ERASMUS UNIVERSITEIT ROTTERDAM

Erasmus School of Economics

Thesis MSc Quantitative Finance

November 3, 2022

Duration Matching in Bond Portfolio Optimization with Interest Rate Swaps

Author: Sven Heeren, 499741sh

Nationale-Nederlanden Supervisors:

drs. Pieter Bouwknegt

drs. Thijs Feenstra

University Supervisor:

dr. Hanno Reuvers

Second Assessor:

dr. Rasmus Löhn

Abstract

This paper introduces duration matching and interest rate swaps to the static bond portfolio optimization problem with government and corporate bonds. It extends the research of Caldeira et al. (2016) and Deguest et al. (2018) on bond portfolio optimization. The bond portfolio problem is highly relevant to insurance companies. In 2021, 82% of the total investment assets in the life insurance business of Nationale-Nederlanden were fixed-income assets. Duration matching is the most popular strategy to minimize the portfolio's interest rate risk. The minimum-concentration, mean-variance and maximum Sharpe ratio portfolios are considered. Constraints to match the duration of assets and liabilities of an insurance company and to limit the required collateral on interest rate swaps are included. This paper finds a positive effect of including interest rate swaps and corporate bonds in the portfolio on portfolio performance. On top of that, optimized portfolios using bond return moments outperform the benchmark minimum-concentration portfolio when a duration constraint is present.

List of Abbreviations

AFNS	Arbitrage-free Nelson-Siegel
CB	Corporate bond
DNS	Dynamic Nelson-Siegel
ENC	Effective number of constituents
EURIBOR	Euro interbank offered rate
FP	Fractional programming
GB	Government bond
IRS	Interest rate swap
LIBOR	London interbank offered rate
MC	Minimum-concentration (portfolio)
MSR	Maximum Sharpe ratio (portfolio)
MV	Mean-Variance (portfolio)
QCQP	Quadratically constrained quadratic program
QP	Quadratic programming
SOCP	Second-order cone program
SR	Sharpe ratio
VAR	Vector autoregression
ZCB	Zero coupon bond

Contents

1 Introduction

Appropriate risk management is a crucial part of managing the assets and liabilities of an insurance company. In that way, an insurance company ensures satisfying the obligations toward its clients, regulators and other stakeholders. Interest rate risk is one of the main types of risk concerning a balance sheet manager. Interest rate risk is the risk of losses on the balance sheet because of the sensitivity of both the assets and liabilities to interest rate changes. The leading measure of interest rate risk is duration. The duration is the price sensitivity of an instrument to changes in the interest rate. Redington (1952) introduced the duration matching strategy to mitigate the interest rate risk. It entails matching the duration of the assets and liabilities of the insurance company. Duration matching assures that the decline in the value of the assets due to increasing interest rates equals the decline in the liabilities' value. Likewise, the rising liabilities due to falling interest rates are accompanied by similarly rising asset values. According to van Bragt et al. (2010), duration matching is a more efficient strategy to mitigate interest rate risk compared with others in terms of having a higher return and lower risk. Therefore, analysing the problem of duration matching on the asset side of the balance sheet in a portfolio optimization problem is one of the main goals of this paper.

Ever since the groundbreaking paper by Markowitz (1952) on mean-variance portfolio optimization, a lot has been written in the literature regarding portfolio optimization in an equity framework. However, the literature on fixed-income portfolio optimization is not as extensive as on equity portfolio optimization due to several inherent problems in using optimized fixed-income portfolios. Nevertheless, the fixed-income portfolio optimization problem plays a significant role in insurance businesses. For example, in 2021, 82% of the total investment assets in the life insurance business of Nationale-Nederlanden were fixed-income assets, whereas only 16% were non-fixed income assets (Nationale-Nederlanden Levensverzekering Maatschappij N.V., 2022). The major problem in optimized bond portfolios is the non-stationarity of bond returns, mainly caused by the decreasing variance over time due to the decreasing time to maturity of the bonds. This problem complicates the estimation of the inputs in the optimized portfolios: the expected return vector and covariance matrix. One method to overcome this problem is estimating these moments by term structure models as done by Caldeira et al. (2016). The incorporation of the duration matching problem in the bond portfolio optimization is an area visited even less in the current literature. In recent years, starting with Deguest et al. (2018), the research on the bond portfolio problems with duration constraints has taken off but is still very limited.

Insurance companies use interest rate swaps to match the duration of their assets and long-term liabilities. For example, the life insurance business of Nationale-Nederlanden mainly uses interest rate swaps to hedge interest rate risk in managing the asset and liability portfolios according to Nationale-Nederlanden Levensverzekering Maatschappij N.V. (2022). The current academic literature, however, does not consider interest rate swaps in bond portfolio optimization problems. Liquidity risk to the insurer arises using interest rate swaps due to swap collateralization, as captured in the technical standards by ESMA (2014). The swap collateralization reduces the credit risk for the counterparty involved in the swap agreement. The liquidity risk is a consequence of changes in the required collateral needed in cash following swap value changes. The possibility that the insurance company cannot meet its collateral requirements due to insufficient liquid funds poses a liquidity risk. Therefore, this liquidity risk must also be considered when choosing the number of interest rate swaps in the portfolio optimization problem. Therefore, in this paper, interest rate swaps and collateral requirements are explicitly considered in the portfolio optimization framework to investigate the influence of interest rate swaps on portfolio performance.

In this paper, the mean-variance and maximum Sharpe ratio portfolios are included as optimized portfolios to investigate their performance compared to the equivalent of the equally-weighted portfolio in the presence of constraints (minimum-concentration portfolio). These portfolios include government bonds, a proxy for corporate bonds and interest rate swaps. One constraint incorporates a target duration to match the duration of assets and liabilities. Another limits the expected required collateral on the interest rate swaps in the next period. The first two bond moments are estimated by the correlated and uncorrelated versions of the dynamic Nelson-Siegel and the arbitrage-free Nelson-Siegel model.

The main finding is that incorporating interest rate swaps and corporate bonds besides government bonds in the static portfolio optimization problem with a duration constraint improves portfolio performance. Hence, in practice, these instruments should be considered as well. Furthermore, the application of optimized portfolios with bond return moments in the objective function is favourable to the investor over the benchmark minimum-concentration portfolio in terms of the Sharpe ratio. In times of sharp interest rate rises, however, the relatively stable minimum-concentration portfolio could benefit the investor by limiting the losses in excess portfolio return. The use of interest rate swaps and corporate bonds is also recommended to limit portfolio losses in these times.

The paper is organized as follows. Section 2 discusses relevant academic literature. The government bond, corporate bond and interest rate swap data are consequently discussed in Section 3. Section 4 describes the different aspects of the portfolio framework, including the portfolios, the different underlying term structure models and the constraints. The performance and behaviour of the portfolios and parameter values are analysed in Section 5. Finally, the paper is concluded in Section 6.

2 Literature Review

The two most influential papers on the bond portfolio optimization problem are Korn and Koziol (2006) and Puhle (2008). Both find promising results in using optimized portfolios requiring the first two moments of bond returns. The estimates of the moments of bond returns are obtained through term structure models. Puhle (2008) believes that, in practice, optimized portfolios are better at dealing with fixed-income risks than the ad hoc approaches mostly applied in practice. However, both papers do not include interest rate swaps and do not consider the duration matching problem.

The main paper focusing on the bond portfolio optimization problem with the presence of duration constraints is Deguest et al. (2018). They introduce multiple portfolio strategies incorporating a duration constraint, such as the maximum Sharpe ratio, the global-minimum variance and the benchmark minimum-concentration portfolio. The duration constraint uses the modified duration of the portfolio. Deguest et al. (2018) find that using portfolio optimization techniques improves the outcome for the investor when compared to ad hoc methods with the presence of duration constraints. However, they do not use term structure models to estimate the moments of bond returns but use relationships between risk-and-return parameters for different bonds consistent with the absence of arbitrage. As term structure models are more popular to use in practice, the practical use of their methods is questionable. Widyatantri and Husodo (2020) also find that optimized portfolios outperform the benchmark on a portfolio consisting of government bonds from Asian emerging markets, like Indonesia, Korea and Thailand. On top of that, Caldeira et al. (2016) find that dynamic term structure models simplify the bond portfolio optimization problem because of the closed-form of the mean and variance in these models. They find a superior performance of optimized mean-variance bond portfolios with term structure models over traditional bond portfolios.

Caldeira et al. (2016) use three different term structure models in their empirical application: the dynamic Nelson-Siegel model as introduced by Diebold and Li (2006), the arbitrage-free Nelson-Siegel model as introduced by Christensen et al. (2011) and a standard Gaussian dynamic term structure model proposed by Joslin et al. (2011). The best term structure model depends on the target duration and the specification of the portfolio optimization problem in general. The literature on term structure models is extensive, and much more models than the aforementioned are available. For example, the multifactor Vasicek-type term structure models in the bond portfolio optimization in Korn and Koziol (2006), the widely used Cox-Ingersoll-Ross model by Cox et al. (1985) and the flexible Hull-White model by Hull and White (1990) could also be considered. In this paper, however, only the dynamic Nelson-Siegel and the arbitrage-free Nelson-Siegel model are considered as these are the most popular in practice.

Compared to existing literature, this paper additionally incorporates interest rate swaps and accompanying collateral into the portfolio optimization problem. Substantial research has focused on modelling interest rate swap yields and swap spreads, for example, Duffie and Singleton (1997) and Grinblatt (2001). However, incorporating interest rate swaps into the portfolio optimization framework is not considered in the academic literature. Regarding the problem of collateral requirements and liquidity risk, Flockermann et al. (2020) propose using stochastic processes and a Monte Carlo approach, which is the starting point of tackling that problem in this paper. Nevertheless, the collateral requirements have not been incorporated previously in the portfolio optimization literature.

