as livestock, home-saving or lending to neighbor or relative. These considerations, of course, do not account for the case of hyperinflation, but the assumption of a more or less stable economic environment is necessary for the model to function. Besides that, when talking about annuitization, pure inflation-linked annuities do not exist in the current financial market, even in the developed counties.

Let's come back to the place of interest rate in the model. In the classical defined contribution scheme, the final outcome for the investor is solely dependent on the investment scheme he chooses and the pension fund itself doesn't bare any responsibility for the risks involved. The pension fund plays only a role of the intermediate implementation vehicle, providing information and realization due to the chosen scheme. In this case fluctuations in the interest rate have the only consequence of lower terminal payment for the investor. Still, investing money solely in a risk-free bank account will provide a small but certainly positive growth of wealth, so the asset stays in principle risk-free.

However, for the retiree in case of defined contribution plan, the changes in the interest rate entail so called conversion risk – the uncertainty in the change rate when converting the accumulated capital into pension. This fact is particularly important for the case of annuity in the second part of the research.

If on the other hand, the financial institution strives to provide some sort of a guarantee or an annuity of the predefined size at the retirement date (so including an element of defined benefit scheme), the importance of the interest rate becomes very obvious. The reason is that the part of client's money invested in the less risky financial instruments (e.g. bonds) will grow depending on the

value of interest rate during every year before retirement. That means that rapid changes in interest rates will cause a large magnitude of the terminal size of investments at the retirement date. The price the pension fund as an intermediate guarantor has to charge for providing the annuity or any minimum guarantee depends directly on the terminal value of the investment portfolio. For this kind of schemes it is basically crucial to take the fluctuations in the interest rate into account.

As in the research I compare the elements of both schemes and investigate the annuity option, I include the process for the interest rate in the scope of the model.

Two conceptually different ways to model the term structure of interest rates exist in the literature (see e.g. Pelsser (2000)). The first one (Cox et al (1985), Vasicek (1977)) uses instantaneous interest rate as a basis for modeling the term structure of interest rates, and the other one (Chan et al. (1992)) relies on real market interest rates as a source of sufficient information. The instantaneous interest rate, used in the former type of models, is a theoretical value, a mathematical concept of a rate earned over infinitesimal period of time and is consequently a non-existing and non-tradable asset. This fact entails a complicated valuation of the real-world instruments and often requires numerical techniques for the estimation. The big advantage of this type of models, however, is the tractability and a simpler setup compared to the market rate models. The advantage of the second type of modeling is that by construction obtained interest rates are created to fit those existing in the market, which makes the model perfectly realistic. This type of models is also particularly useful for valuating exotic interest rate derivatives.

As in the research I stick to a simple setting with a bond as the only basic interest rate asset and do not have an intention to fit the data for any particular country, the advantage of the second type of methodology is eliminated. I therefore choose to use the model belonging to the first category, in particular single factor Vasicek model.

In this model (see Vasicek (1977)) the instantaneous spot rate $r_t$ moves according to the following stochastic differential equation:

$$dr_t = (\alpha - \beta r_t)dt + \sigma_r dW_t^Q \qquad (17)$$

where α and β are fixed model parameters in practice are usually estimated by calibrating the model to the real short rate data, $\sigma_r$ is the volatility of the interest rate and $W_t^Q$ is a Brownian Motion under risk neutral measure $\mathbb{Q}$ .

The Vasicek model incorporates mean reversion, which is an important feature of the interest rates. In difference from, for example, a stock behavior, the interest rate cannot rise indefinitely. This is caused by economic reasons: high interest rates force the economy to slow down as fewer borrowers can afford to pay the interest and fewer investments are made. The consequential

government policy is then to lower the interest rates and in this way to stimulate the economy, and vice versa.

All the previously obtained expressions for the fair valuation of the introduced contracts are done under the assumption of fixed interest rate. Let's see what changes will entail the introduction of the stochastic interest rate.

In case of stochastic interest rate the fair pricing equation (6) takes the following form:

$$L_0 = \alpha A_0 = E^{\mathbb{Q}}[e^{-\int_0^T r_s ds} L_T] \tag{18}$$

The problem that arises immediately is that now the discount rate cannot be taken out of the expectation anymore as in the case of a constant interest rate. The reason is that as the payoff $L_T$ depends on terminal assets value $A_T$, which itself depends on the interest rate value after every period in time. Therefore both terms under parentheses are not independent even if correlation between Brownian Motion terms $W_t^{\mathbb{Q}}$ and $Z_t^{\mathbb{Q}}$ is set to 0.

Recall that solving equation

(18) is necessary in order to obtain the residual parameter $\delta$ for every contract except Merton's. Without knowing it I cannot calculate the contract's payoff and thus cannot evaluate the amount of money the person gets at the retirement date.

The dynamics of the interest rate in equation (17) is described using a different numeraire than that used for the stock and portfolio processes. The choice of the numeraire is based on the mathematical representation of no arbitrage pricing principles for the financial instruments. Appendix 4 provides a basic mathematical setting of the change of measure principle.

In order to calculate the expectation in (18) and price the contracts, I need to change the numeraire to the one which will allows splitting both terms under the expectation. Such a numaraire is a forward neutral measure **T** (see Bjork (1998), p.275).

I now need to change the measure of all assets and interest rate dynamics equations in order to use simulation procedures. The final goal is to obtain the final payoff of the four investigated contracts. Under this new measure the fair pricing equation will take form

$$L_0 = e^{-\int_0^T r_s ds} E^T[L_T] \tag{19}$$

where the payoff $L_T$ for each contract type is calculated over dynamics of the assets $A_T$ under measure **T**.

Also the Vasicek equation for the dynamics of the short rate will change and under measure **T** will take the form:

$$dr_t = (\alpha - \beta r_t + \sigma_r k_t)dt + \sigma_r dW_t^{\mathbf{T}}$$
(20)

where
$$k_t = -\sigma_r B(t,T)$$

$$B(t,T) = \frac{1}{\beta(1 - e^{-\beta(T-t)})}$$

I can then obtain the value of residual parameter δ in the way similar to the case with the constant interest rate. However, changing measure from $\mathbb{Q}$ to **T** implies more complicated mathematics than changing measure from **P** to $\mathbb{Q}$ and includes solving of the expression with double integral.

The alternative approach is to compute δ numerically. The computational principle becomes evident after coming back to the expressions for the parameter δ for every contract when the interest rate is fixed (see Appendix 5).

$$\delta_{ImplPut} = \frac{\alpha A_0 (e^{rT} - e^{gT})}{\alpha E\left[ \left( A_T - A_0 \exp(gT) \right)^+ \right]}$$

$$\delta_{ins} = \frac{\alpha - 1 + E\left[ \left( A_T - \alpha e^{gT} \right)^+ \right]}{\alpha E\left[ \left( A_T - A_0 e^{gT} \right)^+ \right]}$$

$$\delta_{bull} = \frac{A_0 \left( e^{rT} - e^{gT} \right)}{E\left[ \left( A_T - A_0 e^{g_1 T} \right)^+ - \left( A_T - A_0 e^{g_2 T} \right)^+ \right]}$$

Introduction of the stochastic interest rate entails replacing the discount factor $e^{rT}$ in the nominator of Implicit put and Bull spread contracts with the discount factor from the equation (19): $\exp(\int_0^T r_s ds)$ .

All the expressions for the expectation of the assets payoff can be obtained numerically, as the assets dynamics moves according formula (18):

$$A_T = A_0 \exp\left( (r_t + (\mu - r_t)\theta - \frac{\sigma^2 \theta^2}{2})T + \sigma \theta \sqrt{T} z \right)$$

Now interest rate $r_t$ moves according to the stochastic process itself. This means that I can do "simulation within simulation", while running a Monte-Carlo procedure for the interest rate process and another one for the portfolio dynamics dependent on the obtained value of interest rate on every step $dt$. The values of portfolio are averaged on every step and a terminal one is obtained for every interest rate dynamics path. These values are averaged at the end again, for all interest rate dynamics paths. This method has a disadvantage as it requires a lot of computational time, but for

100.000 simulations used in my thesis this tradeoff seems reasonable enough, as the time delay is not dramatic.

## 4.Comparing the contracts: investor perspective

In this part of the report I evaluate the benefit obtained by the investor from payoffs of the contracts described in the previous chapter and choose the optimal one. Recall that the pension contracts represent such characteristic features of the investment contract as cutting downside potential of the portfolio performance (Implicit Put), trading upside potential for the lower price(Implicit Bull Spread) and direct investing (Merton's contract). In order to compare them I use two different approaches: Expected Utility framework and Cumulative Prospect Theory.

### 4.1. Expected utility framework

Expected Utility theory is based on the expected utility hypothesis, which describes the human behavior in the situation when a person has to make a choice regarding few perspectives with uncertain outcome. It was developed by Neumann and Morgenstern (1944) and until recently it was one of the most widespread concepts in modeling peoples' choices. Expected Utility theory states that people make a choice based on the value of the outcome (in terms of money or anything else), the probability of its occurrence and the willingness to take higher risk in order to receive a possible higher benefit in the future (personal risk aversion).

In terms of investment decisions, the goal of the investor is to maximize the function

$$\max U = \max \mathrm{E}\big[u(L_T)\big] \qquad (21)$$

where $U\big(L_T\big)$ is a utility function of the payoff $L_T$.

In the literature there exists a variety of specific forms of utility functions. This form is defined in such a manner that it incorporates realistic aspects of human behavior. For example, it has to have a concave shape, which reflects that every extra unit of the desirable asset brings the person less and less satisfaction. It terms of money, it is definitely more preferable for the investor to obtain 1.001.000 Euro above 1.000.000 Euro, but the initial sum is yet so large that the extra thousand does not make a big difference. At the same time, comparison of extra benefit of 2.000 Euro above 1.000 will result in a significant difference in the obtained satisfaction, although the absolute value of the difference is equal to the first case. In the framework of this research, it also means that people with almost no belongings are happier with the small income than wealthy people, as their initial position on the utility graph is located closer to the left, where marginal utility is higher.

One of the most widespread forms of utility functions is so called constant relative risk aversion (CRRA) function. The particular case of it used in the macroeconomic consumption theory has a following form:

$$U(x) = \frac{1}{1-\gamma} x^{1-\gamma} \qquad (22)$$

where γ is Arrow-Pratt coefficient of relative risk aversion

$$\gamma = -\frac{xU''(x)}{U'(x)} = \frac{-x\gamma x^{-\gamma-1}}{x^{-\gamma}}$$

and x is the outcome which has a certain probability to occur. The above feature gives the function its name: relative risk aversion is constant and can be defined by the choice of the parameter γ.

Risk aversion in general reflects the person's attitude to the risk. A simple concavity of the utility function, although quite intuitive, cannot be taken as an analogous measure. The reason is that different utility functions can have equal second derivatives, and thus this indicator cannot ensure adequate comparison of the represented risk aversion.

As a sufficient indicator the literature uses the Arrow-Pratt coefficient of relative risk aversion

$$R_r = -\frac{xU''(x)}{U'(x)}$$

It is a coefficient of absolute risk aversion multiplied by the level of obtained wealth x (see e.g. Pratt (1964)). The absolute risk aversion is calculated with the help of the formula

$$R_A = -\frac{U''(x)}{U'(x)}$$

As the formula shows, the meaning of an absolute risk aversion is a sensitivity of the curvature of the utility function to the changes in the first derivative, which represents the steepness. Multiplying it with the wealth x gives the possibility to observe the changes in the sensitivity depending on the amount of money obtained. The term 1/γ is interpreted as intertemporal substitution elasticity between consumption of two consecutive periods, or a willingness of the person to set aside a unit of benefit now in order to consume it in the future. This coefficient is an important indicator of the person's vision of the future state of the world: the worse it is the higher the willingness to ensure against this unfavorable situation by keeping some assets for that time (see Cochrane (2005)).

This form of utility function is also often used to study savings behavior and is therefore suitable for the pension framework. The reason for that is that precautionary saving in response to risk is associated with convexity of the marginal utility function, or a positive third derivative of the function $U(x)$ (see Kimball (1990)). Coming back to the specified CRRA utility function in (22), it has a property that

$$U'''(x) = (\frac{1}{1-\gamma}x^{1-\gamma})''' = (x^{-\gamma})'' = (-\gamma x^{-\gamma-1})' = \gamma(\gamma+1)x^{-\gamma-2} > 0$$

The nature of precautionary savings is very similar to the motive of ensuring the old age provision, as a pension effectively represents the savings net against the risk to stay without any income in the future. This consideration supports the choice of CRRA utility function for my research.

### 4.1.1. Expected Utility: fixed interest rate

I calculate the expected utility over the final wealth at the retirement date for the four contract types described in the previous chapter. The procedure is the following: I first simulate the portfolio dynamics and obtain its value at the terminal date $A_T$ ; then compute the payoff for every contract according to its structure depending on the values of portfolio; based on the payoff I calculate corresponding utility that the investor gains, and finally average the obtained values of utility across the amount of simulations for every contract type. This procedure is repeated for different proportions of wealth invested in risky asset with a step 0.01. Appendix 5 shows the graph demonstrating utility delivered by every contract described in the previous chapter for the conventional values of the parameters, based on Doskeland and Nordahl (2007). Out of the four investigated contracts with the unlimited upside potential the highest utility is provided by Merton's contract. The optimal asset allocation is reached at 37% in equity, which equals to the theoretical value $\theta = \dfrac{\mu - r}{\gamma \sigma^2}$ .

For the micro pensions case the values of the parameters change, as financial market characteristics such as stock volatility and return in the developing countries are substantially different from the developed countries. Figure 5 displays the result with the new set of the parameters.
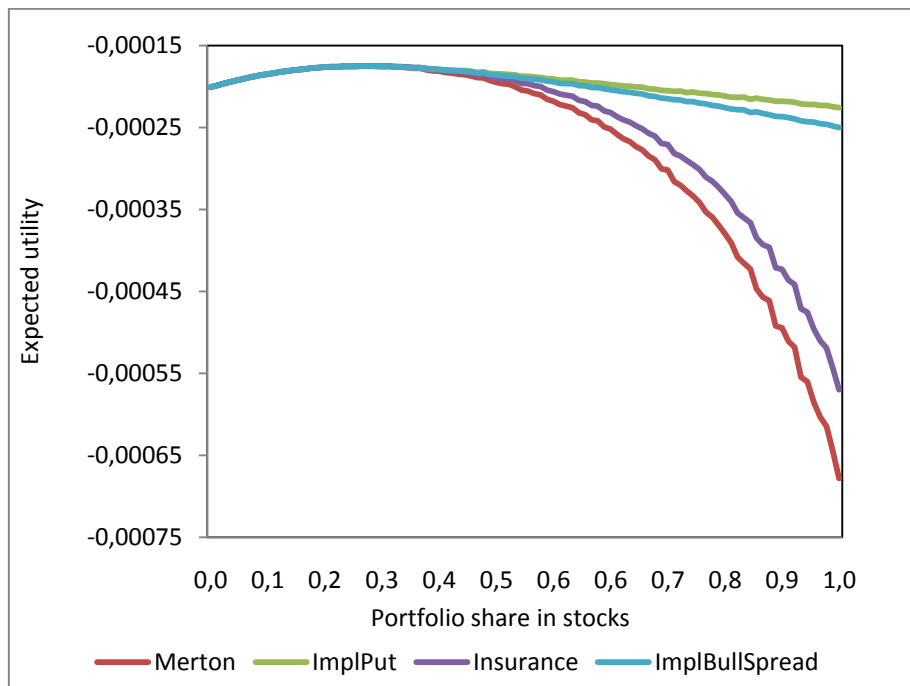
**Figure 5.** Expected utility under power utility function for four contracts: Merton's, Implicit put, Simple insurance and Implicit bull spread for positive θ. **P**arameter values: Initial wealth $A_0 = 5$, interest rate r=12%,expected stock return μ=0.15, volatility of the stock σ=0.3, time until retirement T=20 years, risk aversion γ=3, α=0.9, Implicit Put minimum guarantee rate g=10%, Implicit Bull Spread minimum guarantee rate $g_1 = 10\%$, maximum rate $g_2 = 14\%$

I choose the parameters values based on the existing literature and the underlying assumption that the framework of the model reflects the conditions of the developing countries. As a rule, interest rates in the developing world are much higher than in more stable countries, reflecting the higher relative risk of such an investment. I set the interest rate *r* to 12%, as this is the current rate of return for Grameen Pension scheme (see Appendix 1). This interest rate is actually higher than that of the commercial banks in Bangladesh, which is possible because the financial organizations aimed for development are often aloud to have tax benefits as compared to the commercial banks. Another important indicator of risk is, naturally, the stock volatility, which is higher compared to the developed countries as well. I refer to the states, where the financial market is developed enough to be compared to the developed countries, as, for example, the case of India. This is a necessary assumption as investing in the stock is the option in the initial portfolio formation.

The expected stock return and volatility of the stock are based on findings of Batra (2004), who examined the time variation in volatility in the Indian stock market. I set these parameters' values to 15% and 0.3 respectively.

I compare the results with those obtained by in Doskeland and Nordahl (2007), because the authors use "conventional" values of the parameters (see Appendix 6 for the analogue to the Figure 5. We can interpret these values as representing the situation in the financial market of the developed countries, such as lower expected return and volatility. Another difference is that although the topic of the article is pension scheme design, the authors choose the time horizon 5 years, which is quite short. I set it to 20 years based on the consideration that pension provision is a long term project and on the analysis of the existing micro pension oriented schemes (see Appendix 1).

The main difference that the change of the parameter values entails is that for the small portfolio shares invested in risky asset the difference of the utility delivered by all contract types eliminates substantially. They all perform very similar until 50% of the initial wealth is invested in risky asset. Further the decline in the utility of the contracts that include minimum guarantee (Implicit Put and Bull Spread) is slower than for the other two.

The newly introduced contract, Implicit bull spread, performs relatively poor for the conventional values of the parameters (Appendix 6). Recall that this contract combines the guarantee with the trade off of the upside potential. Implicit bull spread contract's utility declines rapidly in this setting after the maximum and makes the contract unattractive for the investor. For the case of developing markets parameters, looking at all values of portfolio share in risky assets, it performs

slightly worse than Implicit put, but significantly better than the two contracts that imply more risk: Merton's and Simple Insurance. I conclude that for the higher risk and returns characterizing stock behavior in the financial market, which is the case of emerging markets[10], the value of the contracts providing minimum guarantee is higher than in the framework of developed countries. It can therefore be considered as a setting for the pension scheme.

Parameter $\sigma$, which is stock volatility and represents the risk in the model, has a biggest influence on the shape of the graph. With the volatility going up, the value delivered by the Implicit Put and Bull Spread contracts goes up when larger part of the initial investments is put into stocks. The utility delivered by the contracts when a smaller proportion is invested into stocks, becomes closer for all contracts, however, after zooming in, we can see that Merton's contract will always perform better at its maximum, just the difference becomes small and visually unobservable on the current graph.

Returning from theory to practice, the real possibility to implement Merton's contract for the poor people is questionable, at least directly in this form. The biggest advantage of this contract is the opportunity for the investor to pick up the optimal asset allocation given his risk appetite, stock return and volatility and risk-free rate. It is hard to imagine, however, that microfinance institution on the first place will have a capacity to arrange and propose the variety of investment opportunities for their clients, who will most of the time have investments of a very small size. Besides that, poor people are usually not able to make  justified judgments about how much money they want to put, say, in bonds and how much – in stocks, simply because even more wealthy people with a higher educational level are usually lacking a deep  knowledge in this area.

The existing financial product that is designed the closest to the described investments scheme is UTI Mutual fund micro pension scheme in India, where 40% of the assets are allocated to stocks. The question of optimality of this asset allocation for a big extent depends among others on the risk aversion coefficient.

Including this contract in the comparison is quite a theoretical exercise, as a complete implementation of it in reality is not feasible. However, some elements of it like more freedom in investment choice could possibly be used and therefore it's useful to know its place in the overall picture.

### 4.1.2.   Expected Utility: stochastic interest rate

In section 3.4 I described the numerical algorithm for obtaining the values of residual parameter $\delta$, which insures fair pricing of the contracts in case of stochastic interest rate. The simulation proves that the new values of δ will be smaller than those obtained using stochastic interest rate. Recall that δ participates in the payoff function and the higher its value is, the bigger share of the terminal

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[10] Not all developing countries can be considered as emerging markets, but India, for example, belongs to this category.

portfolio the customer gets in case the assets value is high enough. The economic interpretation is that a smaller $\delta$ indicates a higher implicit price that the market charges the investor for providing any sort of guarantee, which is an element of Implicit Put and Implicit Bull Spread contracts. This higher price is caused by additional risk, which the introduction of stochastic interest rate effectively means. However, as the value of parameter $\delta$ is obtained in a "fair" way, the actual risk-reward balance will still be in place, and therefore utility calculation incorporating only this change will not differ much from the case of fixed interest rate.

On the other hand, the payoff itself is also influenced by fluctuations in the interest rate – it gets more volatile as another source of uncertainty is added to the model. This fact is the underlying reason of the conversion risk. This risk is a reflection of the hazard that the payoff of the initial investment will appear not sufficient for the conversion into old-age financial provision due to a small interest rate and thus return.

For the assets dynamics simulation procedure, incorporation of the stochastic interest rate entails running "simulation within simulation", also described in section 3.4. The result of stochastic interest rate participating in assets dynamics is that the distribution of the terminal portfolio values will have a larger variance. Caused by the extremely low values, the benefit from minimum guarantee contracts will go up, but the use of extremely high values is only possible with Merton's and Simple Insurance contracts, thus average relation of contracts remains the same, with the values of the utility going up. The latter fact doesn't have an important meaning, as the utility theory is ordinal, which means that only the relation between the values matter, and not the numbers themselves.

### 4.1.3. Discussion of the minimum guarantee

The interest rate guarantee can be interpreted as a constraint on the investment portfolio of the retiree when using the CRRA utility function for evaluation Jensen and Sorensen (2001). Doskeland and Nordahl (2007) expand the research made by Jensen and Sorensen and obtain similar results. They introduce the structure of annual guarantees versus terminal interest rate guarantee. In this structure the pension fund has to meet its obligations towards the promised guarantee or default at the end of every period. In case of bankruptcy the money extracted from the company (equity of the company) is invested in the risk-free account until the end of period.

For a negative market scenario, when the market performs poorly during the period until maturity and the company managed to meet its obligations and not default – the investor receives the sum equivalent to $A_0 e^{gT}$ at the end of period, where $g$ is the guaranteed rate of return. This is the sum that would be a one-time terminal guarantee. The difference between the annual guarantee and the terminal one is that in case of the former the company has to "report" their balance sheet state at the end of every sub period. In case it doesn't meet the requirements, the contract is finished and the rest of the money is invested in the risk-free rate, which can obviously be not the best strategy for the investor.

In case the market performs well, the part of the return goes to the shareholders (return on equity) and to the bonus reserve, which is set aside for the future guarantees. This reduces the possible return and a potential gain that the investor could have obtained at the end of period. Therefore, in some cases the expected utility of this strategy is lower than that delivered by Implicit Put, which represents a terminal guarantee contract.

Therefore, the main conclusion regarding any type of guarantees in the CRRA utility framework is that it is basically a kind of constraint on the investor's choice of portfolio construction. Sometimes this constraint limits the opportunities of riskier but more profitable investments for the retiree, while it could still be optimal for him given the level of his risk aversion and the situation in the market. Therefore for the conventional choice of parameters the idea of minimum guarantee doesn't appear to be any useful.

This conclusion doesn't seem far-fetched given many recent discussions about the additional value of the guaranteed investment products, especially in the latest volatile market conditions. The objections include the argument about overpriced products caused by the high implied fees. The general advice to investors is that the guarantee as such can only be beneficial in a very limited number of cases – for example when the investor is close to retirement and his risk aversion increases because of the short investment horizon or when he has a clear goal and a time frame to get there (e.g. buying a house).

However, the problem with the CRRA is exactly that it is unable to explain the benefit for the investor even in case the coefficient of the relative risk aversion γ is high enough. Coming back to Figure 5, while testing the contract payoff with different values of γ, the result is that Merton's contract outperforms or is not worse than all other types of contracts implying minimum guarantee no matter what the value of γ actually is. This means that in the expected utility framework with the investor whose decisions are based on CRRA utility function guarantee is actually *never* beneficial in a strictly theoretically defined way.

This naturally leads to the proposition that there exists an alternative approach (at the end, guaranteed interest rate products exist in the market, are provided by many large financial institutions and obviously meet a significant demand, which cannot be simply explained by the means of a marketing).

## 4.2. Cumulative Prospect Theory

The evidence reveals that, when people evaluate risk, they often depart from the predictions of expected utility. One of those is evaluation of the outcomes not in absolute, but in relative terms. Another example is that some people at the same time participate in the lottery (which is an example of gambling, or risk seeking) and buy insurance, in this way demonstrating risk-averse behavior. This may have important implications for an investor's reaction for good versus bad earnings or analyst recommendations (Han and Hsu, 2004).

The approach consistent with Cumulative Prospect Theory (CPT) developed by Tversky and Kahneman  (1996) takes into account some of these violations. Another important aspect of the theory is that real world probabilities of events are weighted in the mind of the person by an S-shaped probability weighting function *w*. This assumption is based on the observation that most people tend to overweight small probabilities of the outcomes and underweight large ones. The core difference between Cumulative Prospect Theory and expected utility is that expected utility framework supposes that people base their choices exclusively on the rational considerations, while CPT assumes that it is not always the case and tries to incorporate some generally observed violations.

In the financial framework, there are few conceptual differences between CPT and expected utility theory, as stated for example in Bernard and Ghossoub (2009) or Rieger and Wang (2008).

- The CPT investor is concerned with the deviation of his wealth from the reference point, whereas the expected utility-maximizing investor is interested only in the final value of his wealth. The reference point is in this case a capital of a certain size that in the mind of investor serves as a benchmark for comparison, a "zero level" of wealth.

- The CPT investor reacts differently towards gains and losses. This is reflected in the introducing of the value function replacing the standard utility function. If all the possible outcomes (payoffs) are ranked from the worst to the best, so from the lowest to the highest: $L_{T,-m} < ... < \Gamma < ... < L_{T,n}$, the form of the value function is the following (see Doskeland and Nordahl (2007)):

$$V(L, p) = V^+(L, p) + V^-(L, p) \tag{23}$$

$$V^+(L, p) = w(p_n)U\left(L_{T,n}\right) + \sum_{k=1}^{n}\left[ w\left(\sum_{j=0}^{k} p_{n-j}\right) - w\left(\sum_{j=0}^{k-1} p_{n-j}\right)\right]U\left(L_{T,n-k}\right) \tag{24}$$

$$V^-(L, p) = w(p_{-m})U\left(L_{T,-m}\right) + \sum_{k=1}^{n}\left[ w\left(\sum_{j=0}^{k} p_{-(m-j)}\right) - w\left(\sum_{j=0}^{k-1} p_{-(m-j)}\right)\right]U\left(L_{T,-(m-k)}\right) \tag{25}$$

Where U is the utility function of the form

$$U(L_T) = \begin{cases} U_G(L_T) = \left(L_T - \Gamma\right)^{\phi}, L_T \geq \Gamma \\ U_L(L_T) = -\lambda\left(\Gamma - L_T\right)^{\phi}, L_T \leq \Gamma \end{cases} \tag{26}$$

*w* is a weighting function, $\Gamma$ is a reference point, related to which the investor evaluates his expectations, *n* – the number of possible positive (relatively to the reference point) outcomes, *m* – number of negative outcomes, $p_i$ -probability of the corresponding outcome $L_{T,i}$. The total number of the possible outcomes is *m+n+1*, where last one is the reference point itself (if reached – meaning that one of the possible payoffs equals the reference point).

Originally Cumulative Prospect Theory was applied in case of finite lotteries for discrete number of outcomes. Of course in case of contracts payoff the outcomes have a continuous distribution, which means that analytical formulas would deviate from those originally proposed by Tversky and Kahneman (1992).

It is possible, however, to use simulation results as a discrete set of outcomes and interpret them similar to "lottery outcomes". After ranking these outcomes from the worst to the best, I calculate the necessary cumulative probabilities and apply formulas (24)–(26). I run 10.000 simulations and the non-cumulative probability of each outcome is then simply 1/10.000.

Equations (24) and (25) reflect the *cumulative* nature of CPT: the weighting function *w* is applied to the cumulative probability of the outcome. Let's look closer at the case of $V_+$ - the value function for gains. If the weights would apply to the regular probability (probability density function), the weighted probability, which is the expression in the square brackets, would simply equal $w\big(p(L_{T,n-k})\big)$. Instead, this cumulative probability is calculated by taking the sum of probabilities of the outcomes equal or better than $L_{T,n-k}$ (the term $\sum_{j=0}^{k} p_{n-j}$ ) the sum of the probabilities of the outcomes strictly better than $L_{T,n-k}$ (the term $\sum_{j=0}^{k-1} p_{n-j}$ ), applying weighting function *w* to each sum and finally taking the difference. The extreme cases $L_{T,n}$ and $L_{T,-m}$ require the usual application of the weighting function (there cannot be any case "strictly better" than the best outcome) and that is the first term in expressions (24)-(25). Similar refers to the loss domain of the function.

The form of the utility function in the equation (26) resembles that one for the utility in Expected Utility theory. The difference is that the expression in the power ϕ is taken over the difference of the outcome towards the reference point. In the losses domain the utility is negative, which is reflected by multiplying the term $\left(\Gamma - L_T\right)$ with –λ.

Figure 7 demonstrates the shape of the new utility function described by formula (26) with different parameter values.
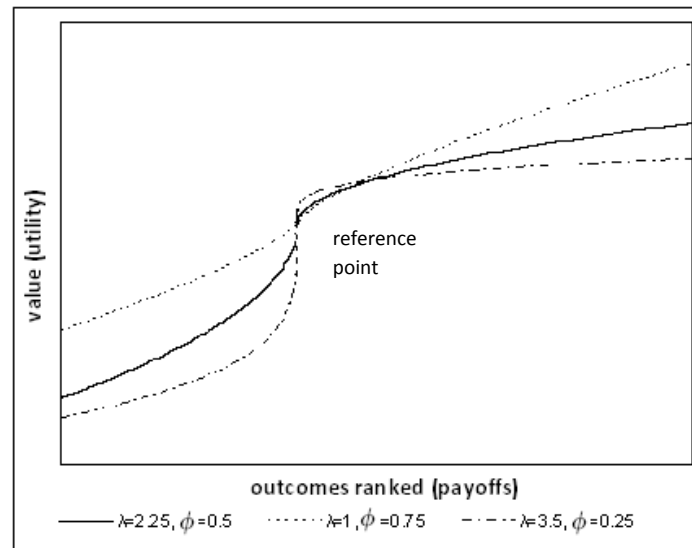
**Figure 7.** Shape of utility function within Cumulative Prospect Theory (described by equation **(26)**). Example built on the portfolio dynamics with the reference point equal to the initial investments.

An inflection point is a reference point and a concave part in the gain domain (a part above reference point) and a convex part in the loss domain (for the payoff less than reference point) capture the risk-averse tendency for gains and risk-seeking tendency for losses. Parameter $\lambda$ defines the additional steepness of the loss domain which reflects the loss aversion feature.

The shape of the upper part of the value function corresponds with the shape of the utility function – both are concave, although with different steepness, as the parameters in the power are not the same. Recall, that the risk attitude in EUT is defined by the mutual location of the span connecting two points and the actual expected utility of getting the money straight away. Risk aversion implies that the utility over the real amount is always higher (above the span) than the expected utility of the gamble. The person prefers to have the real money above playing the lottery. According to CPT, with the losses the situation is exactly the opposite – the people prefer to play a lottery where they have a slight chance to lose less than to have a *definite* loss.  That is why in the loss domain the shape of the value function is mirrored one compared to gains.

Expected utility theory, because of the curvature of the utility function, also implies that the "losses loom larger than gains". When being on the fixed point on the curve, moving down decreases utility faster than moving up adds it. However, this quality also causes a problem, which led to a lot of critics towards the EUT (see e.g. Rabin and Thaler (2001)). With increasing amount of wealth on the bet, the degree of risk aversion may rise unreasonably, because the difference between the corresponding point on the curve and the utility of the gamble increases rapidly. Instead, CPT makes and accent on the loss aversion instead of the risk aversion and helps to eliminate this drawback.

▪ People do not value random outcomes using usual physical probabilities, but apply to them a weighting function of a specified form. This function has typically a narrow S-shape, so that small probabilities are overweighed and large probabilities are underweighted.

The S-shape of the weighting function is based on the so-called "common ratio effect". This effect refers to the observation that more risky of two prospects becomes relatively more attractive when the probability of winning is reduced by equal proportion in both prospects. A person who prefers a sure gain of 100.000 euro over a coin toss for 300.000 euro or nothing, might also prefer a one-in-a-million lottery ticket over two-in-a-million lottery ticket for 100.000 euro. This contradicts the Expected Utility theory, but is in line with the common sense: there is a significant difference between certainty and 50-50 shot; the difference between one or two chances in a million is negligible. This marginal quality of this probability weighting function is in a way similar to that of the utility function, only the impact gets smaller when the wealth amount gets *higher*, while for the probability it gets smaller when the probability number gets *lower* (see Prelec (1998)).

One of the most wide-spread examples in the literature is Prelec's function (Prelec (1998)) which has a following form:

$$w(p) = e^{-(-\ln p)^{\varphi}} \tag{27}$$

Figure 8 demonstrates the shape of the probability weighting function used in CPT.
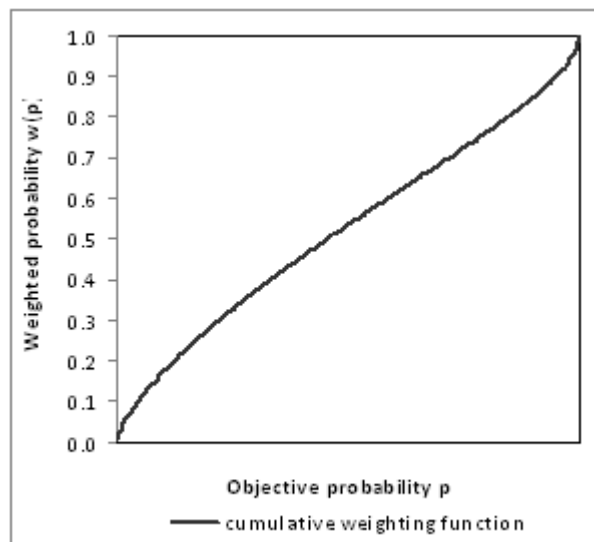


**Figure 8.** Prelec's weighting function applied in CPT to the objective probabilities of the outcomes. Case of Merton's contract, θ=0.1

The popularity of Prelec's weighting function is explained by the fact that the author formalized the anomalies on the probability evaluating in the human mind, and formulated a set of axioms

summarizing these anomalies. The function in formula (27) is selected in such a way that it incorporates the full axiomatic basis. The original idea of probability weighting belongs to Tversky and Kahneman (1992), who also came up with the second most popular probability weighting function form:

$$\Omega(p) = \frac{p^\gamma}{\left[ p^\gamma + (1-p)^\gamma \right]^{1/\gamma}}, 0 < \gamma \leq 1$$

This version of the weighting function has an important drawback: for the particular values of the parameter $\gamma$ it is not always increasing. This feature means that the probability weighting function may assign negative decision weight to some outcomes. This fact in turn implies a counter-intuitive conclusion that that CPT investor sometimes makes choices that give a preference to the outcomes which on expectation are worse than alternative ones[11] (see Ingersoll (2008)). As these two probability weighting functions are the most wide-spread in the literature, the latter shortcoming leads to the choice of Prelec's weighting function for my research.

The problem statement for the investor in this case is the following: maximize value function (including the nonlinear weighting over gains and losses), which is calculated over the utility function over contract's payoff $L_T$ .

### 4.2.1. Cumulative Prospect theory: fixed interest rate

The graph for the value function (analogue of expected utility) calculated for the three contract types is demonstrated on the Figure 9.

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[11] In the literature this fact is often referred to as a violation of the first-order stochastic dominance.
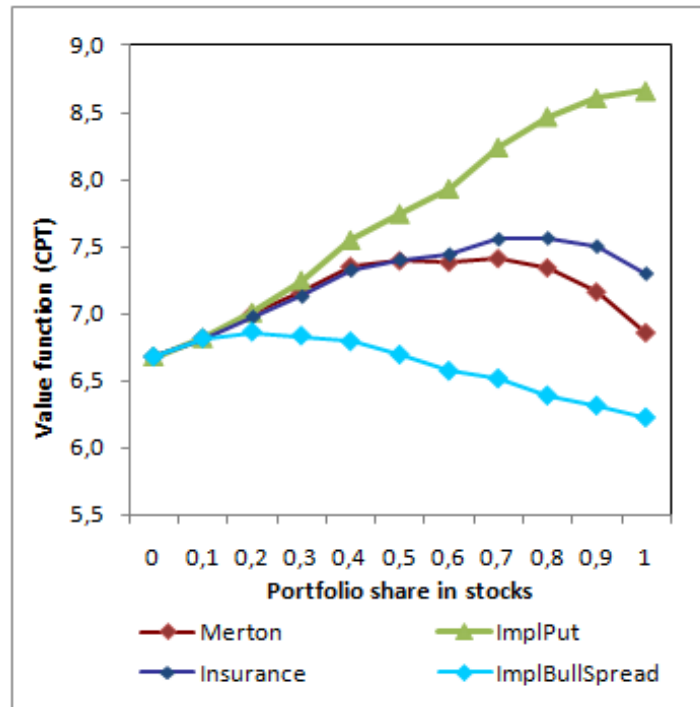
**Figure 9.** Value function for three contract types (the case of constant interest rate) under Cumulative Prospect Theory. Parameters values: Initial wealth $A_0 = 5$, interest rate r=12%, expected stock return µ=0.15, volatility of the stock σ=0.3, time until retirement T=20 years, risk aversion γ=3, α=0.9, Implicit Put minimum guarantee rate g=10%, Implicit Bull Spread minimum guarantee rate $g_1 = 10\%$, maximum rate $g_2 = 14\%$

I obtain the result similar to the one of Doskeland and Nordahl (2007) that in the CPT setting investor finds minimum guarantee a very valuable feature and therefore Implicit Put contract performs the best overall, with the maximum while investing all the initial capital in stocks. Implicit Bull contract, which provides the minimum income guarantee as well, initially also outperforms Merton's and Simple Insurance, but its value quickly goes down due to the limitation of the upper potential. For this type I conclude that the value delivered to the investor by reducing the initial costs is not high enough to cover the loss of utility caused by sacrificing the upside potential.