3 Data

3.1 Dutch Government Bonds

Dutch government bonds are very useful in enhancing the duration of the portfolio. They are also generally less volatile relative to other financial instruments. However, bonds usually generate a lower return than other financial products. The data on the Dutch government bond yields are obtained from the Datastream database incorporated into the Eikon environment.[1] The maturities are 6 months and 2, 5, 10, 20 and 30 years. These are chosen to have a wide dispersed range of government bonds, as the duration of liabilities is not known a priori. Therefore, having a wide

[1] Accessed through the Erasmus Data Service Centre: `https://www.eur.nl/en/library/erasmus-data-service-centre/databases/eikon-datastream`.

range of constituents is advised, because the best choice would depend on the target duration. However, using too many instruments slows down the optimization procedure. Monthly data from February 2011 to June 2022 (137 observations) is obtained. Figure 1 displays the government bond yield series. It is apparent that, in the first six months of 2022, the interest rates suddenly rise substantially as opposed to mainly decreasing or stable and low interest rates in the decade before. However, rising interest rates generally are accompanied by negative bond returns, hence, a poor bond portfolio performance. Therefore, the portfolio performance in the first months of 2022 is analysed to examine the effect of sharp interest rate rises. Table 1 displays the average return, standard deviation and modified duration of the government bonds. The average return of the government bond increases when the maturity of the bond increases but the standard deviation increases as well.

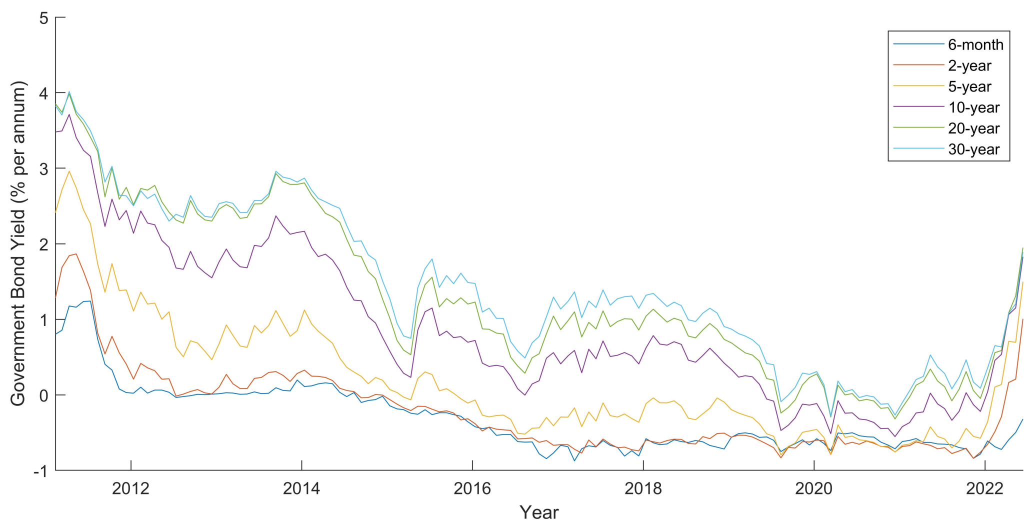


Figure 1: Dutch monthly government bond yields for different times to maturity from February 2011 to June 2022.

On top of these yields, the 3-month Dutch government bond yields are obtained from the Eikon environment. The 3-month Dutch government bond yield is a proxy for the risk-free rate. The computation of excess returns requires the risk-free rate. In empirical research, the 3-month Treasury bill rate is a convenient proxy for the risk-free rate.

3.2 Interest Rate Swaps

Insurers often employ interest rate swaps with the purpose of duration matching. The advantage of an interest rate swap is that it does not require an initial investment but still has a benefit in achieving a target duration. The purpose of interest rate swaps in portfolio optimization is to increase the portfolio duration because the duration of liabilities of insurance companies is often substantial. Therefore, long-term swaps are especially suitable. This research uses the 6-month EURIBOR interest rate swaps with 30/360 day count convention with maturities of 10, 20, 30, 40 and 50 years. The monthly data for these interest rate swap rates are accessed via the Datastream database in the Eikon environment from February 2011 to June 2022. In computations, the 6-month EURIBOR is unnecessary. Only the swap rates and the zero-coupon government bond rates modelled in Section 4.1 are required. These are used to compute the expected value of the swaps in the next period. The expected values appear in a constraint in the portfolio optimization problem, as described in Section 4.3.2. Figure 2 shows the evolution of the swap rates for the different maturities. As can be seen, the swap rates reached a low and became negative in 2020 and 2021 but increased significantly in 2022. This pattern is the same as with the Dutch government bond yields.

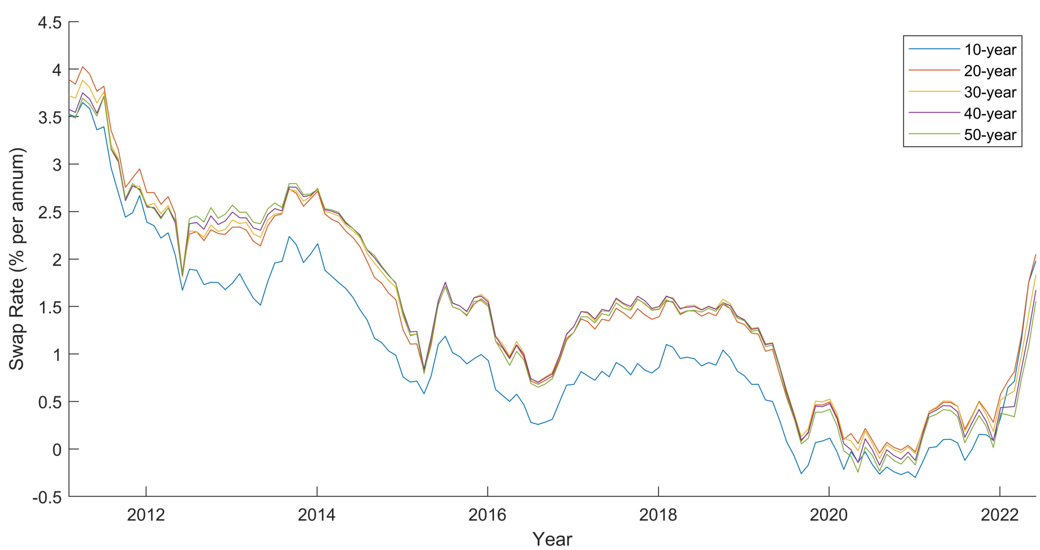


Figure 2: The 6-month EURIBOR interest rate swap rates using 30/360 day count convention for different long-term maturities from February 2011 to June 2022.

3.3 Corporate Bonds

Corporate bonds are included in the portfolio optimization problem as these often generate higher returns than government bonds because corporate bonds have inherent default risk. Default risk is the risk that the company issuing the bond cannot meet its financial obligations and hence depends on its financial situation. Because of the generally higher returns on corporate bonds and lower maturity, the modified duration of corporate bonds is often lower than that of government bonds. Therefore, interest rate swaps are generally more beneficial combined with corporate and government bonds instead of solely with government bonds. Corporate bond indices are included instead of single corporate bonds. The monthly returns and the modified durations of these indices are used in the optimization. As single corporate bonds are only issued on a single date and then have a changing maturity, this would lead to a changing set of investment products over the sample period and overcomplicates the optimization. Therefore, the index is taken as a proxy for the possibilities of investing in corporate bonds. Data on this is obtained from the same database as the government bond and interest rate swap data. It includes monthly data of the liquid tradable iBoxx Euro Corporates index total monthly return and modified duration from February 2011 to June 2022. A separate index is used for the different credit ratings AA, A and BBB, where the credit rating indicates a company's likelihood of defaulting on its debt. The three ratings mentioned before indicate low or medium risk, making them an investment grade. Investment grades are corporate bonds having quality usually required by investors. That is not the case for the higher-risk BB, B and C ratings. Therefore, these bonds are not considered. The highest AAA rating index is also not taken into account as there are no companies in Europe with this rating making them unavailable to investors.

Table 1 shows that corporate bonds with a lower rating, thus a higher default risk, generate higher returns but are also more volatile. It also confirms that corporate bonds generally have a higher return than government bonds as the former have a lower modified duration than the five-year government bond but still generate a higher return. However, the standard deviation of corporate bond returns is also higher because of the higher inherent risk.

Table 1: Average descriptive statistics from February 2011 to June 2022 for the different bonds used in the portfolio optimization.

		Return (%)	Standard Deviation (%)	Modified Duration (%)
Government	0.5 year	-0.26	0.21	0.5016
	2 years	0.00	0.92	2.0044
	5 years	1.36	3.18	4.9914
	10 years	3.67	6.75	9.9130
	20 years	5.56	13.86	19.7482
	30 years	10.40	20.50	29.5822
Corporate	AA rating	1.72	3.38	4.8922
	A rating	1.96	4.10	4.9225
	BBB rating	2.53	4.87	4.6903

4 Methodology

4.1 Term Structure Models

In portfolio optimization strategies, as described in Section 4.4, the first two expected moments of the bond returns are needed. For bonds, this is more complicated than for equity because the distribution of bond returns is non-stationary. One of the reasons for this is the decreasing time to maturity of bonds, according to Deguest et al. (2018). Consequently, the variance tends to go to zero as the bond approaches maturity. To solve this, two term structure models enter the equation to estimate bond returns. These models are very flexible and capture the decreasing time to maturity.