The influence of the additional parameter λ causes the weight of losses in the mind of investor more than twice higher than the weight of gains. This overweighting is the main reason why the minimum income guarantee becomes the most valuable feature of the contract. Another reason is the overweighting the small probabilities with the Prelec's function. Recall, that the investor attaches subjective weights corresponding the cumulative probabilities, so the probabilities of the outcomes "not better than" or "not worse than" the current payoff. This means that the further is the ranked outcome from the reference point, the more extreme it is, the smaller is the cumulative probability of the group of the outcomes that it belongs to in the calculation process. The calculation is processed from the most extreme edges of the ranked group of all the possible outcomes towards the reference point. Therefore, the probabilities of the extreme cases are overweighed in the mind of investor compared to the more moderate payoffs. However, the range of the outcomes may be equally

distributed around the reference point, eliminating this impact. Here the influence of the parameter λ becomes the most evident, as it focuses the subjective benefit evaluation by the future retiree on the losses rather than gains.

### 4.2.2. Cumulative Prospect theory: stochastic interest rate

As discussed in the chapter 4.1.2, the fluctuations in the interest rate influence the magnitude of the payoffs, which become more volatile as additional uncertainty appears in the model. Figure 10 demonstrates the shape of the value function for four contract types after the introduction of the stochastic interest rate.
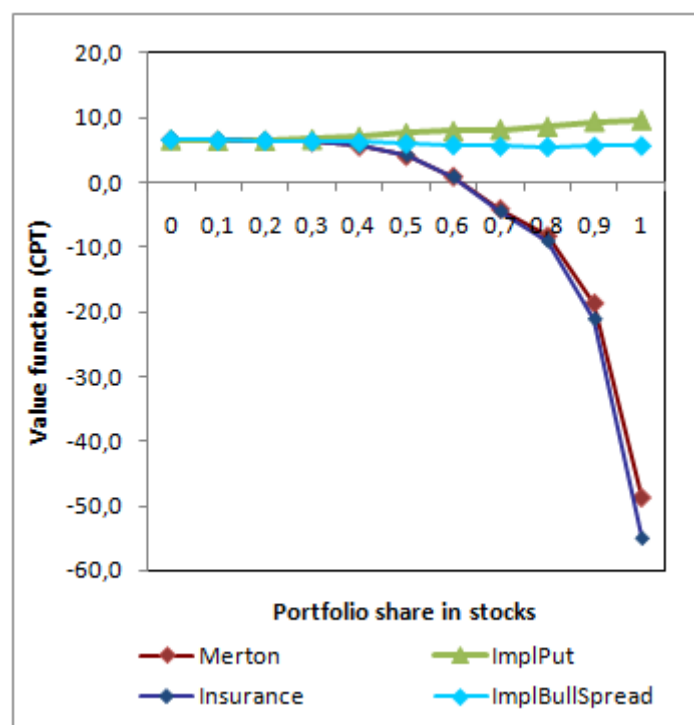


**Figure 10.** Value function for three contract types (the case of constant interest rate) under Cumulative Prospect Theory and under stochastic interest rate. Parameters values: Initial wealth $A_0 = 5$, expected stock return µ=0.15, volatility of the stock σ=0.3, time until retirement T=20 years, risk aversion γ=3, α=0.9, Implicit Put minimum guarantee rate g=10%, Implicit Bull Spread minimum guarantee rate $g_1 = 10\%$, maximum rate $g_2 = 14\%$. Vasicek parameters: a=0,2, b=0,04, initial interest rate $r_0 = 12\%$, volatility $\sigma_r = 0,2$

The minimum income guarantee stays a very valuable feature, and for the Implicit Bull Spread contract its additional value overweighs the price that the retiree needs to pay for it. The opposite happens to the Simple Insurance contract: its additional value eliminates, as its protection from the downside potential is very limited. Therefore the additional price that the retiree needs to pay for it becomes too high for the benefit it provides, which causes its performance to decline lower than Merton's. Facing the risks of the varying interest rates, the investor chooses the contracts that

provide minimum income guarantee, and the Implicit Put stays the most beneficial type of the contract.

The choice of the reference point is based on Doskeland and Nordahl (2007) as a size of the initial capital. However, in the long term scheme it is more realistic that people compare the size of the obtained pension to the amount of money they could potentially get while putting the capital under the risk-free rate. For the reference point it means that the size of it equals the initial capital's potential growth under the risk-free rate during the years before retirement. I conducted the new calculation procedure and found that the shape of the Figure 10 doesn't change, instead it shifts downwards, as the investor evaluates as losses some of the outcomes previously considered as gains. The shift of the reference point does not have a significant influence, as the overall magnitude of the payoffs is high and eliminates the difference of the evaluation. In the setting, where the payoffs are less volatile, I expect the higher influence of this change, due to application of the weighting function and overweighting the losses which with a different reference point were considered as gains.

## *5. Annuity versus Lump sum: calculus versus sentiment*

The previous chapters focused on the accumulation phase of the pension plan. I investigated several contract forms and compared them measuring the benefits they provide for the investor at the date of retirement. In the following part of the research I investigate the payout phase, which is the time after retirement and before the actual death, when the retiree receives the accumulated pension. I compare the annuitization contract with the possibility of taking the accumulated capital as a lump sum. I draw the mapping of the benefits that the retiree obtains in every case, given different length of his life. I introduce a new characteristic -a prediction error, which is a mistake that the person may make when trying to predict his life expectation in order to plan the consumption.  This technique allows calculating the expected utility and value function on average among population and helps to explain the usual antagonism towards the annuity contract.

### 5.1.     Lifelong income: what makes pension a pension

In the setting of the previous chapters the pension model consists of three steps: firstly the pension fund collects a one-time payment of the investor; secondly, the money is invested according to the specified scheme; finally at the retirement date the investor obtains the payoff according to the chosen contract type. This payoff so far in the model is fully distributed to the retiree, meaning that the responsibility of converting the onetime payment to the life time insurance is transferred to the investor himself.

In practice, many pension contracts of the third pillar are designed in this way, which certainly has its advantages. For example, there is no external decision or limitations made for the person – he is free to choose the best option (buy an annuity in the market, keep money on the account with a free accessibility etc.) given every individual situation. However, it is important to remember that third pillar pension schemes in developed countries are usually just an additional arrangement for those retirees who are already covered with the first and second pillar, and therefore serve as an additional financial support, not the primary or exclusive source of income.

The disadvantage of this contract type for investors with lower income is that the freedom of the choice may cause the funds to be used for a purpose other than providing a lifelong financial stability. It is a known fact, for example, that there is a wide-spread practice among Indian rural poor to use (relatively) large sums of money for arranging a marriage of the children or buying a house (Dowla and Barua (2007)).This decision is perfectly fine in a savings framework, but when talking about pensions it obviously violates the ultimate goal of the whole scheme, given that by definition the micro pension will be the only source of income for old people.

The framework for the investment scheme built before, similar to that one of Doskeland and Nordahl (2007) doesn't have yet all the features of the pension scheme. A similar one can basically be used for regular investments or savings account, which provides lump sum payment at the maturity

date. It is essentially a lifelong income feature that makes financial product correspond the definition of the pension. The most obvious alternative for the lump sum payment for the pension fund is therefore to provide a periodical payment during several years, ideally the whole life of the retiree, similar to the first pillar pension schemes, and in this way to insure him against longevity risk.

There are several ways to provide that. The most wide-spread example is purchasing a life time annuity (a periodical payment of a fixed size which investor will receive until his death). Another alternative for the lump sum payment is so called phased withdrawal strategy, when the investor keeps the money invested in the pension fund and withdraws some fraction of them annually. Using this second strategy, the withdrawal rate might be determined according to a fixed benefit level payable until the retiree dies or the funds run out, or it could be set using a variable formula, where the retiree withdraws funds according to a rule linked to life expectancy (Dus et al, 2004). Phased withdrawal is effectively a lump sum contract with the limited access to the accumulated money and externally defined consumption plan. In this research I therefore concentrate on comparing two main contract types: the lump sum withdrawal and the annuity, as they represent conceptually different retirement strategies.

So far I investigated the contracts under the assumption that the period until maturity is definite and the lump sum is paid at the specified retirement date. In a new setting, the payments start at the maturity date, but the date of the last one is not known in advance for every individual investor. The pension fund has to rely on the mortality tables or mortality models in order to predict the length of life in every specific retirement group and country.

## 5.2.    Annuity puzzle

The goal of this research is to look at the annuitization from the investor's perspective, justifying his choice regarding choosing it or otherwise opting for the lump sum payment. In practice the number of retirees who choose to annuitize their accumulated funds is usually much lower than expected in the framework of expected utility. This phenomenon even got a special name – the Annuity Puzzle. Several studies attempt to explain the decisions made by individuals regarding annuitization of their retirement funds, such as Milevsky (1998), Milevsky and Young (2002), Kapur and Orszag (1999) or Blake et al (2003). These studies conclude that depending on the particular features of the model, namely assumption regarding investor's risk aversion (risk-neutral versus risk-averse) or available bequest options, the optimal strategy for the regular investor is to annuitize either immediately or at a particular moment in time after retirement, in some cases gradually, in others fully. Thus, based on existing literature the lump sum payment is never optimal for investor when he has a choice of receiving a life-time annuity. However, all the models in the literature that compare different annuity options use the expected utility framework, and it is interesting to use some elements of Cumulative Prospect Theory or the same research problem. As CPT is based on the

observed principles of people behavior, it can be sufficient to give reasonable explanation for the annuity puzzle, which regular expected utility framework doesn't provide.

Life annuities are nearly the only financial instrument in existence that eliminates the longevity risk for the future retirees. However, annuities also have some disadvantages. First, the purchaser faces loss of liquidity and control over his assets, because the lump sum premium cannot be recovered after purchase of the annuity, irrespective of special needs. Second, in its simplest form, where income payments are contingent on the individual's survival, there is no chance of leaving money for heirs (bequest motive), even in case of the annuitant's early death (Dus et al, 2004). For example, Brown (2000) finds that health status, bequest motive and individual's time horizon for financial decision making are significant determinants of the decision. The absence of the bequest motive gives the investor an insentive to receive the higher benefit while being alive in exchange for giving the pension fund the right to keep part of the funds in case of his death. This is of course very intuitive, as any wealth left after the investor's death doesn't contain any utility value for him.

Other explanations for why individuals in developed countries are reluctant to buy annuities is the ability to pool longevity risk within families (Brown and Poterba, 2000; Kotlikoff and Spivak, 1981), and the presence of other annuitized resources from Social Security or employer-sponsored defined benefits plans (Munnell et al., 2002). The two latter motives should be ruled out in case of micro pensions, because eliminating from the family members responsibility of taking care of older relative is exactly the purpose of the whole scheme. Any other annuitized resources (first and second pillar pension schemes) are by assumption not available for the project target group.  We are left with two parameters that weren't considered in the model before and should be incorporated for measuring the benefits of the choice between lump sum payment and annutization: the  bequest motive and liquidity of the funds (access to accumulated funds in case of emergency). I incorporate only the bequest motive and do not consider the liquidity issue. In practice the micro pension scheme is a next step for the investors already subscribed for micro credit and micro savings program, as for example, a planned pension subscription in DHAN foundation.[12] A pension scheme is naturally a very long-term project and therefore the financial institution arranging the scheme should be confident enough about the ability and willingness of the person to participate in it for many years. The best way to do it is to observe an investor's financial behavior for a certain period of time and to subscribe him or her for the scheme in case he or she is able to set aside part of the income on a regular basis. For example, the Grameen bank in Bangladesh has made Grameen Pension Scheme (a 10 and 5 year savings program, see Appendix 1) - compulsory for its mature members, who have stayed with the bank for many years and have proved their ability to maintain financial stability (as stated in Dowla and Barua (2006)).

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[12] Source: interviews conducted by Caroline van Dullemen with DHAN foundation members.

The availability of saving account or credit option means that in case of crucial emergency there is an accumulated safety stock to be used. Another argument is the results of some interviews with low income investors in Nepal, performed by Krijn de Best, a chairman of the Stichting Nepal foundation, a consultancy for the micro finance institutions in developing countries. During these interviews future subscribers expressed their preference for the features of pension scheme similar to those in developed countries – that is receiving periodical income after the retirement date and use of the funds externally limited for old age provision.[13]

### 5.3. Annuity pricing principle

When talking about annuity, the first question that appears is how to calculate the size of it given the investor's wealth that he is willing to annuitize. I use the actuarially fair pricing principle, as e.g., Brown (2000). According to it the price of an annuity *A* is the solution of the equation

$$L_T = \sum_{t=1}^{T_r - age} \frac{A \prod_{j=1}^{t} (1 - p_j)}{\prod_{i=1}^{t} (1 + r_j)} \tag{28}$$

Where $T_r$ is a retirement age, $p_j$ is one-period mortality hazard, or the probability of dying before the period *t+1* conditional on surviving to period *t*, $r_i$ is the interest rate in the periods prevailing and including *t*, and $L_T$ is amount of wealth accumulated due to retirement date which is supposed to be converted into annuity.

### 5.4. Investor's benefit evaluation: no bequest motive

Staying consistent with the approach of the previous chapters, I first apply Expected Utility theory and then incorporate some elements of Cumulative Prospect Theory in order to evaluate the benefit of annuitization and lump sum withdrawal options for the investor at the retirement date. For the initial setup, I leave out the possibility to transfer a part of accumulated funds to retiree's family upon death and build a primary structure for the comparison to build up further assumptions in the next chapter.

In the case of expected utility, the following assumptions formulate a model framework:

• The retiree has a certain view on the length of his own life. The age at which he predicts his life to end, thereafter denoted by $T_p$, determines his consumption behavior during the years of retirement.

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[13] Source: interview conducted by Krijn de Best with potential clients of micro pension scheme in Nepal

• In case of choosing the lump sum payment, investor consumes every year a fraction of $\dfrac{L_T}{T_p}$ of the funds accumulated due to the retirement date, meaning that he splits the capital in equal shares according to his life length prognosis. [14]

• In case of choosing the annuity option investor receives a yearly payment *A* during the years of his life and no refund for unconsumed money in case of an early death.

• When choosing the lump sum payment, in case of investor's death occurring earlier than predicted age, a certain share of the capital remains not used during the life. The utility of this share is considered to be 0.

• Living longer than predicted in case of choosing the lump sum payment is associated with the loss of utility as retiree stays without any income and is considered to have consumed all the accumulated funds by that date. The size of the loss of utility is calculated based on the level of consumption the investor is "used to" – that is utility of capital share $\dfrac{L_T}{T_p}$ for every year without financial provision.

The assumption that the level of utility obtained in a current period depends on the level of wealth in the previous period seems intuitive enough and is also mentioned the existing literature. Davidoff et al (2005) call it "habit formation". Surely, a rapid decrease of income due to retirement will bring less satisfaction to the retiree who is used to plan his spending according to a highly paid job, than to someone who was anyway receiving a modest social allowance, although they both may end up receiving the same pension in absolute terms. This principle is similar to the concept of a reference point, but reflects subjective feeling of satisfaction changing with time instead.

I first look at the most basic case, assuming that investor has no myopic views (future cash flows are the same value for him as if he would receive this money right now). To start with, I calculate the benefit obtained by investor, using the power utility function defined in the Chapter 2 (formula (22)). Two examples show the utility obtained by investor in case of his own life length prediction equaling to 65 years (below average) and 80 years (above average). Figures 11 and 12 are based on India mortality tables.

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[14] On the first glance, this assumption seems to be too simplified. One can argue that while living longer, the person makes an adjustment of his spending planning, reducing the consumption level watching his funds melting down. However, as in the model there is no correction for the inflation rate, these two might eliminate each other, and the nominal value of annual spending will remain about the same.
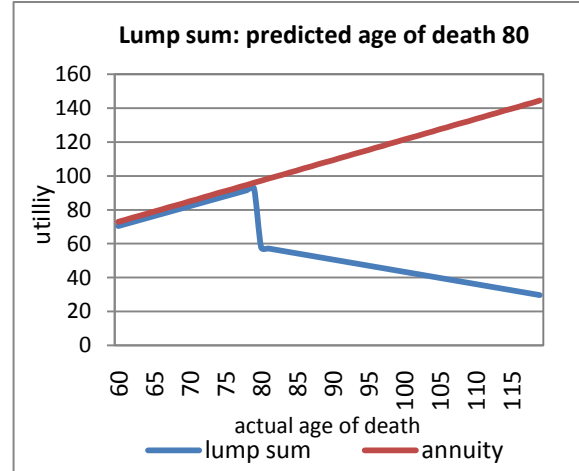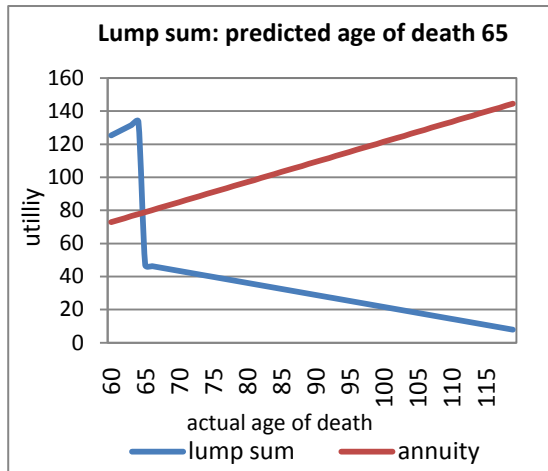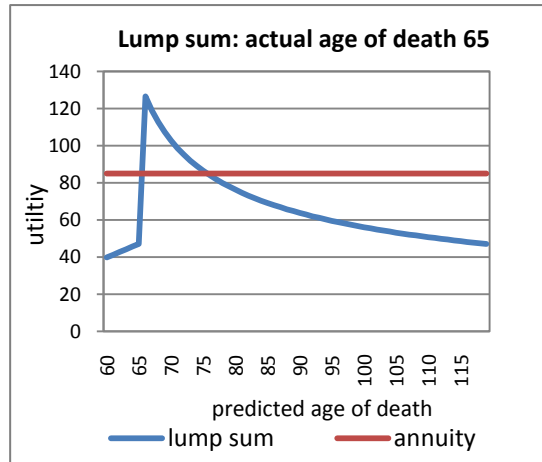
**Figures 11 and 12.** Comparison of CRRA utility of lump sum and annuity options for selected predicted age of death. Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables, subjective discount rate ρ=0.257, India mortality tables

As the pricing of annuity is done in the actuarially fair way, the Figures 11 and 12 unsurprisingly prove that annuitization option is more beneficial for the investors who live longer. However, even those who live shorter and choose to get all the accumulated funds at once, but make a large prediction error regarding their own life length, take a risk of significant utility loss in case of opting for the lump sum alternative.

On the Figure 11 the lump sum option outperforms utility delivered by annuity contract only in case when investor lives less or exactly as much as predicted. The reason behind it is that he splits the accumulated capital in equal shares, and for the short predicted life length these shares are larger than annuity size. However, is he mistaken and lives he longer than predicted, this causes a rapid shortfall of the utility right after the funds are over, and the higher the level of consumption he had before (so the lower the predicted age of death), the higher is the utility loss.

Another example of the importance of investor's subjective life length prediction is given on the Figure 13.

**Figure 13:** Comparison of CRRA utility of lump sum and annuity contracts for selected actual age of death. Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables.

Here the actual age of death of the investor is 65 years, which is five years lower than average, so according to the conventional wisdom, it is not beneficial for him to annuitize. But as we can see on the Figure 13, the chance to outperform the annuity depends solely on the accuracy of his life length prediction, and in case of a large prediction error in one or other direction it is still better to choose the annuity.