The first model is the dynamic Nelson-Siegel (DNS) model as introduced by Diebold and Li (2006). The DNS model is the most popular in practice by participants in the financial market because of its empirical performance, according to Caldeira et al. (2016). The model is described, following the notation by Christensen et al. (2011), as

$$y_t = B(\tau)X_t + \varepsilon_t, \tag{1}$$

$$X_t = (I - \Phi)\mu + \Phi X_{t-1} + \eta_t, \tag{2}$$

$$B(\tau) = \begin{pmatrix} 1, & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1}, & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ \vdots & \vdots & \vdots \\ 1, & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N}, & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix}, \tag{3}$$

where (1) is the Nelson-Siegel specification with the relation between the $N \times 1$ vector y_t with yields on N bonds with different times to maturity τ_i with $i = 1, \ldots, N$ and the underlying three factors X_t. The times to maturity τ_i are stated in years for $i = 1, \ldots, N$. The exponential decay rate λ is a parameter to be estimated. Equation (2) captures the dynamics of the factors as a vector autoregression model where I is a 3×3 identity matrix and Φ is a 3×3 matrix of coefficients. For the errors it holds that $\varepsilon_t \sim N(0, H)$, where H is a diagonal $N \times N$ matrix and $\eta_t \sim N(0, Q)$, where $Q = qq'$ with q being either a diagonal or lower-triangular 3×3 matrix as discussed at the end of this section. This model is in state-space form with (1) the measurement equation and (2) the state equation. Thus this is solved by using the Kalman filter, as described in Appendix A.1.

However, this model is not arbitrage-free. The concept of no-arbitrage is an important theoretical concept in term structure modelling. Therefore, Christensen et al. (2011) introduced the arbitrage-free Nelson-Siegel (AFNS) model, which ensures the model is arbitrage-free by imposing specific restrictions on the DNS model. The AFNS model is an affine term structure model incorporating the DNS structure, making it both theoretically rigorous and empirically successful according to Christensen et al. (2011). The AFNS model examines whether arbitrage restrictions play an important role in term structure models used for portfolio optimization in terms of portfolio performance. In the arbitrage-free Nelson-Siegel model, the stochastic differential equation under the P-measure for the state variables is

$$\mathrm{d}X_t = K[\theta - X_t]\mathrm{d}t + \Sigma \mathrm{d}W_t, \tag{4}$$

according to Christensen et al. (2011). By using the conditional moments of the discrete observations, the obtained state space form of the AFNS model is

$$y_t = -\frac{A(\tau)}{\tau} + B(\tau)X_t + \varepsilon_t, \tag{5}$$

$$X_t = (I - \exp(-K\Delta t))\theta + \exp(-K\Delta t)X_{t-1} + \eta_t. \tag{6}$$

K is a 3×3 matrix of dynamics parameters, θ is a 3×1 vector with drift parameters, $\Delta t = \frac{1}{12}$ as monthly data is considered and $B(\tau)$ is the same as in the DNS model in (3). As in the DNS model, it holds that $\varepsilon_t \sim N(0, H)$, where H is a diagonal $N \times N$ matrix and $\eta_t \sim N(0, Q)$. The Q-matrix has a special structure in the AFNS model as $Q = \int_0^\infty e^{-Ks}\Sigma\Sigma' e^{-K's} ds$. Christensen et al. (2015) show the way to analytically compute the Q-matrix. $A(\tau)$ is a vector with yield-adjustment terms in the AFNS specification to assure the no-arbitrage conditions. The analytical solution for the yield-adjustment term, according to Christensen et al. (2011), is

$$\begin{aligned}
\frac{A(\tau)}{\tau} =& \tilde{A}\frac{\tau^2}{6} + \tilde{B}\left[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau} + \frac{1}{4\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau}\right] \\
&+ \tilde{C}\left[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2}e^{-\lambda\tau} - \frac{1}{4\lambda}\tau e^{-2\lambda\tau} - \frac{3}{4\lambda^2}e^{-2\lambda\tau} - \frac{2}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau} + \frac{5}{8\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau}\right] \\
&+ \tilde{D}\left[\frac{1}{2\lambda}\tau + \frac{1}{\lambda^2}e^{-\lambda\tau} - \frac{1}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau}\right] \\
&+ \tilde{E}\left[\frac{3}{\lambda^2}e^{-\lambda\tau} + \frac{1}{2\lambda}\tau + \frac{1}{\lambda}\tau e^{-\lambda\tau} - \frac{3}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau}\right] \\
&+ \tilde{F}\left[\frac{1}{\lambda^2} + \frac{1}{\lambda^2}e^{-\lambda\tau} - \frac{1}{2\lambda^2}e^{-2\lambda\tau} - \frac{3}{\lambda^3}\frac{1-e^{-\lambda\tau}}{\tau} + \frac{3}{4\lambda^3}\frac{1-e^{-2\lambda\tau}}{\tau}\right],
\end{aligned} \tag{7}$$

where $\tilde{A} = \sigma_{11}^2 + \sigma_{12}^2 + \sigma_{13}^2$, $\tilde{B} = \sigma_{21}^2 + \sigma_{22}^2 + \sigma_{23}^2$, $\tilde{C} = \sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2$, $\tilde{D} = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} + \sigma_{13}\sigma_{23}$, $\tilde{E} = \sigma_{11}\sigma_{13} + \sigma_{12}\sigma_{32} + \sigma_{13}\sigma_{33}$ and $\tilde{F} = \sigma_{21}\sigma_{31} + \sigma_{22}\sigma_{32} + \sigma_{23}\sigma_{33}$. The σ-values are from the matrix

$$\Sigma = \begin{pmatrix} \sigma_{11}, & \sigma_{12}, & \sigma_{13} \\ \sigma_{21}, & \sigma_{22}, & \sigma_{23} \\ \sigma_{31}, & \sigma_{32}, & \sigma_{33} \end{pmatrix}, \tag{8}$$

which is either diagonal or lower triangular depending on whether the model is correlated or uncorrelated. The correlated and uncorrelated cases are discussed at the end of this section. The complete derivation of the AFNS model can be found in Christensen et al. (2011). This model is also in state-space form with (5) the measurement equation and (6) the state equation. Thus this is solved by using the Kalman filter as well, as described in Appendix A.2.

The notation of Caldeira et al. (2016) is partially applied to get the first two moments of the bond returns. For both models it holds that from the Kalman filter, the distribution of yields $y_{t+1|t} \sim N(\mu_{t+1|t}, \Sigma_{t+1|t})$ is obtained. The μ estimate is

$$\text{DNS}: \hat{\mu}_{t+1|t} = \hat{B}(\tau)\hat{X}_{t+1|t}, \tag{9}$$

$$\text{AFNS}: \hat{\mu}_{t+1|t} = -\frac{\widehat{A(\tau)}}{\tau} + \hat{B}(\tau)\hat{X}_{t+1|t}, \tag{10}$$

and the expected variance-covariance matrix for the yields is

$$\hat{\Sigma}_{t+1|t} = \hat{B}(\tau)\hat{P}_{t+1|t}\hat{B}(\tau)' + \hat{H}, \tag{11}$$

for both the DNS and AFNS models. The Kalman filter estimates all components of these equations. After this, the bond returns in the next period are normally distributed with mean

$$\hat{\mu}^r_{t+1}(\tau_i) = \tau_i y_t(\tau_i) - \tau_{i-1}\hat{\mu}_{t+1|t}(\tau_{i-1}), \tag{12}$$

for a bond with a maturity of τ_i where $\hat{\mu}^r_{t+1}$ is the vector with expected bond returns in the next period. The τ_{i-1} maturities are not available in the data set, as these are the maturities available in the data set minus one month. However, the expected yields of these maturities are necessary, as these appear in (12). Therefore, the expected yields for these maturities in the next period are computed by first computing the expected moments for the maturities present in the data set and then using cubic spline data interpolation to get the expected yields of the necessary maturities.

The diagonal elements of the variance-covariance matrix of bond returns Σ^r_{t+1} are given by

$$(\hat{\sigma}^r_{t+1}(\tau_i))^2 = \tau^2_{i-1}(\hat{\sigma}^y_{t+1|t}(\tau_{i-1}))^2, \tag{13}$$

where $(\hat{\sigma}^y_{t+1|t}(\tau_{i-1}))^2$ is the $(i-1)^{th}$ diagonal element of $\hat{\Sigma}_{t+1|t}$ from (11). The off-diagonal elements of Σ^r_{t+1} are given by

$$(\hat{\sigma}^r_{t+1}(\tau_i, \tau_j))^2 = \tau_{i-1}\tau_{j-1}(\hat{\sigma}^y_{t+1|t}(\tau_{i-1}, \tau_{j-1}))^2, \tag{14}$$

where $(\hat{\sigma}^y_{t+1|t}(\tau_{i-1}, \tau_{j-1}))^2$ is the $(i-1, j-1)$ element of $\hat{\Sigma}_{t+1|t}$ from (11).

The DNS and the AFNS model both have a version in which the three factors are uncorrelated (the diagonal case) and one correlated case (the VAR case). For the DNS model, Φ, q and Q are diagonal matrices in the uncorrelated case. Φ is non-diagonal and q is lower triangular when the three factors are correlated. For the AFNS model, K, Σ and Q are diagonal in the uncorrelated case. K is non-diagonal and Σ is lower triangular in the correlated case. Christensen et al. (2011) discuss that both versions of the models perform well in particular situations and that neither of those models is the clear winner in all circumstances. Therefore, the two versions of the models are evaluated for both the DNS and AFNS models as term structure models in the portfolio optimization.

4.2 Interest Rate Swap Value

In practice, interest rate swaps are often used to adjust portfolio interest rate exposure (duration). Therefore in this research, the traditional interest rate swaps are incorporated into the portfolio optimization problem. These swaps exchange a fixed rate (swap rate) for a floating rate (e.g. the LIBOR/EURIBOR). Thus one of the parties pays a fixed rate and receives the floating rate, whereas for the counterparty it is the other way around. In this paper, the zero-coupon bond rate as modelled in the term structure models of Section 4.1 is applied instead of the conventional LIBOR rate in computing the different values needed from swaps. The effect of incorporating an interest rate swap into the portfolio on portfolio duration depends on the position taken on the swap. If one receives the fixed rate it has a positive effect on the portfolio duration, whereas receiving the floating rate has the opposite effect. Section 4.3.1 describes the exact way to compute the interest rate swap duration.

The expected present value of the swap in the next period is needed to compute the expected collateral in the next period. An interest rate swap is equivalent to a portfolio containing a fixed-rate and a floating-rate bond. Hence, the present value of a receive-fixed IRS (V_t^{IRS}) at any point in time is computed by taking the difference between the present value of the fixed-rate bond underlying the swap (V_t^{FI}) and the present value of the floating-rate bond underlying the swap (V_t^{FL}), thus

$$V_t^{IRS} = V_t^{FI} - V_t^{FL}. \tag{15}$$

The present value of the fixed-rate bond underlying the swap is

$$V_t^{FI} = Z \sum_{i=1}^{N} \left(\frac{C}{k} \exp(-y_t(t_i)t_i) + \exp(-y_t(t_N)t_N) \right), \tag{16}$$

where C is the annual swap rate, k is the number of payments per year and Z is the notional amount agreed upon in the swap agreement. t_i ($1 \leq i \leq N$) is the time from t to the moments at which the fixed payments are made and $y_t(t_i)$ is the yield at time t of a zero-coupon bond with time to maturity t_i.

The present value of the floating-rate bond underlying the swap is

$$V_t^{FL} = Z \sum_{i=1}^{N} \left(f_t(t_i, m) \exp(-y_t(t_i)t_i) + \exp(-y_t(t_N)t_N) \right), \tag{17}$$

where $f_t(t_i, m)$ is the m-month forward interest rate in t_i years time at time t. This forward interest rate is computed by

$$f_t(t_i, m) = \frac{(1 + y_t(t_i + \frac{m}{12}))^{t_i + m/12}}{(1 + y_t(t_i))^{t_i}} - 1, \tag{18}$$

where it is assumed that the forward rate for zero-coupon bonds is the floating rate.

4.3 Constraints

The portfolio optimization framework includes two novel constraints in the portfolio management literature. The first constraint has the goal to handle the duration matching problem. The second handles the problem regarding a maximum amount of collateral, thus liquidity risk, on interest rate swaps.

4.3.1 Duration Constraint

Duration is the most popular measure of interest rate risk in the academic literature. Matching the duration of the liabilities and assets of an insurer is one of the main strategies to hedge against interest rate risk. Two types of duration are most prominent, specifically modified and Macaulay duration. Modified duration is the better measure when using duration for hedging purposes, according to Skinner (2004). This is due to the modified duration measuring price sensitivity to changes in the yield to maturity, whereas the Macaulay duration is a relative measure of interest rate risk. Deguest et al. (2018) and Widyatantri and Husodo (2020), the papers investigating duration matching within the portfolio optimization framework, also prefer the modified duration over the Macaulay duration.

The modified duration of a zero-coupon bond with maturity τ_i is

$$D_t^{ZCB} = \frac{\tau_i}{1 + y_t(\tau_i)}. \tag{19}$$

The duration of an IRS is computed by considering it as a portfolio of fixed and floating bonds. Thus a pay-fixed, receive-floating IRS is equivalent to going short in a fixed-rate bond and long in a floating-rate bond. The opposite holds for a receive-fixed, pay-floating IRS. Therefore, the swap duration is computed by taking the difference between the duration of the long and short positions. Thus $D_t^{IRS} = D_t^{FI} - D_t^{FL}$ in case of the receive-fixed, pay-floating IRS. As Smith (2014) describes, a receive-fixed, pay-floating IRS has a positive duration and thus increases the portfolio duration. A pay-fixed, receive-floating IRS has a negative duration and thus reduces the average portfolio duration. For the fixed-rate bond, the duration is computed as the duration of a coupon bond, where the coupon rate is the swap rate. The modified duration of the fixed-rate bond with maturity t_N is

$$D_t^{FI} = \frac{\frac{Z}{P}\left[\sum_{i=1}^{N-1}\left(t_i \frac{C}{k}\exp(-y_t(t_i)t_i)\right) + t_N(\frac{C}{k}+1)\exp(-y_t(t_N)t_N)\right]}{1 + \frac{y_t(t_N)}{k}}, \tag{20}$$

where the bond price P is

$$P = Z \sum_{i=1}^{N-1} \left(\frac{C}{k} \exp(-y_t(t_i)t_i) \right) + \frac{C}{k} + \exp(-y_t(t_N)t_N). \tag{21}$$

The duration of the floating-rate bond is

$$D_t^{FL} = \frac{t_1}{1 + \frac{y_t(\tau_i)}{k}}, \tag{22}$$

where t_1 is the time until the next coupon payment. The floating rate after the current period is not currently known and therefore does not carry any interest rate risk.

After computing the duration of the individual constituents of the portfolio, the portfolio duration is computed by taking a weighted sum of the individual durations. The weights are those assigned in the portfolio optimization to the different constituents. As the goal is to match this portfolio duration to the duration of the liabilities, the duration constraint is

$$w_t' D_t^{Bonds} + n_t' \frac{D_t^{IRS}}{\text{Inv}} = D^{liabilities}. \tag{23}$$

D_t^{Bonds} is a vector with durations of the different bonds, D_t^{IRS} is a vector with durations of the different interest rate swaps and w_t is the vector with weights of the different portfolio constituents. The n_t is a $M \times 1$ vector with the notional amounts in the M swaps and Inv is the total investment in bonds. $D^{liabilities}$ is the duration of the liabilities, thus the target duration. The notional amount in the swap agreement is a decision variable in the portfolio optimization framework. The notional also appears in the computation of the duration of the interest rate swap. The duration of the floating-rate bond in (22) does not depend on the notional amount. On top of that, it holds that

$$\begin{aligned} \frac{Z}{P} &= \frac{Z}{Z \sum_{i=1}^{N-1} \left(\frac{C}{k} \exp(-y_t(t_i)t_i) \right) + \frac{C}{k} + \exp(-y_t(t_N)t_N)} \\ &= \frac{1}{\sum_{i=1}^{N-1} \left(\frac{C}{k} \exp(-y_t(t_i)t_i) \right) + \frac{C}{k} + \exp(-y_t(t_N)t_N)}, \end{aligned} \tag{24}$$

so Z can be removed from the computation of D_t^{FI}. Therefore, the duration of the fixed-rate bond does not depend on the notional amount. Thus the interest rate swap duration does not depend on the notional amount. Consequently, the constraint in (23) is linear in the notionals of the swaps and the weights of the bonds.

4.3.2 Collateral Requirement Constraint

Interest rate swaps contain a default risk, the counterparty risk, which is the risk of default of the swap's counterparty. Collateral enters into the equation in the European Market Infrastructure

Regulation, introduced by the EU securities market regulator, to reduce the default risk.[2] These regulations prescribe the use of collateral following a daily marking-to-market procedure. In this procedure, the interest rate swap is valued daily and the required collateral is based on that value. Johannes and Sundaresan (2007), which investigates the impact of collateralization on swap rates, also applies this procedure. In this report, the required collateral is set equal to the value of the interest rate swap at that moment. Thus the collateral at time t (C_t), combining (15) up to (17), is

$$C_t = V_t^{IRS} = Z \sum_{i=1}^{N} \left(\frac{C}{k} \exp(-y_t(t_i)t_i) - f_t(t_i, m) \exp(-y_t(t_i)t_i) \right) = ZG_t(t_i). \tag{25}$$

As shown in (25), the relationship between the notional amount in the swap agreement and the required collateral is linear for an interest rate swap with a specific time to maturity at a certain point in time with a slope coefficient of $G_t(t_i)$. When the value of the interest rate swap is positive, this is received as collateral. However, when it is negative, one has to pay this value as collateral to the counterparty in the swap agreement.

It is thus required to compute the expected value of the interest rate swap one period ahead of when the portfolio is optimized, as this is the amount of collateral the insurance company needs to keep as liquidity. However, liquidity risk refers to the required liquidity in a worst-case scenario when the required collateral is very high. High collateral requirements happen when the swap value decreases significantly, which occurs when the interest rates increase significantly. To examine the extreme scenarios, a Monte Carlo simulation generates many possible future collateral requirement scenarios, following the idea in a recent study on liquidity risk and collateral by Flockermann et al. (2020).