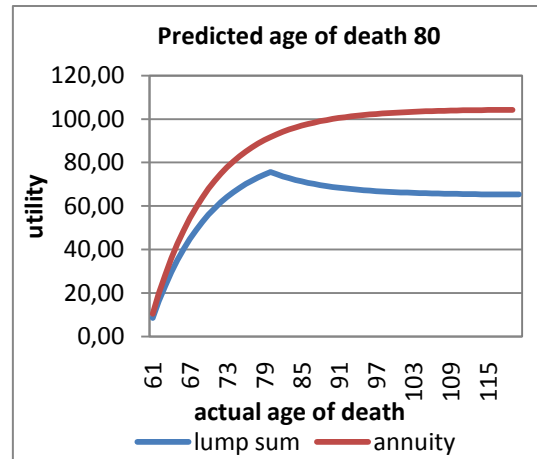## 5.5. Incorporating subjective time discount factor

When the life duration appears in the model, a new characteristic of human behavior comes into play. As future implies some uncertainty about the life conditions and being alive at all, people tend to attach more weight to receiving benefits in the present time then to receiving them in the future. That means that a subjective discount rate should be used to make the expected income less valuable than the income that can be consumed right now. If there are two certain monetary outcomes: $L_T$ at time $T$ and $L_{T+k}$ at time $T+k$, then the following holds:

$$V(L_T) = D_k V(L_{T+k}) \tag{29}$$

where

$$D_k = \frac{1}{(1+\rho)^k}$$

is a discount function, which can be interpreted as the subjective weight that a person attaches to postponing his consumption for $k$ periods, and $\rho$ is the rate that measures preference for immediate consumption over delayed one. Adjusted for the investor's personal time discount rate the previous Figures 14 and 15 will look the following:
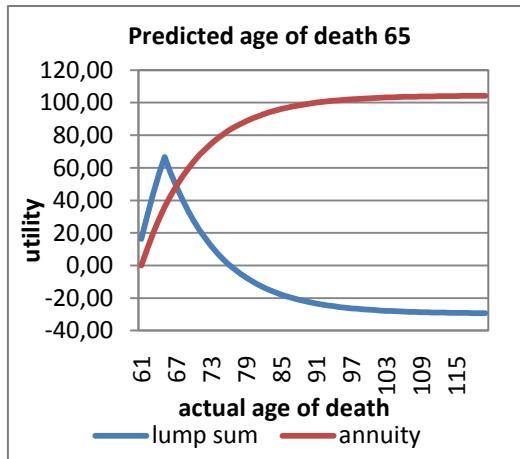
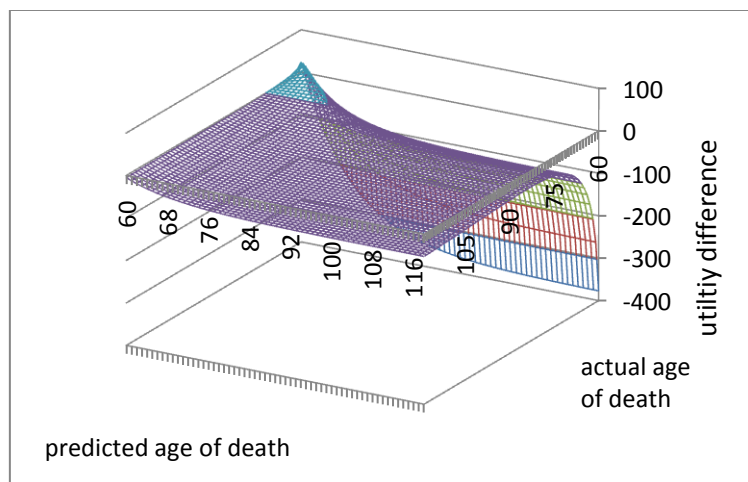**Figures 14 and 15:** Comparison of CRRA utility of lump sum and annuity contracts for selected predicted age of death (incorporating the time doscount factor). Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables, subjective discount rate ρ=0.257

The dependence representing utility versus age of death is not linear anymore, as the discount factor grows exponentially with the time. However, the principle of one contract type outperforming another stays the same: the lump sum option outperforms the utility provided by annuitization only in case of investor living less than average and being accurate enough while predicting his future death age. This latter situation is only possible if the retiree has a very unfortunate health conditions: a terminal illness that he is informed about. This conclusion is quite straightforward and leaves annuitization as a rational choice for all the other situations involving any uncertainty about the future, which is a synonym for the regular life and health conditions.

It is interesting to use the utility provided by annuity as a benchmark and to compare utility provided by alternative lump sum option against this benchmark. Figure 16 shows the relation between two contract options in terms of delivered utility. The larger the value of utility difference on the vertical axes on the Figure, the bigger is the gap between utility provided by annuitization contract and lump sum contract.

The peak of the plot (light blue) appears in the area where predicted age of death is below average (in terms of model that means that investor is going to consume annually a share of capital higher than annuity size) and his actual death happens no later than predicted, so that the "starving years" resulting in the loss of utility do not have a chance to occur. The largest loss of utility (green, red and dark blue) happens when predicted age of death is short enough, but the actual life length is much higher. That can be easily explained by the logic similar to the previous chapter, when investor loses as much as he was consuming and for a longer period.

The following plot demonstrates clearly the area where lump sum option is more beneficial for the investor (the area above zero level on the Figure 17):
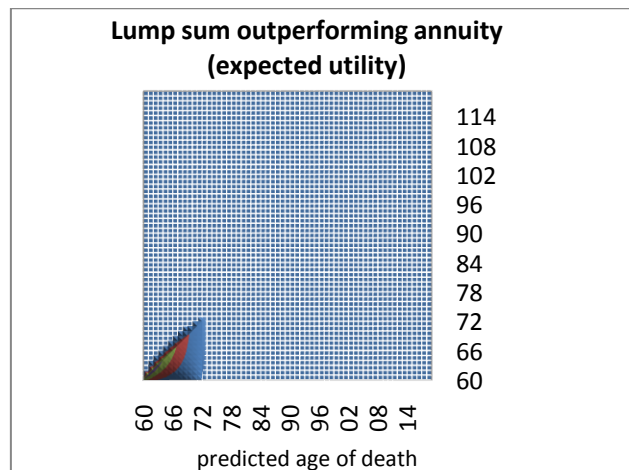


**Figure 17:** Outperformance of the lump sum contract compared to the annuity contract in terms of CRRA utility (area in color). Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables, subjective discount rate ρ=0.257

We can see that with the chosen parameters annuitization is a much better choice in terms of expected utility. The investor gains benefits of the lump sum payment only in mentioned before case of living shorter than average (72 years in this case) and dying earlier or exactly at predicted age. For the selected parameter values it is a very small share of all the combinations of predicted and actual life length.

### 5.6. Prospect Theory: loss aversion in annuitization choice

Existing research indicates that one of the most wide-spread reasons for retirees not to annuitize the accumulated capital is a fear of an early death and not making use of a large part of the money caused by it. In the framework of Cumulative Prospect Theory that means that people tend to compare the overall benefit they will get in the future relative to the size of accumulated capital –

which leads to the conclusion that the amount of money to be annuitized should be set as a reference point. This idea first appears in Cannon and Tonks (2008, p. 191).

In the framework of my model, the following modifications are made.

- The annual fraction of the funds $\dfrac{L_T}{T_p}$ which investor consumes due to his life length prediction when choosing the lump sum option is set as a reference point, or a zero level of consumption in the mind of retiree. The final benefit obtained by investor is not evaluated by its absolute value, but according to its size relative to this reference point.

- In case of early death the utility of the funds not consumed does not equal zero, but is negative, as investor compares zero income to the reference point and considered this to be a loss of maximum available benefits (should he live exactly as long as predicted).

- Similar to chapter 4.2, the power utility function is replaced by a value function of the form stated in formula (26)

$$U(L_T) = \begin{cases} U_G(L_T) = (L_T - \Gamma)^\phi, L_T \geq \Gamma \\ U_L(L_T) = -\lambda(\Gamma - L_T)^\phi, L_T \leq \Gamma \end{cases}$$

which implies loss aversion, and where reference point $\Gamma$ is in this case equal to $\dfrac{L_T}{T_p}$. I fist leave out of the scope the probability transformation part of the cumulative prospect theory and focus on the different shape of the value function. Figures 14 and 15 will take the following form:
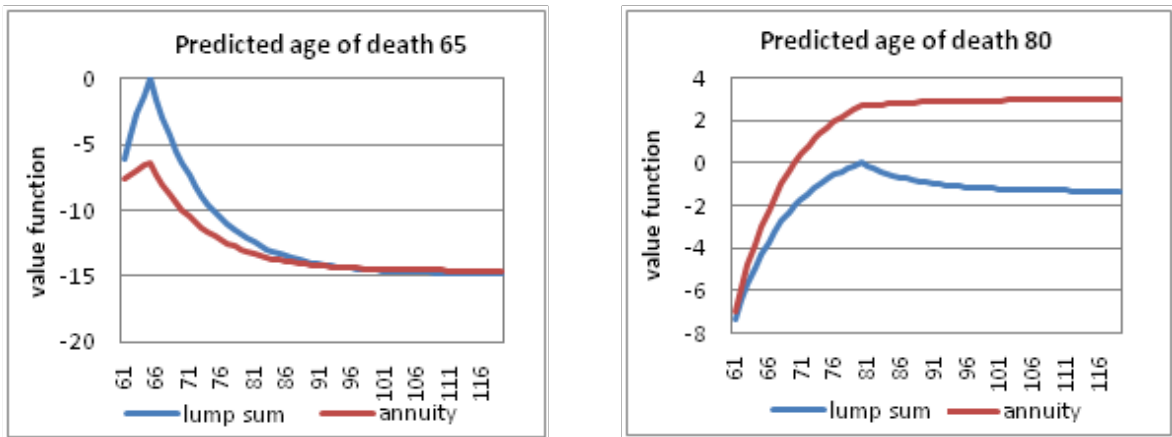


**Figures 18 and 19:** Comparison of value function of lump sum and annuity options for selected predicted age of death (incorporating the time doscount factor). Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables, subjective discount rate ρ=0.257

For the predicted life length 65 years the chance for the lump sum contract to outperform annuity payment is quite high – almost for all possible real ages of death the lump sum payment contract performs better, and for the longest life periods annuity performs better, but the difference is slight. However, for predicted age of death 80 years the situation is worse than in expected utility case – the lump sum contract subjective value is much lower than the one delivered by annuitization option for all cases. This effect is caused by the introduced loss aversion: in the current framework I

assume that for every year of staying without a payment the retiree suffers a loss of utility compared to the reference point, and so the one of the size twice higher than in the expected utility framework (in both cases: either not receiving a part of the accumulated capital due to early death or outliving the predicted age and having no income afterwards.)

Loss aversion versus risk aversion can be characterized in the following way: the person doesn't want to participate in a fair gamble, which in this setting is annuity contract. Being actuarially fair, it represents an insurance against longevity, a more complicated version of "tossing a coin" game, which is often used to demonstrate the concept of a risk aversion. The idea of this game is that a person has a choice between tossing a coin with the outcome 1000 euro in 50% of cases and nothing in the other 50%, and receiving 500 euro. On expectation, the outcome of the game is equal to 500 euro. In this situation a risk averse person would choose to receive the money, risk seeking would participate in the gamble and risk neutral would have no preference. Now, imagine that the potential benefit increases, but the possibility of losing some money is included in the uncertain outcome in such a way that the expectation is now slightly higher than 500 (for example, 50% of winning 2200 and 50% of losing 1000). A loss averse person, similar to risk averse, will opt for the money despite the fact that on expectation the lottery is more profitable, because the loss weights much more in the human mind than a possibility to gain.

In the expected utility theory a common assumption is that individuals are risk averse. When transferring to Cumulative Prospect Theory, loss aversion becomes a dominating characteristic of the behavior. Therefore in this setting annuitization option should be less attractive to the retiree than in expected utility case. Indeed, the area where the lump sum contract outperforms annuity in terms of value function transforms and obtains the following shape:
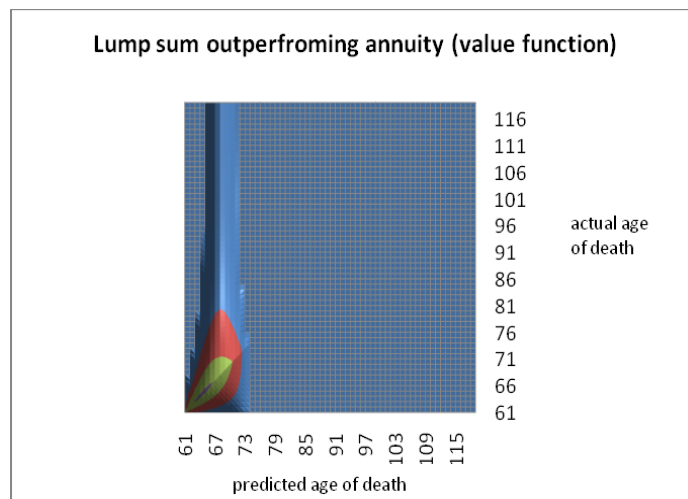


**Figure 20:** Comparison of value function of lump sum and annuity. Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables, subjective discount rate ρ=0.257

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It can be seen that the area of outperformance expanded, and now for the predicted age in certain boundaries around the average annuitization option provides the investor with less benefits than taking accumulated capital as a lump sum. Interestingly, for certain narrow interval of prediction this conclusion is independent from the actual length of the investor's life, which means that if the person prognoses his death at the early stage after retirement and manages to plan his expenses according to that prediction, taking a lump sum is clearly a preferable choice. An important remark needs to be mentioned here. While the Expected Utility framework is a normative theory, which means that it is searching for the optimal decision assuming that people behave rationally, Cumulative Prospect Theory is a descriptive science, which means that it attempts to incorporate and explain the existing violations from the optimal human behavior. It actually states that people behave irrational, but that these unfounded effects are structured, can be categorized and till certain extend predicted. Therefore, results obtained with the use of CPT can help to explain the contradictions between the optimal decision obtained applying Expected Utility theory and real human behavior, and help to create or frame the product in order to overcome these contradictions.

For the people who estimate their life to be short, the subjective value of obtaining accumulated capital at once, splitting it in equal shares and in the best case consuming all of them and not suffering any loss, while in the worst case suffering all the years when the prediction error occurs, is still higher than the subjective value of annuitizing their capital. Each time when investor receives the annuity payment, its size is compared in his mind to the share he would have chosen himself is he given an opportunity to obtain the whole sum of the capital. Therefore each time the annuity payment is lower, he estimates this as a loss of the benefit. When accumulated loss of the benefit associated with annuity is higher than the loss during all the years of "starving" when the prediction error occurs, the final preference is given to the lump sum option.

In terms of a real world interpretation, the result above means the conclusion similar to that one obtained in the Expected Utility framework: in cases when people possess information which states that their life expectation is much lower than average and is effectively just few years after retirement they will only lose if they subscribe for the annuity contract. For the pension provider, this implies that the people who have (or even just think that they have) some superior information about, for example, their health, will be less likely to wish to enter the annuity contract. This entails the following problem, called *adverse selection*: the pension fund gets only the clients who are likely to live longer as they do not expect to die early. Recall that the value of the annuity is set in actuariy fair way, which means that on average the sum of the discounted cash flows that the whole group of subscribed people receives in the future equals the sum of the collected funds. During this process the average life tables are used, but when the people who tend to live shorter and know about it expel themselves from the coverage, this causes violation of the mortality probabilities among the subscribed group. This is called asymmetric information. Now there are more people to receive the

payment as they are still alive than there are those who die earlier and "leave" their funds for the other subscribed members who live longer. Then for the annuity provider the funds received are not sufficient anymore to ensure the payments to the surviving members of the scheme. From the side of the clients, it is a beneficial situation, as they now receive higher annuity than they should expect in this situation.

However, this on the first glance beneficial situation contains an ultimate danger for the sustainability of the whole scheme, as it might lead to the circumstances where the pension provider has no funds left to ensure the payment of the annuity, and therefore in general the insurance companies make a correction for the adverse selection in their calculations. After this correction the price that the average person would pay for the annuity contract rises, making it actually not actuarially fair anymore in a more general average sense, but fair for the corrected group. In this case the average investor might find the annuity contract too expensive and that will encourage the choice in favor of the lump sum.

For the micro pensions scheme this question is particularly actual, as the information asymmetry is usually one of the biggest problems for the Micro Finance Institutions. In practice, for the micro credit and savings schemes it is solved in a manner called joint liability – compulsory subscription of the whole or a part of the community. In general people in the small communities have superior information about each other compared to the financial institutions simply due to absence of any official records (sometimes even the personal identification documents). By subscribing the whole group and introducing particular risk sharing mechanisms among them the financial institution can partially overcome this problem. The most famous real life example of the group subscription is the Grameen bank in Bangladesh.

This solution might enable the pension provider and the client to be in the symmetric situation where they both benefit in an equal way. However, in case of pensions, a compulsory subscription has to be reconsidered. Enforcing the average mortality statistics among the client group seems necessary for the sustainability of the scheme, but the fairness of this way for the people who have certain information about their early death is arguable. In my opinion some exceptions should be made for the people who have a confirmed diagnosis of a terminal illness. An optional solution can be introduction of the hybrid product – an annuity contract together with the health insurance covering medical expenses in case of need. This can be a very attractive contract for the clients, as it essentially provides a benefit in any possible outcome, but the nature of all the pricing calculations will change completely, and therefore this option is out of the scope of current research though definitely worth a further investigation.

### 5.7. Incorporating the bequest motive

In the previous chapter all the calculations were done under the assumption that the investor has no intention to leave any funds to his heirs or spouse after his death. This is a crucial assumption as it forces the retiree to focus the consumption on the period of his life and dramatically reduces the subjective value of the contract proportional to the funds remaining due to the date of death.

In the micro pension setting, this assumption is not completely unrealistic. About 20% of beneficiaries of social pension scheme in Nepal, for example, are living alone[15]. Being expelled of any support of the family or partner makes the retiree need old age financial provision in fact more desperate, therefore people in this category are more valuable clients for the micro pension provider in terms of potential impact.

However, a large share of the poor population has a family actually bigger than the average in developed countries, meaning that even if bequest motive is not a primary obstacle for choosing annuity contract, it cannot be completely ignored in the research. Thus on the next step I change valuation principles in order to associate some positive value with the money remaining after investor's death in case when he takes accumulated capital as a lump sum. This means that the retiree is receiving some (obviously non-financial nature) benefits during his life with the idea that after his death some money is left for the use of his family.

In terms of valuation principles for the lump sum contract the modification is made that the utility of the funds remaining after investor's death is now not considered to be zero anymore. I assume that all the money remaining from the investor's accumulated capital goes to his family after his death. Therefore the retiree is much less afraid to die earlier and lose a large share of pension–because even in this case the money will be consumed by his family members.

Recall the calculation of investor's benefits in every particular combination of predicted age and actual time of death. Now for those cases where retiree dies earlier than predicted ($T_i < T_p$) while being subscribed for the lump sum contract, the expected utility as well as value function in CPT framework will have a higher value, due to the additional utility of the inheritance left. The risk of outliving the accumulated funds is still there and therefore annuitization option remains providing its important insurance against longevity. In the pure annuity contract there is no possibility for the retiree to leave any inheritance upon his death, as this is the price he needs to pay to possibly receive the survival benefits. Therefore for the annuity contract introducing a bequest motive in fact doesn't change anything in terms of total utility.

In the existing literature, there are few suggested options regarding valuating the benefit of the inheritance of a known size. Most common approach is to apply utility function of the form used to evaluate pension benefits with possibly modified parameters. It is usually multiplied with the coefficient representing the strength of bequest motive from the side of the retiree compared to his

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[15] Source: http://www.helpage.org/Resources/Researchreports

wish to receive regular income from the accumulated capital (see for example Blake et al. (2003)). Feinstein and Lin (2006), on the other hand, argue that it is not necessary for the bequest function to have the same shape as the utility function itself. They explore different possible shapes of the latter and discover that the main influence the shape of bequest function has is related to the investment behavior of the retiree during his retirement ages. However, as in my research I consider the fraction of the initial capital invested in risky asset constant during the years before retirement and no investing after retirement date, the shape of the bequest function will not have a strong impact on the utility obtained by investor in current framework. Therefore I choose to evaluate the remaining after the date of death wealth with the same utility function as the pension itself, multiplied by parameter $b$, which represents the strength of the bequest motive. Parameter $b \in [0;1]$, with $b=0$ we are back to the situation when leaving bequest has no value at all (for example, if the person is completely lonely, see Figure 17). In case $b=1$ leaving the pension as inheritance for the family is exactly the same valuable for the retiree as if he would use it himself. For the illustration purposes I choose the value of the parameter $b=1$. This extreme value gives an opportunity to evaluate the limit case for the lump sum contract, as it stresses its main advantage over the annuity contract.

The following plot demonstrates modified relation between utility delivered by the lump sum and annuity contracts when bequest motive is incorporated.
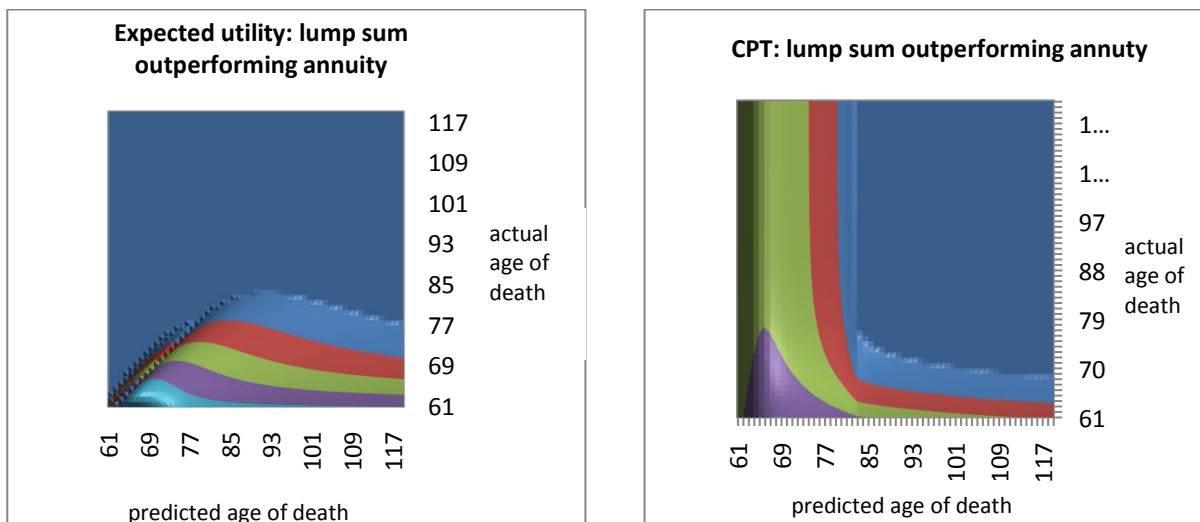


**Figures 21 and 22:** Comparison of utility and value functions of lump sum and annuity when bequest motive is incorporated. Parameters: retirement age: 60, maximum age: 119, accumulated capital=500, interest rate r=12%, risk aversion coefficient γ=0.54, India mortality tables, subjective discount rate ρ=0.257, bequest motive b=1

Comparing Figure 21 to the Figure 17, we can see that the area where the lump sum contract outperforms annuity (in colors other than blue) expands for all the predicted ages if the retiree dies earlier. This is quite intuitive, as an early death entails a large inheritance for the heirs, and depending on the value of the coefficient which weights the benefits for retiree himself from such a future, this value adds to the previous utility. At the same time, the annuity contract is equally attractive as in the

case of no bequest motive in absolute values, as it never provides the person an opportunity for inheritance. Similar holds for the Figure 22, representing the CPT approach to calculating the benefit. Additionally to the area similar to Figure 20 where the lump sum contract outperforms the annuity contract, the area where the retiree dies at early age after retirement also becomes more attractive for those taking money as a lump sum.

### 5.8.    Parameters influence

Figures 21 and 22 incorporate the particular values of the parameters based on the existing literature. Every parameter has a certain interval that due to its place in the model and objective reality it can belong to. For the current research the choice of the exact values of these parameters is not crucially important for two reasons. Firstly, I do not believe that any type of experiments can actually define the exact equivalent of, say, risk aversion coefficient. These experiments would always be heavily influenced by the way the experiment is held, for example, framing of the questions asked or a benefit suggested as a money equivalent. Also, it is hard to imagine that lottery winnings or money outcomes can provide a close approximation of people's behavior when they have to make choices as serious as planning their retirement time. All the conclusions are anyway drawn by the assumption of the representative agent – a hypothetical person with some characteristic behavior, which can represent all the people involved in the pension scheme, and this assumption is not completely realistic on its own. Therefore I'm convinced that this type of research can help to make some general conclusions, and not in any case based on the particular parameter values.

Secondly, the examination of the present model shows that the changes in each of the parameters within plausible boundaries do not influence the overall picture and do not change the principle conclusions made before in the current framework of research. Having said that, still in order to demonstrate the overall outcome portrayed by the model, I need to choose particular parameter values and investigate the sensitivity to them.

*Bequest motive b*. The higher is the value of *b* coefficient (measuring the strength of bequest motive), the higher is the benefit from leaving inheritance to a family for a retiree during his life, and the wider is the area for the early age of death where the lump sum option outperforms annuitization. In the calculations I use the value of b=1, just in order to be able to see the results in the extreme case, but more plausible is to take a middle value, such as 0,5, because the case when the person is that eager to leave inheritance more corresponds to the case of savings. For the case of pension framework the main thing is to have an opportunity to leave the inheritance in principle, because that's one of the main benefits of taking the lump sum capital compared to the annuitization.

*Time discount factor ρ.* The value of the subjective time discount rate can naturally have a significant influence on the interaction of the benefit obtained with annuity or lump sum contract, as the main value that annuitization provides to the subscriber is the insurance against longevity. Hypothetically, if the value of $\rho$ is high enough, the value of the annuity contract is quickly declining, as the person then doesn't appreciate the insurance it provides as much as before, and strives to use all the money in the nearest future.

In case of people with lower income, the value of the subjective discount rate is expected to be higher than in case of regular investors. The life of poor people is constantly affected by risks and uncertainties, which people with the more stable and secure life style do not experience. They see the future as a less clearly defined perspective with unpredictable conditions, which has immediate implications for pension scheme, as it has a very long time horizon. However, the difference in these values is not significant enough to influence the overall picture. Laurence (1991) compares the intertemporal preferences of the households with significantly different income level in the United States and indeed finds that rich people are more "patient" than the poor. However, she estimates the difference in subjective rates of time preferences to be seven percentage points as a maximum.

In practice the approximate value of this parameter is calculated by conducting experiments with the people of the particular interest group, usually in the form of survey questions which include multiple choices of the payoff tables with alternative effective interest rates. The conventional values of the time discounting vary between 0 and 1, similar to the interest rate discount factor.

In the present model the time discounting factor has to be extremely high in order to have any notable influence on the shape of the outperformance area, so I can make a conclusion that for the case of micro pensions this parameter doesn't play any significant role.

The value chosen for the demonstrational Figures equals 0,536 and is based on the findings of Lawrance (1991), who conducted some field experiments among rural poor in India, Uganda and Ethiopia. A very close value (0.54) is obtained in the PhD thesis of Lammers (2008), who conducted similar experiments among people infected with AIDs in South Africa. This latter study can be interpreted as an investigation of the extreme case of a short time horizon, when people know for sure that the possess a disease dangerous for the life and are thus expected to have a high time discount rate. However, we can see that the value of the parameter doesn't differ much from the conventional one.

*Annuity size* has a certain influence on the shape of the Figure 17, making annuity contract attractive to more people as the value of the annuity rises. As it is obtained in a strict dependence with the accumulated capital size applying actuariary fair pricing principle, the size of the latter one doesn't matter for the outcome of the research. This means that what is actually playing an important role are life expectancy probabilities, as they directly enter the calculations of the annuity size.

*Interest rate r.* In the formula (28) calculation of the annuity size is made using fixed interest rate. Incorporating the model for the stochastic interest rate used in the previous chapters (Vasicek equation) pushes the annuity size down, as it assumes more risk and uncertainty for the pension provider. However, I evaluate the choice between annuity and the lump sum option at the date of retirement, when the accumulated during the retirement capital $L_T$ is converted in one of the two contracts. The uncertainty that the stochastic interest rate entails has therefore more impact in the first part of this paper, where the retiree can expect a certain protection from this risk from the side of the pension fund (minimum income guarantee). Although we can interpret the annuity as a kind of a guarantee as well, it implies a different principle strategy. For the minimum income guarantee at the retirement date the retiree pays from his initial investment, while annuity has an implied indirect fee, which is loss of the money in case of an early death. In case of taking the contract as a lump sum, the person can only invest this money in the risk free bank account *under the same interest rate* that the annuity capital of the whole pension fund is invested. This fact leaves only mortality premium as a more significant distinguishing component. In the current setting the size of the annuity is calculated at the retirement date given the outcome of the investment strategy. For the calculations I used the interest rate of 12%.

*Life tables.* The influence of the life expectancy factor is quite straightforward: the shorter live the people in the covered group, the easier it is for the pension fund to ensure the cash stream during the old age for those remaining, and the higher can the value of the annuity be set. Age of retirement and maximum age of death influence the age for which the division line for the annuity and lump sum benefit will be drawn. For people who live longer than a particular age, annuity is a better option, but the exact value of this age depends on the life tables.

## 5.9. Expected average benefit calculation

So far in the framework of the model I looked at every possible combination of subjectively predicted life length and the real time of death, calculating the benefit obtained by retiree in every particular situation. It is interesting and important, however, to compare the value obtained by the average investor that is to calculate the expected utility or value function across the whole population. In the current framework there is a certain difficulty in obtaining the quantitative result for this value. Let's first take a close look at it in case of expected utility.

In the Figures which compared the two contracts in the previous chapters, the benefit obtained by investor choosing the lump sum option crucially depends on his subjective evaluation of the future life length, because he plans his consumption due to this prediction. When evaluating the group of

retirees as a whole, in order to calculate the overall expected utility I need to know the distribution of the predicted ages for every real life length.

The following tables represent calculation process ($T_r$ –age of retirement, $T_p$ –predicted age of death, $T_i$ – actual age of death):

| | | Tp (predicted age of death) | | | | |
|---|---|---|---|---|---|---|
| | | $T_r$ | $T_{r+1}$ | $T_{r+2}$ | ... | $T_{max}$ |
| **Ti (actual age of death)** | $T_r$ | $U_{11}$ | $U_{12}$ | ... | ... | ... |
| | $T_{r+1}$ | $U_{21}$ | ... | ... | ... | ... |
| | $T_{r+2}$ | ... | ... | ... | ... | ... |
| | ... | ... | ... | ... | ... | ... |
| | $T_{max}$ | ... | ... | ... | ... | $U_{max\ age\ max\ age}$ |

| | | Tp (predicted age of death) | | | | |
|---|---|---|---|---|---|---|
| | | $T_r$ | $T_{r+1}$ | $T_{r+2}$ | ... | $T_{max}$ |
| **Ti (actual age of death)** | $T_r$ | $p_{11}$ | $p_{12}$ | ... | ... | ... |
| | $T_{r+1}$ | $p_{21}$ | ... | ... | ... | ... |
| | $T_{r+2}$ | ... | ... | ... | ... | ... |
| | ... | ... | ... | ... | ... | $p_{max\ age}$ |
| | $T_{max}$ | ... | ... | ... | ... | $_{max\ age}$ |

**Tables 1 and 2**. Left-hand side: utility obtained by investor in every outcome of actual age of death and predicted age of death; right-hand side: probabilities of the outcomes

Table 1 in calculation process demonstrates the utility obtained by investor for every particular combination of predicted and actual death age. In order to calculate the general expected utility I need to know the probability of each of the outcomes. Life tables available from different sources which are used in actuarial calculations, provide only the actual probabilities of death occurring at the certain age – *p(Ti)*. What is missing here, is the distribution of peoples' subjective predictions for every given age of death, that is *p[$T_p$|$T_i$=const]*. For example, in order to calculate the expected utility obtained by people who died at the age of $T_{r+2}$, which is two years after retirement, I need to know the values of the probabilities in the corresponding row of Table 2. This probabilities distribution effectively reflects the ability of people to be precise about their prediction – in case everyone would predict his age of death with no error, Table 2 would consist of ones on the diagonal and 0 elsewhere. Of course, that is not the case in real life and therefore most of the people will have some prediction error, which will be the reason of the respective loss of utility.

In practice it is highly unlikely that there is an opportunity to obtain the exact shape of this distribution. What can be claimed based on the intuition is that it will be a modification of the real world probabilities, as people apply individual information they have regarding their own health conditions, genetic aptitude, life level etc. available only for them to the general life length in the region. I expect the error to decline towards the edges of the distribution, meaning that more people will be right about their life length and less will make a significant miscalculation. The existing research regarding subjective death probabilities based on the polls and experiments conducted in US (see, e.g. Hurd and McGarry (1997) or Mirowsky (1999)) is only conclusive about the major correctness and a positive bias in the predictions, meaning that in most of the cases people make a correct but one-two years shifted to the older age prognosis.

Real average life expectancy of the population has a probability distribution which can be obtained directly from the demographic statistics, and therefore is known. In case of subjective

probabilities, I consider every person's prediction about his own death to be based on the following information:

a) General knowledge – people are informed about the mortality rates in their country and age group. This means that they have in mind the real probability values from the life table. In practice it is not that direct, of course, but the assumption that people know the average life length in their region and approximate maximum age of death is reasonable enough.

b) Every person has some individual information about his/her health status, chronical diseases present in the family, life style etc. Therefore for every particular age of death the probability distribution of the age of death predictions will be a transformation of the real probability distribution from the life tables cause by the extra personal information. The transformation mechanism cannot be defined explicitly, but some intuition behind it can be described.

Select, for example, the group of people who actually died at the age of 75. If these people would have no individual information regarding their own conditions of life, the best information they could rely on while planning their consumption (in the absence of any more specific data) would be general life table. This means that they would pick up the average value of the life length and keep it in mind while planning their retirement spending. Now, when the individual information comes into play, the person obtains a different prediction for the length of his/her life, and this prediction will be closer to the real situation than the conclusion made based solely on the death probabilities obtained for the whole population. Coming back to the chosen example, out of the selected group, there will be a larger share of people who will expect their life to end approximately at 75 years than it would be just according to life tables. In terms of probability distribution transformation, people will give a higher weight to the probabilities to die at the age of 75 or few years earlier or later. I assume that the larger is the mistake regarding prediction, the smaller is the chance to make it, which means that people are reasonable enough to use the information they have in order to improve the chance of their prognosis to be closer to reality, and not vice versa.

When choosing the distribution type for the subjective death probabilities that people in a particular death age group tend to prescribe to every age in their mind, I was guided by the following intuitive characteristics of this decision making process. Firstly, there is usually not one particular value that seems to be most probable, rather there is a certain interval, when the person assumes the probability of death to be higher than for other intervals. In terms of conventional understanding – people, while being asked about their presumed life length, generally tend to think in terms of year intervals, and not one particular age. For example, if someone has information based on observing his older relatives regarding the inheritable diseases that may cause an earlier death, he may give an interval of ages 68-73 a higher probability than the older ages. Contrary, if in someone's family

average age until all people live is approximately 90 years, this person will attach a higher probabilities to the higher edge of life tables data.

The plausible assumption is also that the distribution will have a symmetric shape – meaning that people use the individual information they have to define the highest probability density age interval. After defining it they cannot say more in addition than that the probability of the other ages, younger or older, are declining more the further from this interval the particular age is, with the same speed to the left or to the right. These considerations are yet describing explicitly the subjective information interpretation, and on the later stage the information regarding average life length will be taken into account by the person as well. I will incorporate it by combining the subjective probabilities distribution with the objective probabilities from the life tables. In the conventional interpretation that means that the person will adjust the highest probability density intervals by incorporating the data from the life tables. An example of consideration could be the following: "In my family the average age until which men usually survive is approximately 70-75 years, and the average life length for my generation in my country is about 78. I am leaving a very healthy life style and have never suffered any chronic diseases, so my condition is definitely not worse than average. I can state that the highest probability for me has an interval of 77-82 years".

Given the considerations above, I make an assumption that the subjective death probabilities have a generalized normal probability distribution, in particular error (exponential power) distribution, which reflects the intuitive characteristics of people's prognosis. This distribution is bell-shaped, unimodal and looks the following (with the tails heavier than normal when β<2, normal when β=2 and lighter than normal when β>2):
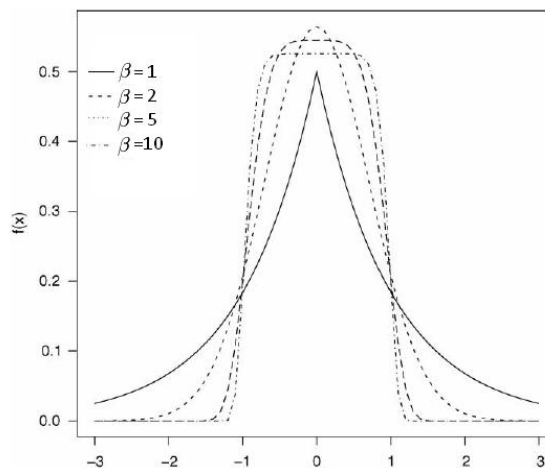


**Figure23.** Illustration of the shape of generalized normal distribution for different parameter β values.

Two parameters define the shape of the distribution: β and standard deviation. Parameter β defines the tail behaviour and the meaning it has in my model is the number of people who were close enough to guess the real life length. The higher is β, the more "flat" is the interval around the median, and the more people have made a few years prediction error. Standard deviation defines the

interval were major mass of the probabilities belongs to, so that the lower value it has, the smaller is the average error people make regarding their life prognoses, and the higher is the concentration of the outcomes around the median (which is set as an actual age of death).

The choice of the rule according to which people combine the official statistics and the individual information they have is based on the article of Clemen and Winkler (1999). The authors look at the different possibilities of the combination of two experts' forecasts expressed in terms of probability distributions. The situation with the death probabilities is in principle very similar, the only difference lies in the categories represented by those probabilities.

The combination approach uses multiplicative averaging and is sometimes called a logarithmic opinion pool. The combined probability distribution is of the form

$$p(\theta) = k \prod_{i=1}^{n} p_i(\theta)^{w_i}$$

where

$$k = \frac{1}{\sum_{i=1}^{n} p_i(\theta)}$$

is a normalizing constant ensuring that the obtained values will satisfy the condition necessary to represent probabilities, namely that their sum will equal one; $w_j$ are the weights that satisfy some restrictions to ensure that $p(\theta)$ is a probability distribution of each outcome $\theta$, and $n$ – number of forecasts available. Typically, the weights are restricted to sum to one. The meaning of them is the importance the individual attaches to the average statistics compared to the personal information.