Simulations are not made of the required collateral to keep the linearity from (25) in the constraints. Instead, $G_t(t_i)$ is simulated, as the notional amount is a decision variable. As the liquidity risk is concerned in the near-worst-case scenario, the 99.5^{th} percentile of the negative simulations is considered. The steps to obtain the 99.5^{th} percentile are:

1. Generate N_{sim} independent simulations from $N(0, \hat{H})$ and $N(0, \hat{Q})$. These are simulations for $\varepsilon_{1,t}, \ldots, \varepsilon_{N_{sim},t}$ and $\eta_{1,t}, \ldots, \eta_{N_{sim},t}$, respectively.
2. Apply (1) to (3) in the DNS case and (5) and (6) in the AFNS case with the estimated parameters to get the simulated values of $\hat{y}_{1,t}(t_i), \ldots, \hat{y}_{N_{sim},t}(t_i)$.

[2] `https://www.esma.europa.eu/databases-library/interactive-single-rulebook/clone-emir/article-11-0`

3. Combine (18) and $\hat{y}_{1,t}(t_i), \ldots, \hat{y}_{N_{sim},t}(t_i)$, to get the simulated values for $\hat{f}_{1,t}(t_i, m), \ldots, \hat{f}_{N_{sim},t}(t_i, m)$.

4. Compute $\hat{G}_t(t_i) = \sum_{i=1}^{N} \left(\frac{C}{k} \exp(-\hat{y}_t(t_i)t_i) - \hat{f}_t(t_i, m) \exp(-\hat{y}_t(t_i)t_i) \right)$ for all N_{sim} simulated values of $\hat{y}_t(t_i)$ and $\hat{f}_t(t_i, m)$ to get $\hat{G}_{1,t}(t_i), \ldots, \hat{G}_{N_{sim},t}(t_i)$.

5. Order the simulations for the slope coefficient multiplied by -1 as $G_{(1),t}(t_i), \ldots, G_{(N_{sim}),t}(t_i)$, then the 99.5^{th} percentile is $G_{(\lceil 0.995 N_{sim} \rceil),t}(t_i)$.

The simulations are multiplied by -1 in step 5 as the worst-case scenario in terms of losses is needed and not the best-case. The worst-case scenario is the almost highest value of the slope coefficient since this leads to the almost highest required collateral. The total required collateral over all the swaps in the next period must be below some benchmark set by the insurer. The maximum amount of collateral as a constraint in the portfolio optimization is

$$n_t' G_{(\lceil 0.995 N_{sim} \rceil),t} \leq B_c, \tag{26}$$

where n_t is a $M \times 1$ vector with the notional amount for the different interest rate swaps. These notionals are decision variables in the portfolio optimization. $G_{(\lceil 0.995 N_{sim} \rceil),t}$ is a $M \times 1$ vector containing the 99.5^{th} percentile of the simulation for the different interest rate swaps and B_c is the benchmark set by the insurer, treated as given in the portfolio optimization. The constraint results in an acceptable worst-case scenario for the insurance company and ensures the liquidity risk is at an acceptable level. The constraint is linear in the decision variable n_t. The upcoming section describes different optimized portfolios combined with the previously described constraints.

4.4 Portfolio Optimization Framework

This research considers myopic single-period portfolio optimization as, in practice, financial institutions mainly use myopic portfolio optimization, according to Brandt (2010). He states that this is because the expected utility for the investor is often lower with a dynamic than with a myopic strategy because of high estimation errors in the first two bond return moments in the former. Lan (2015) also shows that the utility gain of implementing a dynamic portfolio is either negative or positive but small and insignificant compared to the myopic case. Hence, the computational benefits of the myopic approach compared to the dynamic outweigh the utility benefits of using the latter. Several different static constrained portfolios are implemented. All optimized portfolios in the next section do not account for the risk and return of interest rate swaps in the objective function as these do not require initial investments. Therefore, the interest rate swaps only alter the portfolio duration with the weight based on their notional value relative to the total bond investment.

In the portfolio optimization, the set of constraints $\mathcal{C}$ at time t is

$$\mathcal{C} = \begin{cases} 1_N' w_t = 1, & (27) \\ w_t' D_t^{Bonds} + n_t' \dfrac{D_t^{IRS}}{\text{Inv}} = D^{liabilities}, & (28) \\ n_t' G_{(\lceil 0.995 N_{sim} \rceil),t} \leq B_c, & (29) \\ w_t \geq 0, \quad n_t \geq 0. & (30) \end{cases}$$

The constraint in (27) ensures wealth to be fully invested in the portfolio and (28) and (29) are described in Section 4.3.1 and 4.3.2, respectively. Constraint (30) ensures that the notional amount invested in an interest rate swap is positive and prevents short positions in the bonds. Previous literature, like Jagannathan and Ma (2003), shows that weight constraints can substantially increase portfolio performance. Puhle (2008) finds extreme bond weights in the optimized bond portfolios when weight constraints are absent. Extreme positions in bonds are undesirable in practice. Therefore, he proposes introducing short-sale constraints to get useful results in practice.

Combining (28) and (29), it can be concluded that the absolute values of Inv and B_c are not the relevant parameters. When multiplying both Inv and B_c with a factor x, the same outcome is obtained by multiplying the nominal amount vector n_t with the factor x. This does not affect the optimal value of the objective function as the objective functions do not depend on the nominals in the interest rate swaps. Hence, instead of regarding Inv and B_c as two separate parameters, the ratio of the two is important. In the remainder of the paper, this ratio is called the liquidity risk ratio. It describes the maximum amount of collateral on interest rate swaps allowed in the next period as a fraction of the total investment in bonds. Therefore, the total amount of investment in bonds is fixed at 1,000,000 and B_c is adjusted to get the desired ratio. All constraints in $\mathcal{C}$ are linear in the decision variables.

4.4.1 Minimum-Concentration Portfolio

The benchmark portfolio is the minimum-concentration (MC) portfolio, which is the natural extension of the equally-weighted portfolio when constraints are present according to Deguest et al. (2018). This method uses a portfolio concentration measure. Such a measure quantifies the diversification of the portfolio. The precise measure is the effective number of constituents, which is

$$ENC(w_t) = \frac{1}{||w_t||^2}, \tag{31}$$

where w_t is the vector with portfolio weights for the bonds. The ENC has a minimum of 1, which means that the portfolio is fully concentrated in one instrument. It has a maximum of N, the number of instruments in the portfolio, being the equally-weighted case. The MC portfolio intends to diversify the portfolio as much as possible. It maximizes the ENC subject to the constraints, which leads to

$$\max_{w_t} \quad ENC(w_t) \quad \text{subject to} \quad \mathcal{C}. \tag{32}$$

However, maximizing the ENC is the same as minimizing its reciprocal, leading to

$$\min_{w_t} \quad ||w_t||^2 = w_t' w_t \quad \text{subject to} \quad \mathcal{C}, \tag{33}$$

since the square of the Euclidean norm of a vector equals its dot product. The objective function in (33) is a convex quadratic function. The convexity follows from writing the objective function in the quadratic form $w_t' w_t = w_t' I_N w_t$ where I_N is the $N \times N$ identity matrix. The identity matrix is positive definite (all eigenvalues are one), so the objective function is a convex quadratic function. As all constraints are linear in the decision variables and the objective function is convex and quadratic, the MC portfolio problem is a quadratic programming (QP) problem. This problem is solved by the sparse interior-point-convex method following the basic algorithm of Mehrotra (1992).[3] This method is convenient for solving QP problems as it is efficient and widely applicable according to Pearson and Gondzio (2017).

4.4.2 Maximum Sharpe Ratio Portfolio

Deguest et al. (2018) find that the maximum Sharpe ratio (MSR) portfolio is superior when duration constraints are present. The Sharpe ratio is

$$SR_t = \frac{w_t' \tilde{\mu}_t}{\sqrt{w_t' \Sigma_t w_t}}, \tag{34}$$

where $\tilde{\mu}_t$ and Σ_t are the excess return vector and the covariance matrix, respectively, of the returns of the portfolio instruments. It holds that $\tilde{\mu}_t = \mu_t^r - R_t^f$ where R_t^f is the 3 month Dutch government bond yield at time t which is a proxy for the risk-free rate. These moments are estimated using term structure models described in Section 4.1. The MSR portfolio maximizes the expected Sharpe ratio

[3] The *quadprog* package in the Matlab Optimization Toolbox contains this algorithm.

of the portfolio with respect to the portfolio weights subject to the constraints. The optimization problem is

$$\max_{w_t} \quad SR_t \quad \text{subject to} \quad \mathcal{C}. \tag{35}$$

However, this problem is non-convex as the objective function is not convex. The problem classifies as a fractional programming (FP) problem as its objective function involves a ratio of mathematical functions in the decision variables. Non-convex problems are generally NP-hard problems. Transforming the problem into a convex QP problem, if possible, is generally the way to solve FP problems. A specific class of FP problems, so-called concave-convex FP problems, are solved using a transformation. The Schaible transform by Schaible (1974) is one of the transformations to solve such problems.

Theorem 1 (Schaible transform) *Introduce a concave-convex fractional programming problem*

$$\max_{x} \quad \frac{f(x)}{g(x)} \quad \text{subject to} \quad x \in \mathcal{S}, \tag{36}$$

where $f(x)$ is a concave function, $g(x)$ is convex and $\mathcal{S}$ is a compact convex set. The Schaible transform introduces two new variables $y = \frac{x}{g(x)}$ and $t = \frac{1}{g(x)}$, hence, $x = \frac{y}{t}$. The optimization problem then becomes

$$\begin{aligned} \max_{y,t} \quad & t f(y/t) \\ \text{subject to} \quad & y/t \in \mathcal{S} \\ & t g(y/t) \leq 1 \\ & t > 0. \end{aligned} \tag{37}$$

Then this problem is a convex optimization problem and is equivalent to (36)*.*

To simplify the use of the Schaible transform, the maximum Sharpe ratio portfolio problem is rewritten combining the two vectors of decision variables as $W_t = [\, w_t \; n_t \,]'$. Also rewrite the excess return vector as $\tilde{\mu}_{full,t} = [\, \tilde{\mu}_t \; 0_{M\times1} \,]'$, where $0_{M\times1}$ is a $M \times 1$ vector with zeros and the covariance matrix as $\Sigma_{full,t} = \begin{bmatrix} \Sigma_t & 0_{N\times M} \\ 0_{M\times N} & 0_{M\times M} \end{bmatrix}$. Then the problem is

$$\begin{aligned} \max_{W_t} \quad & \frac{W_t'\tilde{\mu}_{full,t}}{\sqrt{W_t'\Sigma_{full,t}W_t}} \\ \text{subject to} \quad & \begin{bmatrix} 1_N & 0_{M\times 1} \end{bmatrix}' W_t = 1 \\ & W_t' \begin{bmatrix} D_t^{Bonds} \\ D_t^{IRS}/\text{Inv} \end{bmatrix} = D^{liabilities} \\ & W_t' \begin{bmatrix} 0_{N\times 1} \\ G_{(\lceil 0.995 N_{sim}\rceil),t} \end{bmatrix} \leq B_c \\ & W_t \geq 0. \end{aligned} \tag{38}$$

It holds that $W_t'\tilde{\mu}_{full,t}$ is a concave function, as it is linear and linear functions are both convex and concave. $\Sigma_{full,t}$ is a diagonal block matrix. A diagonal block matrix has eigenvalues equal to the eigenvalues of the matrices on the diagonals. The eigenvalues of $\Sigma_{full,t}$ are all nonnegative because of the positive semi-definiteness of Σ_t and the eigenvalues of a null matrix are zero. Hence $\Sigma_{full,t}$ is positive semi-definite. Then the square root of a quadratic term, $\sqrt{W_t'\Sigma_{full,t}W_t}$, is a positive convex function according to Landsman (2008). The Schaible transform introduces the two decision variables $y_t = \frac{W_t}{\sqrt{W_t'\Sigma_{full,t}W_t}}$ and $h_t = \frac{1}{\sqrt{W_t'\Sigma_{full,t}W_t}}$. Applying the Schaible transform delivers the new optimization problem

$$\begin{aligned} \min_{y_t, h_t} \quad & -y_t'\tilde{\mu}_{full,t} \\ \text{subject to} \quad & \begin{bmatrix} 1_N & 0_{M\times 1} \end{bmatrix}' y_t - h_t = 0 \\ & y_t' \begin{bmatrix} D_t^{Bonds} \\ D_t^{IRS}/\text{Inv} \end{bmatrix} - h_t D^{liabilities} = 0 \\ & y_t' \begin{bmatrix} 0_{N\times 1} \\ G_{(\lceil 0.995 N_{sim}\rceil),t} \end{bmatrix} - h_t B_c \leq 0 \\ & y_t'\Sigma_{full,t} y_t \leq 1 \\ & y_t \geq 0, \quad h_t > 0. \end{aligned} \tag{39}$$

This problem is convex as the objective function and all constraints are convex in the decision variables. The constraints are convex as these are either linear in y_t and h_t or quadratic in y_t with positive semi-definite matrix $\Sigma_{full,t}$. The optimization problem (39) is equivalent to the problem in (35) and it is convex. The final weights are obtained by using the relationship $W_t = \frac{y_t}{h_t}$. Due to the convexity, the problem only has one optimal solution: the global optimum. This problem is called a quadratically constrained quadratic program (QCQP). QCQPs are a subset of the second-order cone programs (SOCPs), according to Alizadeh and Goldfarb (2003). To transform the QCQP into a SOCP, perform

$$y_t'\Sigma_{full,t}y_t = y_t'A'Ay_t = (Ay_t)'Ay_t = ||Ay_t||^2 \leq 1, \tag{40}$$

which is a second-order cone constraint. Then, the problem can be solved efficiently as a second-order cone problem using the primal-dual interior-point method for conic quadratic optimization following Andersen et al. (2003).[4] They show that the algorithm can solve problems very robustly and efficiently.

4.4.3 Mean-Variance Portfolio

Markowitz's mean-variance portfolio subject to the linear constraints is also incorporated, following Brandt (2010). The mean-variance (MV) portfolio is written as the expected utility maximization problem to capture the trade-off between the expected risk and return of the portfolio as adequately as possible for a specific level of risk aversion. It consists of minimizing the difference between the expected variance of the portfolio scaled with a coefficient of risk aversion and the expected excess return of the portfolio, so the optimization problem is

$$\min_{w_t} \quad \frac{\gamma}{2}w_t'\Sigma_t w_t - w_t'\tilde{\mu}_t \quad \text{subject to} \quad \mathcal{C}, \tag{41}$$

where γ is the level of relative risk aversion. A higher γ indicates a higher level of risk aversion of the investor. The objective function in (41) is a quadratic function in the decision variable w_t. In combination with the linearity of the constraints, this problem is a QP problem. As Σ_t is a covariance matrix, this matrix is symmetric positive semi-definite. Hence, the quadratic objective function is convex, meaning this problem is a convex QP problem with linear constraints. As a result, finding a local minimum also means finding the global minimum. This problem is solved using the algorithm for optimizing the MC portfolio, as described in Section 4.4.1.

4.5 Performance Evaluation

The evaluation of the different optimized portfolios with optimized weights $\hat{w}_t$ at time t is done using several performance measures. First, the annualized average excess return is considered. This

[4] The primal-dual interior-point method for conic quadratic optimization is the core algorithm behind the *coneprog* package which is in the quadratic programming package in the Matlab Optimization Toolbox.

measure is computed by computing

$$R_{p,t} = \hat{w}'_{t-1} R_t - R_t^f, \tag{42}$$

for every out-of-sample period. $R_{p,t}$ is the return over the last month of the portfolio at time t and R_t^f is the risk-free rate at time t. R_t is the vector with returns over the previous period for the individual constituents of the portfolio computed using the observed yields at times $t-1$ and t. Then

$$\hat{\mu}_p = \frac{12}{T} \sum_{t=1}^{T} R_{p,t}, \tag{43}$$

where $\hat{\mu}_p$ is the annualized average out-of-sample excess portfolio return and T is the number of out-of-sample periods.

The annualized standard deviation of these out-of-sample portfolio returns is the second measure of portfolio performance as that is an indication of the risk of the portfolios. This is computed by

$$\hat{\sigma}_p = \sqrt{\frac{12}{T-1} \sum_{t=1}^{T} (R_{p,t} - \hat{\mu}_p)}. \tag{44}$$

The third measure is the annualized Sharpe ratio which provides insight into the risk-return ratio of the portfolio. The annualized Sharpe ratio is computed by

$$SR = \frac{\hat{\mu}_p}{\hat{\sigma}_p}. \tag{45}$$

The statistical significance of the differences in the Sharpe ratios of different portfolios is tested using the HAC inference test by Ledoit and Wolf (2008) as described in Appendix B.

Furthermore, the average portfolio turnover is analysed as

$$\text{Turnover} = \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{N} |w_{j,t} - w_{j,t-1}|, \tag{46}$$

where it holds that a lower turnover is better as that leads to lower transaction costs and a more stable portfolio.

Finally, a measure quantifying the number of interest rate swaps in the portfolio is considered. The measure for this purpose is the average total notional amount in the interest rate swaps as a fraction of the total investment in bonds and will be called the IRS Ratio. Hence, it holds that

$$\text{IRS Ratio} = \frac{1}{T} \sum_{t=1}^{T} \frac{\sum_{i=1}^{M} n_{t,i}}{\text{Inv}}. \tag{47}$$

A moving window is utilized to evaluate the performance of the different portfolio optimization techniques. In the moving window, the in-sample period consists of 8 years of historical data (96 observations). Different possibilities for the length of the in-sample period have also be considered but these changes do not affect the results of the term structure models significantly. The optimized portfolios are evaluated one month later. After the evaluation, the window is shifted by one month and the first month of the prior in-sample period is discarded. Portfolios are then optimized and evaluated again. The portfolios are thus re-balanced monthly. This process continues until the end of the available data, so June 2022 ($T = 41$ out-of-sample periods). Figure 3 displays the moving window framework.

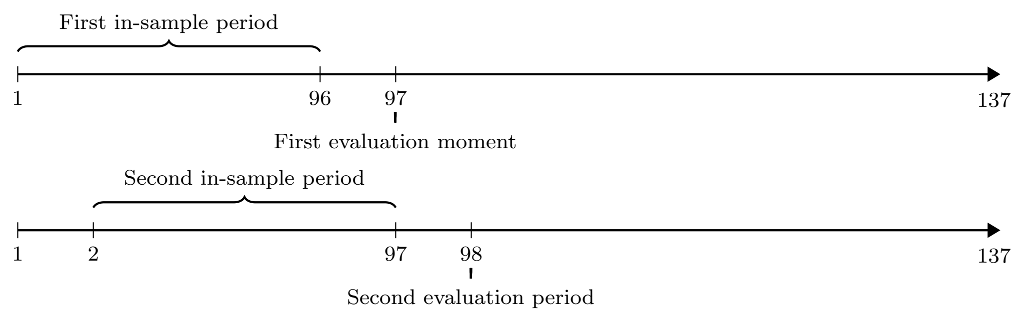


Figure 3: Illustration of the rolling window to evaluate the performance of the different portfolio optimization techniques with eight years (96 months) of in-sample data and evaluating the portfolio one month later.