In the case of two probabilities distributions – objective and subjective, the combination rule will take the form:

$$p(\theta) = k(p_s(\theta)^w p_o(\theta)^{1-w})$$

and the value of weight $w$ can range from 0 to 1. Weight $w=1$ when the person considers his subjective information to be so important compared to the average statistics, that the latter one should not be taken into account at all. Weight $w=0$ when the opposite holds – the person has no particular information to rely on, and therefore builds the prediction based on the mode of the average life tables. Picking up a particular value for $w$ means making a strong assumption about how much importance all the individuals attach to the personal information they possess. As the exact value is not possible to obtain in practice, throughout the research I take different values of $w$ and evaluate their impact on the benefit obtained by the retiree.

Depending on the values of parameters α and β, for the people who died at the age of 70, for example, the distribution of their prognoses based on the individual information will look in the following way.
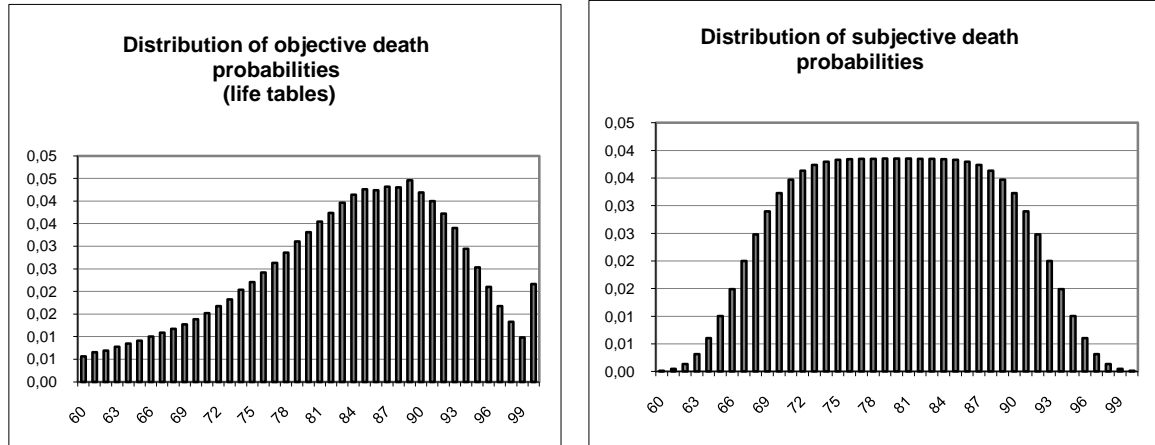


**Figures 24 and 25.** Left-hand side: distribution of the real death probabilities from the life tables. Source: Life tables: UK 2005-2007. Right-hand side: distribution of objective probabilities of death built for the actual age of death 80 years. Parameters of the distribution: standard deviation *d=10, β=5.*
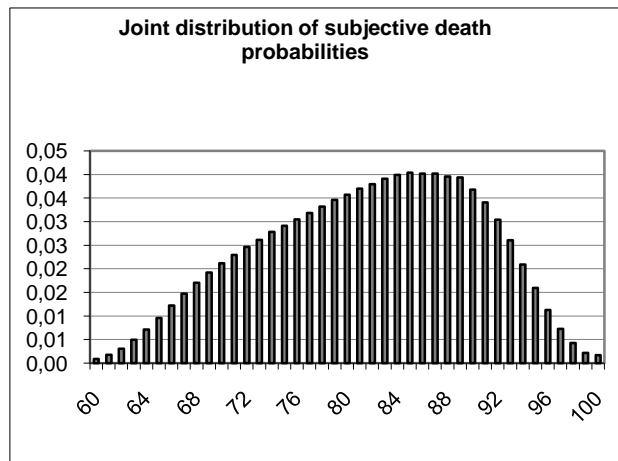


**Figure26**. Joined distribution of subjective death probabilities for the group of people with actual age of death 80 years. Distribution is based on objective and subjective information from Figures 24 and 25.
Life tables: UK 2005-2007, weight assigned to the subjective information *W=0.5*

I calculate the expected utility value for the Lump Sum contract type for different values of the standard deviation and β=4 and 8 (the value delivered by annuity contract is constant and known for every particular age of death as it doesn't depend on the life forecast of the person). The results are displayed in Tables 3 and 4.

| Objective probability weights | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Std | 1 | 0,9 | 0,8 | 0,7 | 0,6 | 0,5 | 0,4 | 0,3 | 0,2 | 0,1 | 0 |
| 20 | 7,04 | 7,05 | 7,09 | 7,14 | 7,21 | 7,28 | 7,36 | 7,44 | 7,51 | 7,58 | 7,64 |
| 10 | 7,97 | 8,01 | 8,05 | 8,09 | 8,12 | 8,14 | 8,15 | 8,15 | 8,12 | 8,02 | 7,75 |
| 2 | 8,33 | 8,36 | 8,39 | 8,42 | 8,45 | 8,47 | 8,49 | 8,52 | 8,55 | 8,59 | 8,52 |

**Table 3.** Values of the expected utility for different subjective probability weights and standard deviation values. *β=4.*
Expected utility of the annuity contract payoff = 9,24

| Objective probability weights | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Std | 1 | 0,9 | 0,8 | 0,7 | 0,6 | 0,5 | 0,4 | 0,3 | 0,2 | 0,1 | 0 |
| 20 | 6,97 | 6,99 | 7,04 | 7,11 | 7,19 | 7,28 | 7,36 | 7,45 | 7,53 | 7,60 | 7,64 |
| 10 | 8,00 | 8,06 | 8,11 | 8,16 | 8,21 | 8,25 | 8,29 | 8,31 | 8,33 | 8,31 | 7,73 |
| 2 | 8,33 | 8,36 | 8,39 | 8,42 | 8,44 | 8,46 | 8,48 | 8,50 | 8,52 | 8,54 | 8,65 |

**Table 4.** Values of the expected utility for different subjective probability weights and standard deviation values. *β=8.*
Expected utility of the annuity contract payoff = 9,24

Table 5 demonstrates the expected values of Value function for annuity (EV ann) and Lump sum contract (EV LS) replacing utility function in CPT theory. In this case the value function for the annuity still depends on the life expectancy forecast made by person, as the reference point is calculated based on it.

| | | Objective probability weights | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Std | | 1 | 0,9 | 0,8 | 0,7 | 0,6 | 0,5 | 0,4 | 0,3 | 0,2 | 0,1 | 0,0 |
| 20 | EV LS | -0,03390 | -0,03016 | -0,02678 | -0,02377 | -0,02112 | -0,01883 | -0,01688 | -0,01529 | -0,01414 | -0,01381 | -0,03195 |
| | EV ann | -0,11853 | -0,10422 | -0,09131 | -0,07978 | -0,06958 | -0,06062 | -0,05283 | -0,04615 | -0,04061 | -0,03659 | -0,06266 |
| 10 | EV LS | -0,03595 | -0,03171 | -0,02791 | -0,02451 | -0,02150 | -0,01885 | -0,01652 | -0,01451 | -0,01284 | -0,01175 | -0,01579 |
| | EV ann | -0,12654 | -0,11109 | -0,09715 | -0,08469 | -0,07363 | -0,06387 | -0,05531 | -0,04783 | -0,04140 | -0,03614 | -0,03721 |
| 2 | EV LS | -0,03595 | -0,03171 | -0,02791 | -0,02451 | -0,02150 | -0,01885 | -0,01652 | -0,01451 | -0,01284 | -0,01175 | -0,01579 |
| | EV ann | -0,12654 | -0,11109 | -0,09715 | -0,08469 | -0,07363 | -0,06387 | -0,05531 | -0,04783 | -0,04140 | -0,03614 | -0,03721 |

**Table 5.** Values of the expected Value function for different subjective probability weights and standard deviation values.
β=4

I make the following conclusions based of these findings.

- In the expected utility theory, even in case the parameters are chosen maximally in favor of the lump sum option (such as *b=1* and *std=3*, which can be interpreted as the fact that almost all the people were right about the subjective prognosis they made and thus the error is minimal) the lump sum contract is always on average less beneficial for the well-being of the investor and thus not advisable.

- In the CPT case, the opposite holds – expected value of the annuitization is on average always lower than that of the lump sum. This difference has two underlying reasons: the subjective fear of losing the capital while dying at the early age and the overweighting of the probability of this to happen. As mentioned before, CPT is a descriptive behavioral theory and thus has a value in reflecting the real people behavior. It points to the fact that a loss of the accumulated capital has a significant influence on the voluntary choice in annuitization. Therefore the pension provider has either to introduce a compulsory subscription, as discussed before, or to change the framing of the pension product, namely to introduce some sort of capital guarantee, a partial bequest option or a hybrid contract between health insurance and annuity, which also has a potential to reduce the fear associated with the loss.

## 5.10. Real world challenges: mortality tables and hybrid products

The only known example of the microfinance institution that has tried to implement an annuity product in reality is a Centre's of Agricultural Research and Development (CARD) pension benefit addition to their Members Mutual Fund (MMF) scheme (this example is solely based on (Microinsurance compendium (2006)). MMF was designed to provide loan balance coverage plus burial assistance in the case of borrower death, so it was a hybrid between a loan and life insurance. The scheme was implemented in 1996 in Philippines and gained an immediate initial success. Confident after the bright start, the management decided to expand the product coverage and offered a pension benefit to members reaching sixty-five years of age for only US$0.05 more per week. When the client reached sixty-five, or became permanently disabled, the new product offered a lifetime monthly pension between US $5.45 and US $10.90, depending on a how long the annuitant had been a CARD member. Under this arrangement, it took 14 months of monthly premiums of US$0.40 from a member to accumulate the lowest pension amount of US$5.45. There was no minimum participation period before the pension was available; members just had to turn sixty-five years old. During the 1998 audit, CARD's external auditors advised management that the pension situation was financially unsustainable and the liability was a very serious threat. Even though the average age of a CARD member was 43.6, the potential volume of soon-to-be pensioners would quickly deplete CARD's capital. CARD eventually managed to extricate itself from its liability and shut the scheme down repaying all premiums to the existing members.

The main problem with this annuity product was implementation without testing and without actuarial input. This story doesn't mean, therefore, that the possibility of implementing the annuity in the future is ill-fated by default. Insurance products for poor people are a wide-spread practice among the microfinance institutions, which means that they possess enough actuarial information to maintain them. Usual practice is to use old mortality tables from developed countries like US (for example, 1958 and 1980, see Microinsurance, demand and market prospects, Indonesia by Allianz AG, (2006)). These proxies, because of the lack of accurate information, lead to the application of loadings to protect the insurer, but at the same time they increase the premium prices for the potential clients.

One of the main pitfalls that the pension provider might meet on the way of designing and implementing the annuity contract for micro pension scheme is the accurate actuarial data on mortality statistics. As Microinsurance compendium (2006) states, besides the fact that sometimes even the age of the clients can be difficult to pin down, small changes in the life style can have dramatic effects on the long-term life span. The authors give an example that improvements in the provision of clean water and sanitation, or a successful vaccination or mosquito net campaign can dramatically improve average life spans. Besides that, usually a majority of the MFI clients are female (for example, 95% for Grameen Bank, see Dowla and Barua (2006)), and the mortality data for men and women can differ substantially. Another difference that can occur is the range of mortality rates in different areas. This raises two issues: obtaining accurate life expectancy tables relevant for the covered target group and potential legislation constrains, as using the differentiated approach to the customers might be interpreted as discriminative rates.

Assuming that the microfinance institution can develop its own mortality tables from the group of clients (given that it is large enough, has sufficient coverage and exists in the market for a certain amount of years), it can eliminate the unsystematic mortality risk. However, the risk of the mortality statistics to have dramatic changes in time (systematic risk) will still stay in place. In the developed countries this problem is solved by calculating so called cohort, or generational life tables, which incorporate the statistical life expectancy trends in the society. For developing countries this method cannot be applied because the smooth character of life expectancy changes can be interrupted by some significant improvements in the life style. These latter events are of course extremely desirable from the point of view of the society, but for the financial organisation they represent the risk of being not able to meet its obligations and thus default.

One of the solutions to partially against systematic risk (as there is absolutely no possibility to predict it in case of poor people) is transferring it to the capital market and creating mortality-linked securities similar to the principle of interest-rate hedging securities. However, as at the current moment this type of securities does not exist in the market, there is no practical opportunity to

implement this strategy, although the potential demand is estimated to be huge due to a large volume of life insurance and pension liabilities outstanding.

Another solution to partially hedge against systematic mortality risk is so called natural hedge. The principle implies a simultaneous sale of the products, which will eliminate the risks that the annuity carries for the pension fund, should the life expectancy improve. An example of such a product is life insurance. On average, in case of life conditions improving and people living longer, the company will win on the delivering life insurance, but lose on the annuity contract. On one hand, this will help the pension fund/insurance company to protect itself from the rapid changes in the mortality statistics. On the other hand, it can also help to eliminate the decision obstacles on the way for the pension subscribers to enter the annuity contract, as it hits both the fear to die early and not to make any use of the accumulated funds as well as a fear not to leave any money for the family. (See for example Murtaugh, Spillman and Warshavsky (2001)). This suggestion is based on the intuitive empirical observation that people who experience health problems usually have shorter life expectancy. If insurance companies simultaneously sold both an annuity and health insurance they would be selling two insurance products whose risks were off-setting (so the two products are a natural hedge). The drawback is that even for regular investors such markets are very thin and the costs are very high (Cannon and Tonks, 2008).

This solution also requires two underlying conditions to be fulfilled:

1. The clients are willing to purchase both the annuity pension and the life insurance contract;

2. The microfinance institution is capable to provide both products.

The second condition in some cases can be relaxed as there is also an opportunity for two MFIs providing annuity and life insurance contracts to exchange the risks they have, similar to the principle of the swap contract. Cox and Lin (2001) call this type of arrangement a "mortality swap". They show that such an approach can also lower the overall costs that the customer bears and give a competitive advantage to the financial institution.

The first condition can sound counterintuitive, as it implies that people are afraid at the same time of the two opposite situations: living too short or too long. However, most of the people are not certain regarding how long their life is going to be, except for those who have particular confirmed information about their health, like a terminal illness. Therefore it is not an improbable situation that the clients are interested in both products. Besides that, the idea is that not *every* client needs to purchase both products – the financial institution just needs to reach a sufficient coverage selling both so that the coverage could represent the population mortality trends.

The following table illustrates the main risks that can cause poor people to have an immediate need of financial resources (source Financial Access Initiative, 2009)). We can see that people with lower income are exposed to the health and unexpected death risks till great extend.

| Bangladesh<br>42 households | | India<br>48 households | | South Africa<br>152 households | |
|---|---|---|---|---|---|
| Event | % | Event | % | Event | % |
| Serious injury or illness | 50 | Serious injury or illness | 42 | Funeral of family outside the household | 81 |
| Did not receive expected income | 24 | Loss of crop or livestock | 38 | Serious injury or illness | 10 |
| Fire/loss of home or property | 19 | Loss of regular job | 10 | Funeral of member of the household | 7 |
| Loss of crop or livestock | 7 | Theft | 4 | Theft | 7 |
| Business failure | 7 | Abandonment or divorce | 4 | Violent crime | 4 |
| Cheated/cash loss | 7 | Serious harassment by officials | 4 | Fire/loss of home or property | 3 |

**Table 6.** Most frequent events causing a financial emergency, by country, with the percentage of country sample affected at least once during the study year.

To my knowledge, at the moment annuity is not a product provided by any microfinance institution and the life microinsurance is a product for which the demand significantly exceeds the supply [16], so the prospective of this hybrid product are definitely worth further investigation.

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[16] Source: International Labour Organisation, http://www.ilo.org/global/About_the_ILO/lang--en/index.htm

## 6. Conclusions

In this thesis I investigated the most beneficial features for the pension scheme for people with lower income. I introduced two elements: different investment strategies before the retirement and the choice between annuity and a lump sum withdrawal at the retirement date. I use two approaches: the Expected Utility theory (based on Von Newann and Morgenstern (1944)) and Cumulative Prospect theory (CPT), which belongs to the behavioral finance (based on Tversky and Kahneman (1992)).

The first part of the research covers an investment period of the pension scheme, or working years of the retiree, when the future retirement capital is accumulated. I introduce four contract types, three of them based on Doskeland and Nordahl (2007) and the fourth replicating the so called Bull Spread investment strategy. Merton's contract performs technically the best at its maximum, but the overall performance for different portfolios is higher for the contracts which contain minimum income guarantee element. Merton's contract is a form of a direct investment in the financial market, similar to Defined Contribution scheme. The high delivered utility is caused by the complete freedom of investment choice for the investor and absence of any kind of limitations, which for example is a minimum income guarantee. However, after modifying parameters to the case of micro pensions and developing countries and given the long-time horizon of the scheme, the advantage of this type of the contract becomes insignificant and for different portfolio combinations is overall dominated by the contracts containing the elements of the minimum income guarantee. This is the principle of the Defined Benefit pension schemes. Capital protection also proves to be an optimal scheme in the behavioral framework, as it provides people with a feeling of being insured against loses, which for the CPT investor have a higher impact than the gains of the same magnitude.

The second part of the thesis is focused on underlying reasons and consequences for the retiree regarding her decision to annuitize the accumulated capital at the retirement date. The conclusion that annuitization is a necessary choice for the average investor with lower income is the strongest in the research. Applying the Expected Utility theory shows (in line with the existing literature) that annuity is clearly beneficial for the average retiree who makes her choice rationally. I focus on the expectations of people regarding subjectively expected length of their life and demonstrate that even if these expectations on average have only a few year error, it may cause a significant loss of utility in case the person takes the capital as a whole and spends the money relying on her prediction. Detailed mapping for different ages of real age of death and expected age of death demonstrates that the choice of annuity is preferable for the vast majority of people except a small group of those possessing a terminal illness and knowing about it. This conclusion stays unchanged after incorporating the modifications appropriate for the micro pensions case. Using Cumulative Prospect theory helps to find a behavioral explanation for the motivation of the majority of people who opt for the lump sum withdrawal of the contract and rely on themselves for the further financial

planning. Within CPT framework the most important arguments against annuity are a fear of the person to lose the accumulated wealth in case of an early death and a will to leave something for the family (bequest motive). From the side of the pension fund or microfinance institution the biggest challenge in implementing annuity can be an accurate data regarding mortality for the covered group of people. Hybrid products, like annuity plus health or life insurance, can help to overcome both drawbacks. Other modifications of pure annuity contract, as behavioral analysis shows, for example an annuity with a capital protection guarantee or annuity with the partial withdrawal of funds upon premature death of the retiree, can help to eliminate the disadvantages of the contract in the mind of the investor.

The following table demonstrates the outcome of the research and the interpretation in the real world situation. Despite the fact that the model is quite theoretical, I also draw some possible recommendations for the implementation of the micro pension scheme.