4.6 Incorporation of Corporate Bonds

The portfolio optimization also includes corporate bonds besides government bonds and interest rate swaps to examine the effect of these bonds on the portfolio performance. As described in Section 3.3, the performance of three corporate bond indices is used as a proxy for the returns of corporate bonds instead of single corporate bonds. These are treated the same as government bonds in the constraint set $\mathcal{C}$, so their weights are in the weight vector w_t. The modified durations and the monthly returns of the indices are directly obtained from the database. The historical sample moments over the in-sample period are used as expected return and variance in the next period. Using historical sample moments introduces estimation errors that possibly affect the portfolio performance. The downside of estimation errors in the inputs of portfolio optimization problems is acknowledged by Michaud (1989). He states that mean-variance optimizers tend to act as statistical error maximizers.

5 Results

5.1 Parameter Values

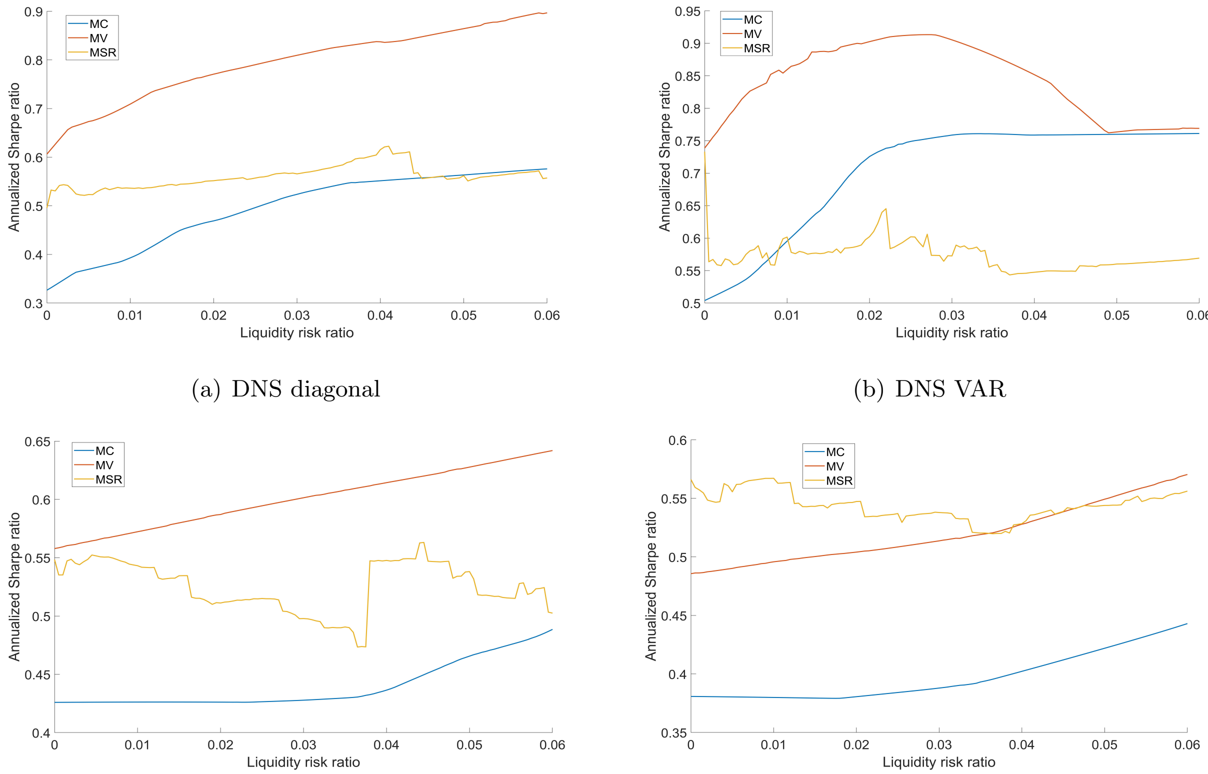
In the optimization, the target duration, relative risk aversion and liquidity risk ratio are fixed. In practice, however, the values of these parameters vary across insurance companies and can be set to the appropriate value in using the optimization. The robustness of the results when changing the parameter values is examined in Section 5.3.

Target duration. The target portfolio duration in the optimization problem is such that it matches the duration of the liabilities of the insurance company leading to a duration match between the assets and liabilities of the insurance company. Consequently, the duration matching strategy reduces the interest rate risk to the insurer. This duration of the liabilities varies across insurance companies in different countries and businesses. In this report, the duration of the liabilities is 13.4 years which is the average modified duration of the liabilities of insurance companies in The Netherlands in 2019, according to a report by EIOPA (2019), the European Insurance and Occupational Pensions Authority.

Relative risk aversion. The level of relative risk aversion γ indicates the investor's risk appetite. A higher γ indicates a higher level of risk aversion and hence a lower risk appetite. According to Gandelman and Hernández-Murillo (2015), the level of relative risk aversion ranges from 0.2 up to 10 and occasionally higher. Their analysis of the relative risk aversion of inhabitants of countries all around the world indicates the level of relative risk aversion to vary closely around 1. Therefore, in this report, as insurance companies represent a country's inhabitants, γ is set to 1 in the analysis.

Liquidity risk ratio. The final parameter in the portfolio optimization is the liquidity risk ratio explained in Section 4.4. In practice, the insurance company should set this parameter considering the maximum amount of collateral the insurer can handle in the next period in the near-worst-case scenario as a fraction of their investment in bonds. Figure 4 displays the relation between the liquidity risk ratio and the Sharpe ratio of the three optimized portfolios for the four underlying time series models with government and corporate bonds and interest rate swaps. This relation is positive for most models and portfolios. An explanation of that relation is the ability to use more interest rate swaps matching the duration of assets and liabilities when the liquidity risk ratio increases. Subsequently, one can invest more in bonds generating a higher return without considering the duration. After a certain value for the ratio, the positive effect of increasing the liquidity risk ratio on the Sharpe ratio diminishes, disappears or even backfires shown in Figure 4.

This mostly happens after a ratio of about 0.05 (5%) for the DNS models. This percentage is also a value which is reasonable to use in practice. Hence, that value is used in the remainder of the report.



(a) DNS diagonal

(b) DNS VAR

(c) AFNS diagonal

(d) AFNS VAR

Figure 4: The effect of the liquidity risk ratio on the annualized Sharpe ratio for the four underlying term structure models for the three optimized portfolios with government and corporate bonds and interest rate swaps.

5.2 Portfolio Performance

Tables 2 and 3 present the performance measures of the portfolios with multiple specifications of the term structure models excluding and including the corporate bonds. Table 2 shows the performance when interest rate swaps are included in the portfolio. The portfolios have average annualized excess returns ranging from 3.41% to 7.80% and annualized standard deviations from 6.06% to 9.84%. These values lead to Sharpe ratios in the range of 0.3733 to 0.8906. Table 3 shows the performance when interest rate swaps are not included in the portfolios. This results in average annualized returns of the portfolios in the range from 3.87% to 6.12% and annualized standard deviations from 9.24% to 15.41%. The Sharpe ratios are between 0.2550 and 0.6063.

5.2.1 Effect of Interest Rate Swaps and Corporate Bonds

First, including and excluding interest rate swaps, most portfolios incorporating corporate bonds outperform portfolios without corporate bonds in terms of the Sharpe ratio. This outperformance happens in 82% of the portfolios with interest rate swaps and all cases without them. The portfolio returns generally increase and are less volatile when corporate bonds are added to the possible set of investment instruments. The outperformance shows the significance to an investor of incorporating corporate bonds in a bond portfolio besides government bonds. The turnover increases in less than half of the portfolios (45.83%) due to the corporate bonds. This result implies that the additional use of corporate bonds does not inherently come with higher transaction costs in the portfolio, which is positive for an investor.

On top of that, the average increase in the Sharpe ratio due to the incorporation of corporate bonds is 29.57% when interest rate swaps are present. However, this increase is only 24.21% when interest rate swaps are not present. This means that the advantage of corporate bonds is more prevalent combined with the presence of interest rate swaps. This finding confirms the statement in Section 3.3 regarding the higher benefits of interest rate swaps in combination with corporate and government bonds together over government bonds only. The IRS ratio indicating the use of interest rate swaps in the portfolio is also larger in 75% of the portfolios with corporate bonds in contrast to those without them. It also shows the higher need for interest rate swaps in combination with corporate bonds due to the generally smaller modified duration of these bonds than the duration of the government bonds.

Furthermore, interest rate swaps improve the portfolio performance regardless of the in- or exclusion of corporate bonds. In 87.50% of the portfolios in Table 2 and 3, the Sharpe ratio of the portfolio with interest rate swaps is higher than the Sharpe ratio of the one without interest rate swaps. However, the same percentage of portfolios with interest rate swaps have a higher turnover. Thus it depends on the magnitude of the transaction costs to know if it is profitable to invest in interest rate swaps. Even with that drawback, interest rate swaps in bond portfolios are promising.

All in all, both the use of interest rate swaps and corporate bonds are promising in the context of a bond portfolio optimization problem in combination with government bonds. This also explains the widespread use of these instruments in practice, e.g. in insurance companies. Therefore, in the remainder of this paper, the cases with both interest rate swaps and corporate bonds are analysed (the results in the 'Including Corporate Bonds' section of Table 2).

Table 2: Performance of the bond portfolios including interest rate swaps for the out-of-sample period.