**Table 7.** Alternative elements of the micro pension scheme and some recommendations

| Element | Theory: outcome of the research | Practice: Micro pensions | | Conclusion/ recommendation |
|---------|-------------------------------|-------------------------|---|---------------------------|
| | | **Advantages** | **Disadvantages** | |
| Defined Contribution scheme: direct investing | (Merton's contract) Is the most beneficial type of contract for regular investor because of providing no restrictions | No default risk for the pension fund; easy to implement – collecting the money and investing them as predefined | All the risk transferred to investor; not affordable for poor people who in most cases do not possess necessary knowledge | Too dangerous for the people with lower income, they need more guidance and protection from the risk |
| Defined Benefit scheme: adding minimum guarantee | (Implicit Put contract) Beneficial from the CPT perspective due to loss aversion | Underprivileged person is protected from the downside potential | More difficult to implement for MFI- needs large financial institution as a partner; can in practice be too expensive to implement | Overall preferable feature |
| Cutting upside potential | Performs worse than other contracts | Makes the contract cheaper for the investor and thus makes the guarantee more affordable | Similar to the previous: difficulties in implementation | Inconclusive |
| Annyitizing accumulated money (pure annuity) | Most beneficial from the *rational* point of view; not optimal within CPT framework | The only way to provide long term financial security | Not possible to leave money to the family upon death; no liquidity and control over the accumulated funds; requires accurate life tables | Strongly recommended option; requires risk-pooling mechanism and thus coverage of a group; requires hedging against rapid improvement in life expectancy |
| Lump sum contract | Optimal choice in the *mind* of average investor (CPT framework ) | Control over accumulated money; eliminates fear of the loss of the capital | Possible misuse of the funds; danger of making a wrong life prognosis and occurring of the "starving years" | Not optimal option: converts pension scheme into savings scheme, thus makes no sense for those who already are clients of micro savings scheme; |

| | | | | overall uncertainly ensures old age financial provision |
|---|---|---|---|---|
| Bequest option (inheritance) | By assumption is an important motivation (b=1) No bequest option pushes investor towards choosing annuity contract | Adding this feature eliminates one of the strongest arguments against annuitization, rings against loss aversion; makes the contract much more attractive in the mind of investor | Makes the contract more expensive as intervenes into risk pooling (early death doesn't leave all the capital the rest of the group) | Useful feature for the annuity contract; possibly to leave the share of the capital for the family in case of early death |
| Compulsory/ community subscription | Necessary for the annuity contract; in other cases is a tool to encourage savings behavior | In case of annuity eliminates adverse selection; encourages financial discipline | Not possible for the people with confirmed terminal illness; in a way limits a freedom of choice | In case of annuity contract needs to be introduced in some way; should come in its order after savings products |
| Hybrid product | Necessary in case of annuity, otherwise no hedge against improving average life expectancy | Can reduce the price of the package for the poor person; life insurance+annuity contract provides a natural hedge | Not always possible to subscribe the person for both contracts simply due to the lack of money. Can still be too expensive while otherwise the person would subscribe for at least one of them | Annuity and life insurance contract, either together from the same provider or hedging in the partnership with another MFI ("mortality swap") |
| Fixed term | No direct conclusion, existing schemes have this feature most of the time | Overcomes the problem with not known age of the people; makes calculations and implementation easier for the pension fund | Puts people of the different age in the same situation | Is not a crucial feature; makes the scheme easier to implement and thus recommended |

## 7. Further research suggestions

The research covers a period during investment phase and the payout phase, which means that the period of collecting the money is out of the scope. I assume that initial investment is made as a lump sum, and it is a very powerful simplification, especially in the context of micro pensions, as one of the biggest problems that the pension provider can face in real life may be an unorganized or irregular payments from the side of the clients. The ways to ensure a stable stream of money and encourage financial discipline among underprivileged people can be a very interesting and relevant extension to this research. One of the possible solutions could be introduction of the group liability, and there are enough practical examples of this system being used in micro crediting.

Inflation is left out of the scope of the research for two reasons: simplifying the model and the fact that in developing countries losses caused by inflation can in the long term still be lower than those associated with the physical assets. However, there are ways to incorporate inflation in the model, for example, by investing inflation-linked securities or purchasing commodities for part of the funds. Another approach could be to model inflation in the initial financial setting by adding one more source of risk and similar to the stochastic interest rate.

Another interesting and important part to investigate for the pension scheme is the age diversity in the group of subscribers. In the current research, the initial invested capital sum is fixed, which means that all the clients in the scheme are investing approximately the same amount of years. In other words, the scope of the research is covering only the group of people of the same age. In the real pension fund, however, new clients are constantly entering the scheme and in case of annuity or any other type of contract involving insurance elements, this group expansion needs to be taken into account.

I mention hybrid products only as one of the opportunities to overcome the difficulties with mortality tables for the pension provider and making the package of combined financial products cheaper for the client. This direction of product development requires additional research in the area of insurance, and was therefore left out of the scope of my thesis. I believe, however, that it is a way to move forward and implement such complicated product as pension for the people with lower income. It is therefore more certainly worse further investigation.

There might be more to explore in the Implicit Bull Spread contract. Its unique feature to make the pension cheaper for the clients by cutting the upside potential intuitively sounds very valuable in the micro pensions setting. An area to explore it further can be to optimize the relation between the lower and upper strikes and in this way to improve performance.

## 8. *Literature*

Basu, P., 2006. Improving access to finance for India's rural poor. The International Bank for Reconstruction and Development/ The World Bank.

Batra, A., 2004. "Stock Return Volatility Persistence in India: 1973-2003", Working paper, ICRIER, New Delhi, India.

Baxter,M., Rennie, A., 1996. Financial Calculus. An introduction to derivative pricing. Cambridge university press.

Bernard, C., Ghossoub, M., 2009. Static Portfolio Choice under Cumulative Prospect Theory. Munich Personal RePEc Archive.

Bjork, T., 1998. Arbitrage theory in continuous pricing. Oxford university press.

Blake, D., Cairns, A. J. G., Dowd, K., 2003. PensionMetrics 2: Stochastic Pension Plan Design During the Distribution Phase. Insurance: Mathematics and Economics 33 (2003) 29–47.

Brennan, M.J., Schwartz,E.S., 1976. The pricing of equity-linked life insurance policies with an asset value guarantee. Journal of Financial Economics 3 (1976) 195-213.

Brown J. F., 2000. Private pensions, mortality risk and the decision to annuitize. J.F.Kennedy School of Government and NBER.

Brown, J., Poterba, J.M., 2000. Joint Life Annuities and Annuity Demand by Married Couples. Journal of Risk and Insurance 67: 527-556.

Brown, J., Kling, J.R., Mullainathan, ., Wiens, G.R., Wrobel, M.V., 2009. Framing, reference points, and preferences for life annuities. TIAA-CREF Institute.

Butler, M., Teppa, F., 2007. The choice between an annuity and a lump sum: Results from Swiss pension funds. Journal of Public Economics 91 (2007) 1944-1966.

Cannon, E., Tonks, I., 2008. Annuity markets, p. 191. Oxford university press.

Chang, F-R., 2003. Life insurance, precautionary saving and contingent bequest. Mathematical Social Sciences 48, 55-67.

Chan, K., Karoly,A., Longstaff,F., Sanders,A., 1992. An impirical comparison of alternative models of the short-term interest rate. The Journal of Finance, Vol 47,1209-1227.

Cochrane, J. H., 2005. Asset pricing. Princeton university press.

Cox,J., Ingersoll,J., Ross, S., 1985. A theory of the term structure of interest rates. Econometrica, Vol 53, No2, 385-407.

Cox, S.H., Lin, Y. 2001. Natural hedging of life and annuity mortality risks. Department of Risk Management and Insurance.

Davies, G.B, Satchell, S.E., 2004. Continuous Cumulative Prospect Theory and Individual Asset Allocation.

Doskeland, T. M., Nordahl, H.A., 2008. Optimal pension insurance design. Journal of Banking and Finance 32, 382-392.

Dowla, A., Barua, D., 2006. The poor always pay back. Kumarian press.

Dus, I., Maurer, R., Mitchell, O.S., 2004. Betting on Death and Capital Markets in Retirement: A Shortfall Risk Analysis of Life Annuities versus Phased Withdrawal Plans. University of Michigan, Michigan Retirement Research Center.

Financial Access Initiative, 2009. How do poor people deal with risks? Lessons from portfolios for the poor: how the word's poor live on $2 a day. Wagner Graduate School.

Gianadda, S, 2007. Micropensions.

Han,B., Hsu, J. 2004. Prospect Theory and its Applications in Finance.

Hardy, M. 2003. Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance. John Wiley and Sons, Inc., Hoboken, New Jersey.

Hull, J. C., 2006. 6th edition. Option, futures and other derivatives. Pearson/Prentice Hall.

Islam, T., 2007. Microcredit and poverty alleviation. Ashgate publishing.

Jensen, B. A., Sorensen, C., 2001. Paying for minimum interest rate guarantees: Who should compensate who? European Financial Management 7, p. 183-211.

Kimball, S. M., Precautinary saving in the small and in the large. Econometrica, 58 (Jan.1990), p.53-73.

Kapur, S., Orszag, M., 1999. A portfolio approach to investment and annuitization during retirement. Working Paper. Birkbeck College, London.

Kotlikoff, L., Spivak, A., 1981. The Family as an Incomplete Annuity Market. Journal of Political Economy 89: 372-391.

Ledgerwood, J., 2000. Microfinance Handbook: an Institutional and Financial Perspective. The World Bank.

Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: the continuous-time case. Review of Economics and Statistics 51, 247-257.

Merton,R.C., 1983. On consumption-indexed public pension plans. In: Bodie, Z., Shoven, J. (Eds.), Financial Aspects of the US Pension Plan System, University of Chicago Press, Chicago.

Milevsky, M., 1998. Optimal asset allocation towards the end of the life cycle: to annuitize or not to annuitize? Journal of Risk and Insurance 65, 401–426.

Milevsky, M., Young, V., 2002. Optimal asset allocation and the real option to defer annuitization: It's not now or never. Working Paper. York University, Toronto, and University of Wisconsin-Madison.

Murtaugh, C., Spillman, B., Warshavsky, M., 2001. In sickness and health: An annuity approach to financing long-term care and retirement income. The Journal of Risk and Insurance, 68(2): 225-253.

Pellser, A., 2000. Efficient methods for valuing interest rate derivatives. Springer finance.

Protecting the poor. A microinsurance compendium. International Labor Organization, 2006. Munich Re Foundation.

Rieger, M.O., Wang, M., 2008. Prospect theory for continuous distributions. Risk Uncertainty (2008) 36:83–102.

Shankar,S., Asher, M. G., 2009. Developing micro pensions in India: issues and challenges. Submitted for consideration to Association for Public Policies and Management.

Tversky, A., Kahneman, D., 1992. Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and Uncertainty 5, 297–323.

Vasicek, O. An equilibrium characterization of the term structure. Journal of Financial Economics 5 (1977) 177-188.

Wright, G.A.N., Mutesasira, L., 2001. The relative risks to the savings of the poor people. MicroSave, Nairobi.

Yaari, M., 1965. Uncertain lifetime, life insurance and the theory of the consumer. Review of Economic Studies 32, 137–150.

# Appendix 1

## Some existing examples of pension schemes: India and Bangladesh

| Scheme | Structure | Conditions | Target group | Frequency / min amount | Distribution | Flexibility | Investments | Status/ coverage |
|---|---|---|---|---|---|---|---|---|
| **UTI Retirement Benefit Pension Fund micropension scheme** | DC scheme based on individual accounts | Voluntarily, funded. Lump sum, annuity or combination of the two (percentage, frequency and amount totally flexible). Ijoining at the age 18-55, investing until 55 (58 years). | Was designed by adjusting the "Retirement Benefit Pension Fund", an open-ended balanced fund, for low income investors. SEWA Bank: all clients including men and SHG | Monthly/starting from 50Rs to 2000Rs for SEWA and from 200Rs for Mann Deshi Bank; 1000Rs for SHEPHERD | SEWA Bank, Mann Deshi Bank, SHEPHERD (Self-Help-Group Promotion for Health and Rural development), Paradeep Port and Dock Labour Union | No sanctions apply in case participant can't make the monthly contribution; no entry load; investors not allowed to withdraw part of the fund, even in case of emergency; they can close the account and receive lump sum payment minus penalty | Maximum equity allocation 40%, balance in debt instruments. Dividends reinvested. During the payout phase the funds continue to be invested as in the accumulation phase. | about 100.000 individuals; SEWA Bank - Gujarat |
| **Invest India Micropension Services (IIMP) - advisor to NPS** | "national level micropensions marketplace" | A channel for low-income workers to invest retirement savings in the stock market via existing equity funds. Savings based market-linked scheme. | low-income workers (milk sellers, vendors and housewives linked to SEWA) | Whatever amount monthly | Network of MFIs, cooperatives and associations of low-income earners, non-members to be reached through "pilot" projects in partnership with government departments, post offices and NGOs | n/a | n/a | Via SEWA Bank: 45.000 in Ahmedabad, total coverage 100,000. |
| **SEWA bank in-house micropension product** | recurring deposit | A defined sum invested on a monthly basis for a fixed rate of return 7%. Deposit had a set tenure after which the customer retrieves its principal along with the interest | Old-age women | Monthly/starting from 30Rs | bank in-house product | n/a | n/a | more than 3.000 initially + 5.700 for the modified version |

| Mann Deshi Bank's plan of an in-house micropension product | planned as deposit | Investment period 10, 15, 20, 25 and 30 years. Floating interest rate system rather than a fixed rate. | Initially supposed to target wealthier clients | Monthly / 50-200Rs | bank in-house product | Each late payment would result in 1% penalty of amount outstanding. Premature redemptions were planned to result in an exit load. Early withdrawals only aloud with a penalty charge and in a limited number of circumstances | The bank was not allowed to invest the collected funds into securities (instead of regular loans) and thus could not obtain a higher return rate than on savings products. | Has never been implemented due to regulatory challenges |
|---|---|---|---|---|---|---|---|---|
| Activists of Social Alternatives (ASA) initiative | pension/social security scheme | Fixed amount deposit for a predefined number of years: 5, 10, 15, 20. Except for the 5-year plan all other plan provide pension for life and also lump sum bonus: once 1000Rs for 10-year plan, twice 1000Rs for 15 | Members of ASA | 10 Rs weekly or 45 Rs monthly | n/a | Can be taken in a lump sum or monthly installments | Grameen model: the funds are used to provide microcredits to its members | Research and development of the project |

| Scheme | Structure | Conditions | Target group | Frequency / min amount | Distribution | Flexibility | Investments |
|---|---|---|---|---|---|---|---|
| *Grameen pension products* | | | | | | | |
| Grameen I | deposit | Compulsory for all group members; savings collected on group account; | n/a | Daily from the first week/1 taka | Grameen's network | Funds not freely accessible. Members could borrow from the collective fund after the group approval with 5% tax. Free access in case of dropping out. | Various use of savings funds, including repayment of loans with proceeds from the group fund |

| | | | | | | |
|---|---|---|---|---|---|---|
| **Grameen Pension Scheme (GPS)** | pension | Compulsory for mature Grameen bank members. For new members only if borrowing more than 8000 taka. 10 and 5 year programs only, roll-over possible. 12% and 10% yearly interest rate respectively; accumulated principal and interest released as a lump sum or as monthly income. | Members of the Grameen bank, employees of the bank | Monthly/100 or more taka for mature members ("Green" GPS); 50 taka for members borrowing 5000 taka or less ("Red" GPS) | Grameen's network | If the borrower leaves the bank, she can still maintain the GPS with the bank. In case of failing to deposit committed installments regularly, the account is closed and the interest is paid at the reduced rate. | n/a |
| **Personal Savings Account** | deposit | Voluntarily; savings collected on individual account until the group becomes recognised by the bank. Savings amount positively linked to loan size: loan ceiling increases with the savings size. 8.5% interest | Members of Grameen bank, general public | Daily/ 5 taka until the group becomes formally recognised by the bank; afterwards various amount depending on the size of the loan, along with 2.5% of all loans | Grameen's network; deposits made at weekly "center" meetings, withdrawals made at branch offices | Accessible at will | Insurance for the member, distributing loans to other members |
| **Special savings account** | deposit | Voluntarily;  8.5% interest | Members of Grameen bank | 2.5% of all loans | Grameen's network | Until three years old is illiquid; after 3 years withdrawals can be made subject to minimun balance 2000 taka | Insurance for the member, distributing loans to other members |

**Comments to Appendix 1**

There are few examples of the microfinance institutions and banks implementing savings or deposit schemes which resemble micro pensions. These schemes have few main common features.

- Most of the schemes are constructed for the determined amount of years (e.g. 10,15,20,25,30) for two reasons:

1) many customers do not know their age and do not have any official document stating it;

2) determining the size of annuity depending on the amount of years till retirement can be too complicated and confusing for the clients.

- The frequency of payments is monthly or weekly: in case of less frequent, like yearly, missed or delayed payments would influence the pension to a great extent; also this frequency encourages more discipline from the side of clients in their repayment periodicity.

- The minimum amount invested depends on a large extend on the distributor's structure and transaction costs (a good example is the difference between SEWA and Mann Deshi Banks subscribers for the same UTI MF scheme – 50 and 200 Rs minimum respectively).

- There are currently no pension schemes proposing a guarantee of any sort for the clients (if invested not with fixed interest rate). This fact is due to the difficulties of adequate annuity size estimation, and seems to be a matter of big concern for the clients.

- Making the scheme mandatory or voluntarily is a matter of an open discussion. Quite some problems are observed with voluntary contributions due to lack of discipline among investors and sometimes family pressure (wide-spread in case of women - they are expected rather to give money away for the common household or to lend them to relatives). Due to these facts Grameen Bank, for example, made their pension scheme subscription mandatory in order to receive a loan of a certain size.

- Early withdrawal or misuse of the collected funds is also a wide-spread problem. Possible solutions vary from complete forbidding using the funds (unless closing the account) or significant penalties, to providing the loans secured by deposit size in order to satisfy immediate and crucial financial needs.

Appendix 1 is based on the following sources: Gianadda (2007), Fowla and Barua (2006), D. Uthira and H.L. Manohar, "Economic Implications and Sustainability of Micropensions in the Era of Pension Reforms in India".

Internet sources:

www.iimp.in,

http://www.asadev.com/publication/ratingreport/asa-mcrilratingreport05.pdf

# Appendix 2

## Formal definition of Brownian motion

The definition of the Brownian Motion can be found, for example, in Baxter and Rennie (1996).

The process $W = W_t^{\mathbb{Q}}, t \geq 0$ is called Brownian Motion under measure $\mathbb{Q}$ if it satisfies the following conditions:

I.   $W_t$ is continuous and $W_0 = 0$

II.  $W_t \sim N(0, t)$

III. Its increments are normally distributed $W_{s+t} - W_s \sim N(0, t)$ under measure $\mathbb{Q}$, and are independent of the behavior of the process up to time *s*.

# Appendix 3

## *Derivation of bond and stock equations*

The form of the bond process can be derived from equation (1).

$$dD_t = rD_t dt$$

$$\frac{dD_t}{D_t} = r dt$$

$$\ln(D_t) = rt + c$$

$$D_t = \exp(rt + c)$$

$$D_t = D_0\, e^{rt}$$

I need to find a solution for stochastic differential equation (2) for the stock price

$$dS_t = \mu S_t dt + \sigma S_t dZ_t^P$$

$$S_0 = D_0 = 1$$

It has the following form:

$$S_t = S_0 \exp\left( (\mu - \frac{\sigma^2}{2})t + \sigma Z_t^P \right) \tag{30}$$

I use Ito's formula to verify that (30) is indeed a solution.

For the stochastic process $X_t$ satisfying

$$dX_t = \mu_t dt + \sigma_t dZ_t$$

and $f(X_t)$ being a deterministic twice continuously differentiable function, Ito's formula looks the following (see Baxter and Rennie (1996)):

$$d\left(f(X_t)\right)=\left(\sigma_t f'(X_t)\right)dZ_t^P+\left(\mu_t f'(X_t)+0.5\sigma_t^2 f''(X_t)\right)dt$$

In case of a stock the function

$$f(X_t)=\exp(X_t)$$

where

$$dX_t=\mu_1 dt+\sigma dZ_t^P \qquad (31)$$

$$\mu_1=\mu-\frac{\sigma^2}{2}$$

$$f'(X_t)=f''(X_t)=f(X_t)=S_t$$

Indeed,

$$d\left(f(X_t)\right)=\left(\sigma f'(X_t)\right)dZ_t^P+\left(\mu_1 f'(X_t)+\frac{\sigma^2}{2}f''(X_t)\right)dt=$$

$$=(\sigma S_t)dZ_t^P+(\mu_1 S_t+\frac{\sigma^2}{2}S_t)dt=\sigma S_t dZ_t^P+\left((\mu-\frac{\sigma^2}{2})S_t+\frac{\sigma^2}{2}S_t\right)dt=$$

$$=\mu S_t dt+\sigma S_t dZ_t^P$$

## *Derivation of asset portfolio dynamics equation (under P)*

**a)** Firstly, let's look again at the portfolio dynamics definition (eq. (4))

$$dA_t=A_t\left(\theta\frac{dS_t}{S_t}+(1-\theta)\frac{dD_t}{D_t}\right)$$

$$\frac{dS_t}{S_t}=\mu dt+\sigma dZ_t^P$$

$$\frac{dD_t}{D_t}=rdt$$

$$\Rightarrow dA_t=A_t\theta(\mu dt+\sigma dZ_t^P)+A_t(1-\theta)rdt$$

$$dA_t=A_t\left((\theta\mu+(1-\theta)r)dt+\theta\sigma dZ_t^P\right)$$

$$dA_t=A_t\left((r+\theta(\mu-r))dt+\theta\sigma dZ_t^P\right)$$

**b)** I find solution for the equation (5) for the assets dynamics under real world measure P in the way similar to stock process.

The solution becomes obvious after denoting

$$r + (\mu - r)\theta = \mu_A$$
$$\theta\sigma = \sigma_A$$

Then the differential equation (5) takes the form similar to (31) with the notation difference

$$dA_t = A_t \left( \mu_A dt + \sigma_A dZ_t^P \right)$$

and thus the solution to it is

$$A_t = A_0 \exp\left( (\mu_A - \frac{\sigma^2}{2})dt + \sigma_A Z_t^P \right)$$

Replacing $\mu_A$ and $\sigma_A$

$$A_t = A_0 \exp\left( \left( r + (\mu - r)\theta - \frac{\sigma^2\theta^2}{2} \right)dt + \sigma\theta Z_t^P \right)$$

# Appendix 4

## Choice of numeraire and fair pricing principle

This Appendix is based on the following sources: Baxter and Rennie (1996), p.74-78 and Pelsser (2000), p. 11-14.

The fair pricing principle is done under the measure $\mathbb{Q}$, which is the only measure which turns the relative price process for the stock into martingale. The original stock price process will then have a form:

$$dS_t = S_t \left( \mu dt + \sigma(dZ_t^{\mathbb{Q}} - \frac{\mu - r}{\sigma} dt) \right) = rdt + \sigma dZ_t^P$$

where the change of drift is made based on Girsanov's theorem. The assets portfolio dynamics equation will also change form.

$$dA_t = A_t \left( \theta \frac{dS_t}{S_t} + (1-\theta)\frac{dD_t}{D_t} \right)$$

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t^P$$

$$\frac{dD_t}{D_t} = rdt$$

$$dA_t = A_t \left( \theta(rdt + \sigma dZ_t^P) + (1-\theta)rdt \right)$$

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$$dA_t = A_t \left( rdt + \theta\sigma dZ_t^{\mathbb{Q}} \right)$$

# Appendix 5

**Derivation of residual parameter δ for the three contract types under fixed interest rate**

As fair pricing is done under the risk neutral measure $\mathbb{Q}$, here and further expectation is taken under this measure and the index is dropped for the notation simplicity.

Calculations in this part are based on the following formulas:

i) Fair pricing equation

$$L_0 = \mathrm{E}\left[e^{-rT}L_T\right]$$
$$L_0 = e^{-rT}\mathrm{E}\left[L_T\right]$$
$$L_0 e^{rT} = \alpha A_0 e^{rT} = \mathrm{E}\left[L_T\right]$$

ii) Equation for the assets portfolio under measure

$$A_T = A_0 \exp\left((r - \frac{\sigma_A^{\,2}}{2})T + \sigma_A \sqrt{T}z\right)$$

iii) The mean value of the random normal variable $X \sim N(\mu, \sigma)$ can be calculated as

$$\mathrm{E}[X] = \int_{-\infty}^{+\infty} y \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

If $g : \mathbb{R} \to \mathbb{R}$ is an integrable function, we can calculate the expected value of with the following formula:

$$\mathrm{E}[Z] = \mathrm{E}[f(X)] = \int_{-\infty}^{+\infty} f(y) \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy \qquad (32)$$

iv) The fact that normal distribution is symmetric.

### 1. *Implicit Put*

$$L_T = L_0 e^{gT} + \alpha\delta_{ImplPut}\left(A_T - \frac{1}{\alpha}L_0 e^{gT}\right)^{+}$$

Taking expectation

$$E[L_T] = E\left[ L_0 e^{gT} + \alpha \delta_{ImplPut} \left( A_T - \frac{1}{\alpha} L_0 e^{gT} \right)^+ \right] =$$

$$= \alpha A_0 e^{gT} + E\left[ \alpha \delta_{ImplPut} \left( A_T - \frac{1}{\alpha} L_0 e^{gT} \right)^+ \right] =$$

$$= \alpha A_0 e^{gT} + E\left[ \alpha \delta_{ImplPut} \left( A_0 \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - A_0 \exp(gT) \right)^+ \right] =$$

$$= \alpha A_0 e^{gT} + \alpha \delta_{ImplPut} A_0 E\left[ \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - \exp(gT) \right)^+ \right]$$

$$\Rightarrow \delta_{ImplPut} = \frac{\alpha A_0 e^{rT} - \alpha A_0 e^{gT}}{\alpha A_0 E\left[ \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - \exp(gT) \right)^+ \right]}$$

$$\delta_{ImplPut} = \frac{e^{rT} - e^{gT}}{E\left[ \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - \exp(gT) \right)^+ \right]}$$

$$\delta_{ImplPut} = \frac{1 - e^{gT - rT}}{E\left[ \left( \exp\left( (-\frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - \exp(gT - rT) \right)^+ \right]} \tag{33}$$

Let's look into the expression in the denominator.

$$E\left[ \left( \exp\left( (-\frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - \exp(gT - rT) \right)^+ \right] = \int_{-\infty}^{+\infty} g(z) \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz \tag{34}$$

where $g(z) = \exp\left( (-\frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - \exp(gT - rT)$ and $z \sim N(0,1), \mu = 0, \sigma^2 = 1$

and expression under integral in (34) exists only for $g(z) \geq 0$.

$$g(z) \geq 0 \Leftrightarrow \exp\left((-\frac{\sigma_A^2}{2})T + \sigma_A\sqrt{T}z\right) - \exp(gT - rT) \geq 0$$

$$\exp\left((-\frac{\sigma_A^2}{2})T + \sigma_A\sqrt{T}z\right) \geq \exp(gT - rT)$$

$$(-\frac{\sigma_A^2}{2})T + \sigma_A\sqrt{T}z \geq gT - rT$$

$$z \geq \frac{(g - r + \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}} = -d_2$$

$$d_2 = \frac{(r - g - \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}$$

Denote
$$d_1 = d_1 + \sigma_A\sqrt{T} = \frac{(r - g + \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}$$

Coming back to (34)

$$\int_{-\infty}^{+\infty} g(z)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz = \int_{-d_2}^{\infty}\left(\exp\left((-\frac{\sigma_A^2}{2})T + \sigma_A\sqrt{T}z\right) - \exp(gT - rT)\right)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz =$$

$$= \underbrace{\int_{-d_2}^{\infty}\exp\left((-\frac{\sigma_A^2}{2})T + \sigma_A\sqrt{T}z\right)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz}_{(A)} - \underbrace{\int_{-d_2}^{\infty}\exp(gT - rT)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz}_{(B)}$$

Evaluating first and second integral separately,

A)

$$\int_{-d_2}^{\infty}\exp\left((-\frac{\sigma_A^2}{2})T + \sigma_A\sqrt{T}z\right)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz = \int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(\sigma_A\sqrt{T}z + (-\frac{\sigma_A^2}{2})T + \frac{z^2}{2}\right)dz$$

$$= \int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z - \sigma_A\sqrt{T})^2\right)dz =$$

$$\begin{bmatrix} v = z - \sigma_A\sqrt{T} \\ dv = dz \\ z = -d_2 \Rightarrow v = -d_2 - \sigma_A\sqrt{T} = -d_1 \end{bmatrix}$$

$$= \int_{-d_1}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv - \int_{-\infty}^{-d_1}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv =$$

$$= 1 - N(-d_1) = N(d_1)$$

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B)
$$\int_{-d_2}^{\infty} \exp(gT-rT)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz = \exp(gT-rT)\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz =$$

$$= \exp(gT-rT)(1-N(-d_2)) = \exp(gT-rT)N(d_2)$$

Coming back to (33),

$$\delta_{ImplPut} = \frac{1-e^{gT-rT}}{N(d_1)-e^{gT-rT}N(d_2)}$$

## 2. Simple Insurance contract

The payoff of the contract looks the following :

$$L_T = A_T - \left(A_T - L_0 e^{gT}\right)^+ + \alpha\delta_{ins}\left(A_T - \frac{1}{\alpha}L_0 e^{gT}\right)^+$$

By analogy with the Implicit Put case

$$\alpha A_0 e^{rT} = E[L_T] = E\left[A_T - \left(A_T - L_0 e^{gT}\right)^+ + \alpha\delta_{ins}\left(A_T - \frac{1}{\alpha}L_0 e^{gT}\right)^+\right] =$$

$$= A_0 e^{rT} - E\left[\left(A_0\exp\left((r-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha A_0 e^{gT}\right)^+\right] +$$

$$+\alpha\delta_{ins}E\left[\left(A_0\exp\left((r-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\frac{1}{\alpha}\alpha A_0 e^{gT}\right)^+\right]$$

$$\alpha e^{rT} = e^{rT} - E\left[\left(\exp\left((r-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT}\right)^+\right] +\alpha\delta_{ins}E\left[\left(\exp\left((r-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\frac{1}{\alpha}\alpha e^{gT}\right)^+\right]$$

$$\alpha -1 = \alpha\delta_{ins}E\left[\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-e^{gT-rT}\right)^+\right] - E\left[\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}\right)^+\right]$$

$$\delta_{ins} = \frac{\alpha -1+\overbrace{E\left[\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}\right)^+\right]}^{(C)}}{\underbrace{\alpha E\left[\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-e^{gT-rT}\right)^+\right]}_{(D)}}$$

I calculate expression (C) while (D) was obtained in Implicit Put contract.

$$E\left[\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}\right)^+\right]=\int_{-\infty}^{+\infty}g(z)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz=$$

$$=\int_{-\infty}^{+\infty}\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}\right)^+\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz \qquad (35)$$

$$g(z)=\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}$$

$$g(z)\geq 0 \Leftrightarrow \exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)\geq\alpha e^{gT-rT}\Leftrightarrow$$

$$\Leftrightarrow (-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\geq T(g-r)+\ln\alpha \Leftrightarrow$$

$$\Leftrightarrow z\geq\frac{(g-r+\frac{\sigma_A^2}{2})T-\ln\alpha}{\sigma_A\sqrt{T}}=-d_2{}'$$

$$d_2'=-d_1'-\sigma_A\sqrt{T}=\frac{(g-r-\frac{\sigma_A^2}{2})T-\ln\alpha}{\sigma_A\sqrt{T}}$$

$$d_1'=-d_2'-\sigma_A\sqrt{T}=\frac{(r-g-\frac{\sigma_A^2}{2})T+\ln\alpha}{\sigma_A\sqrt{T}}$$

Coming back to expression (35)

$$\int_{-\infty}^{+\infty}\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}\right)^+\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz=$$

$$=\int_{-d_2'}^{+\infty}\left(\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)-\alpha e^{gT-rT}\right)^+\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz=$$

$$=\int_{-d_2'}^{+\infty}\exp\left((-\frac{\sigma_A^2}{2})T+\sigma_A\sqrt{T}z\right)\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz-\int_{-d_2'}^{+\infty}\alpha e^{gT-rT}\frac{1}{\sqrt{2\pi}}\exp(\frac{z^2}{2})dz=$$

$$=N(d_1')-\alpha e^{gT-rT}N(d_2')$$

$$\Rightarrow \delta_{ins}=\frac{\alpha-1+N(d_1')-\alpha e^{gT-rT}N(d_2')}{\alpha\left(N(d_1)-e^{gT-rT}N(d_2)\right)}$$

### 3. Implicit Bull Spread contract

The payoff of the contract (equation (13)(13)):

$$L_T = L_0 e^{g_1 T} + \alpha \delta_{bull} \left( A_T - \frac{1}{\alpha} L_0 e^{g_1 T} \right)^+ - \alpha \delta_{bull} \left( A_T - \frac{1}{\alpha} L_0 e^{g_2 T} \right)^+$$

Where $g_1 < r < g_2$

$$\alpha A_0 e^{rT} = E\left[ L_0 e^{g_1 T} + \alpha \delta_{bull} \left( A_T - \frac{1}{\alpha} L_0 e^{g_1 T} \right)^+ - \alpha \delta_{bull} \left( A_T - \frac{1}{\alpha} L_0 e^{g_2 T} \right)^+ \right] =$$

$$= \alpha A_0 e^{g_1 T} + E\left[ \alpha \delta_{bull} \left( A_T - \frac{1}{\alpha} L_0 e^{g_1 T} \right)^+ - \alpha \delta_{bull} \left( A_T - \frac{1}{\alpha} L_0 e^{g_2 T} \right)^+ \right]$$

$$A_0 e^{rT} = A_0 e^{g_1 T} + \delta_{bull} E\left[ \left( A_T - A_0 e^{g_1 T} \right)^+ - \left( A_T - A_0 e^{g_2 T} \right)^+ \right]$$

$$A_0 \left( e^{rT} - e^{g_1 T} \right) = \delta_{bull} E\left[ \underbrace{\left( A_T - A_0 e^{g_1 T} \right)^+}_{(F)} - \underbrace{\left( A_T - A_0 e^{g_2 T} \right)^+}_{(G)} \right]$$

$$\delta_{bull} = \frac{A_0 \left( e^{rT} - e^{gT} \right)}{E\left[ (F) - (G) \right]} \tag{36}$$

Because $g_1 < r < g_2$, the following holds:

If $A_T > A_0 e^{g_2 T}$, both *(F)>0* and *(G)>0*, if $A_T < A_0 e^{g_1 T}$ both *(F)<0* and *(G)<0* and for $A_0 e^{g_1 T} \le A_T \le A_0 e^{g_2 T}$ $(F) \ge 0$ and $(G) \le 0$, besides that $(F) \ge (G)$, thus

$$E\left[ (F) - (G) \right] =$$

$$= \int_{-\infty}^{+\infty} \left[ \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - e^{g_1 T} \right)^+ - \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - e^{g_2 T} \right)^+ \right] \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz =$$

$$= \int_{-\infty}^{+\infty} \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - e^{g_1 T} \right)^+ \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz -$$

$$- \int_{-\infty}^{+\infty} \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - e^{g_2 T} \right)^+ \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz$$

where expressions under integral are analogous to the case of Implicit Put contract.

Rewriting (36) in the form

$$\delta_{bull} = \frac{A_0 \left( 1 - e^{gT - rT} \right)}{e^{-rT} E\left[ (F) - (G) \right]}$$

I can immediately (by analogy) write a solution:

$$\int_{-\infty}^{+\infty} \left( \exp\left( (-\frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - e^{g_1 T - rT} \right)^+ \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz = N(d_1) - e^{g_1 T - rT} N(d_2)$$

Where

$$d_1'' = \frac{(r - g_1 + \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}$$

$$d_2'' = \frac{(r - g_1 - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}$$

$$\int_{-\infty}^{+\infty} \left( \exp\left( (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} z \right) - e^{g_2 T} \right)^+ \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz = N(d_3) - e^{g_2 T - rT} N(d_4)$$

$$d_3 = \frac{(r - g_1 + \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}$$

$$d_4 = \frac{(r - g_1 - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}$$

Thus

$$\delta_{bull} = \frac{1 - e^{g_1 T - rT}}{N(d_1'') - e^{g_1 T - rT} N(d_2'') - N(d_3) + e^{g_2 T - rT} N(d_4)}$$

# Appendix 6

## Expected utility calculation with conventional parameters and analytical and numerical parameter δ

Expected utility calculation for the conventional values of the parameters (parameters value is based on Doskeland and Nordahl (2007)). The optimal asset allocation is reached at 37% investment in equity, which equals to the theoretical value $\theta = \dfrac{\mu - r}{\gamma \sigma^2}$. This means that for the risk aversion coefficient smaller than one in the current set of parameters the optimal strategy includes leveraging of the stock.
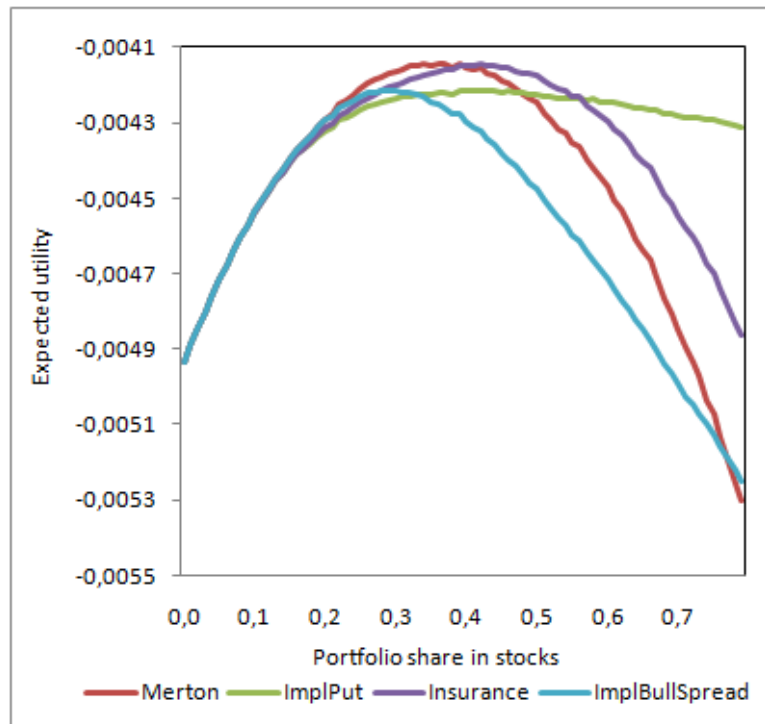


Expected utility under power utility function for four contracts: Merton's, Implicit put, Simple insurance and Implicit bull spread for positive portfolio share in stocks and convention values of the parameters. Parameter values: Initial wealth $A_0 = 5$, interest rate r=4%, expected stock return μ=0.065, volatility of the stock σ=0.15, time until retirement T=20 years, risk aversion γ=3, α=0.9, Implicit Put minimum guarantee rate g=2%, Implicit Bull Spread minimum guarantee rate $g_1 = 2\%$ , maximum rate $g_2 = 6\%$