		$\hat{\mu}_p$ (%)	$\hat{\sigma}_p$ (%)	SR	Turnover (%)	IRS Ratio (%)
Excluding Corporate Bonds						
MC	DNS diagonal	3.94	8.49	0.4638	1.93	9.99
	DNS VAR	4.26	8.24	0.5167	0.65	9.69
	AFNS diagonal	3.98	8.49	0.4683	0.52	10.86
	AFNS VAR	3.82	8.77	0.4354	2.28	9.74
MV	DNS diagonal	5.54	9.16	0.6051	104.86	10.14
	DNS VAR	6.73	7.56	0.8906	52.99	14.23
	AFNS diagonal	4.31	9.74	0.4422	78.75	3.82
	AFNS VAR	4.61	9.52	0.4838	90.38	9.38
MSR	DNS diagonal	3.64	9.74	0.3733	83.71	4.40
	DNS VAR	4.92	9.60	0.5133	38.34	2.03
	AFNS diagonal	4.05	9.57	0.4233	65.94	1.86
	AFNS VAR	3.82	9.84	0.3879	64.01	1.90
Including Corporate Bonds						
MC	DNS diagonal	3.82	6.76	0.5656	5.23	19.51
	DNS VAR	4.60	6.06	0.7598	1.62	19.81
	AFNS diagonal	3.53	7.53	0.4694	2.30	15.69
	AFNS VAR	3.41	8.12	0.4200	5.99	13.47
MV	DNS diagonal	7.80	9.03	0.8637	103.15	12.39
	DNS VAR	6.08	7.91	0.7694	45.85	18.54
	AFNS diagonal	5.79	9.17	0.6314	78.57	4.20
	AFNS VAR	4.72	8.63	0.5472	78.45	10.79
MSR	DNS diagonal	5.28	9.34	0.5655	101.28	4.54
	DNS VAR	5.15	9.26	0.5559	25.80	1.37
	AFNS diagonal	4.96	9.19	0.5394	62.20	1.08
	AFNS VAR	4.98	9.13	0.5455	61.44	0.96

Table 3: Performance of the bond portfolios excluding interest rate swaps for the out-of-sample period. The measure IRS ratio is always zero, as no IRSs are used.

		$\hat{\mu}_p$ (%)	$\hat{\sigma}_p$ (%)	SR	Turnover (%)
Excluding Corporate Bonds					
MC	DNS diagonal	3.93	9.92	0.3963	1.47
	DNS VAR	3.96	9.96	0.3981	0.19
	AFNS diagonal	3.98	9.95	0.3998	0.13
	AFNS VAR	3.97	9.95	0.3985	0.18
MV	DNS diagonal	4.46	10.37	0.4302	87.72
	DNS VAR	4.49	10.11	0.4438	40.85
	AFNS diagonal	3.87	10.18	0.3804	68.33
	AFNS VAR	4.71	10.23	0.4606	82.19
MSR	DNS diagonal	3.93	15.41	0.2550	118.59
	DNS VAR	4.41	10.19	0.4329	30.88
	AFNS diagonal	4.13	10.06	0.4106	47.68
	AFNS VAR	4.02	10.27	0.3916	59.78
Including Corporate Bonds					
MC	DNS diagonal	4.02	9.53	0.4218	1.51
	DNS VAR	4.06	9.56	0.4244	0.23
	AFNS diagonal	4.07	9.55	0.4260	0.18
	AFNS VAR	4.06	9.56	0.4248	0.21
MV	DNS diagonal	6.12	10.10	0.6063	83.78
	DNS VAR	4.95	9.24	0.5352	39.22
	AFNS diagonal	5.21	9.35	0.5579	37.41
	AFNS VAR	5.24	9.41	0.5569	69.41
MSR	DNS diagonal	4.81	10.91	0.4408	119.43
	DNS VAR	4.64	9.70	0.4785	27.12
	AFNS diagonal	5.01	9.64	0.5200	65.71
	AFNS VAR	4.63	9.45	0.4906	53.08

5.2.2 Comparison of Portfolio Specifications

Now including the interest rate swaps and corporate bonds, it is relevant to consider which portfolios work best in terms of portfolio performance.

The MV portfolio has superior performance in terms of the average annualized excess return and the Sharpe ratio compared to the other two portfolios. It also outperforms the second optimized portfolio strategy, the MSR portfolio, regarding the standard deviation. On top of that, the MSR portfolio has a higher average annualized excess return than the MC portfolio for all four term structure specifications but has a much higher standard deviation. However, the MSR portfolio still manages to beat the MC portfolio in terms of the Sharpe ratio in two of the four cases and equals one. Table 4 displays the p-values of testing the statistical difference between the Sharpe ratios of the optimized portfolios with the MC portfolio. This table shows that the Sharpe ratio of the MC portfolio is statistically lower than that of the optimized portfolios in five of the eight cases. Thus in most portfolio specifications, the optimized portfolios using the bond return moments in their objective functions significantly outperform the benchmark MC portfolio. The MC portfolio is the equivalent of the equally-weighted portfolio when additional constraints are present. Therefore, using optimized portfolios instead of benchmark portfolios is worthwhile in the bond portfolio problem for an investor. This finding was known for decades in the equity case but is now observed in bond portfolios.

Table 4: The p-values to test the statistical differences of the MV and MSR portfolios with the MC portfolio including IRSs and corporate bonds using the HAC inference test.

MV	DNS diagonal	DNS VAR	AFNS diagonal	AFNS VAR
p-value of Δ_{Sharpe}	0.0000*	0.9572	0.0001*	0.0001*
MSR	DNS diagonal	DNS VAR	AFNS diagonal	AFNS VAR
p-value of Δ_{Sharpe}	0.9979	0.0575	0.0079*	0.0000*

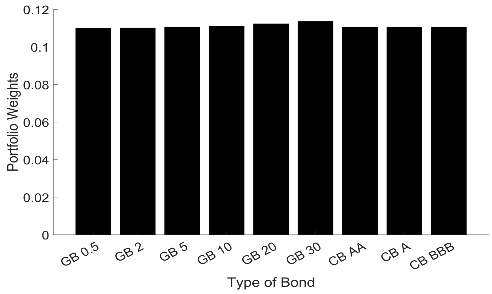
The p-values are of the test $H_0 : \Delta_{Sharpe} = 0$. * $p < 0.05$

One drawback to the optimized portfolios is that the turnover in these portfolios is often much higher compared to the stable MC portfolio. In the case of the MV portfolio, the turnover is twenty to fifty times the turnover of the MC portfolio, as displayed in Table 2. Thus the MV portfolio could be significantly more costly than the MC portfolio due to its transaction costs. These transaction costs should be taken into account by an investor when choosing the preferred portfolio strategy.

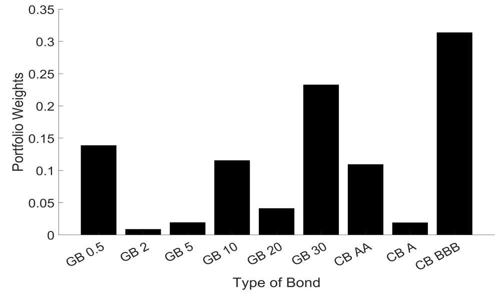
Another interesting question is whether introducing the no-arbitrage conditions in the Nelson-Siegel model enhances the portfolio performance. In all three types of portfolios, Table 2 shows that the performance of the AFNS specification is worse than that of the DNS specification in terms of the Sharpe ratio. In the MSR portfolio, this difference is small and statistically insignificant with Sharpe ratios of 0.5394 and 0.5455 as opposed to 0.5655 and 0.5559. However, for the other portfolio strategies, this difference is more significant. Specifically, the Sharpe ratios are 0.4694 and 0.4200 compared to 0.5656 and 0.7598 for the MC portfolio. The p-values using the HAC inference test to test the statistical significance of the difference between these Sharpe ratios are 0.0026 and 0.0000, respectively. Thus for the MC portfolio, both Sharpe ratios of the DNS specification are statistically higher than that of the AFNS specification. For the MV portfolio, the Sharpe ratios are 0.6314 and 0.5472 for the AFNS compared to 0.8637 and 0.7694 for the DNS. Here, the p-values are 0.0000 and 0.1509, respectively, meaning that for one of the two cases the Sharpe ratio is significantly higher. Therefore, the imposition of no arbitrage does not enhance portfolio performance. As the AFNS model complicates computations leading to higher computing times, applying this model is not advised.

Overall, the best-performing portfolio when interest rate swaps and corporate bonds are present is the MV portfolio with the DNS term structure model with uncorrelated underlying factors. It has a Sharpe ratio of 0.8637 but has a high portfolio turnover of 103.15%. The notional amount on the interest rate swaps in this portfolio is 12.39% of the total investment in bonds.

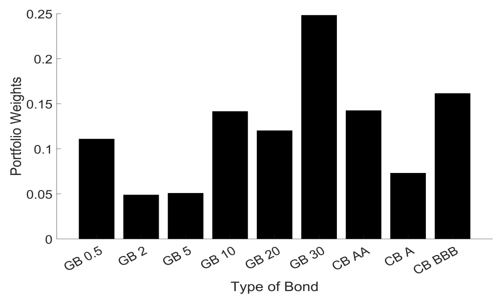
To get more insight into the portfolio composition for the different types of portfolios, Figure 5 displays the average portfolio weights for the three types of portfolios over the out-of-sample periods. The term structure specification is the one which performed the best for the specific portfolio type. As shown in Figure 5(a), the MC portfolio is indeed an equivalent of the equally-weighted portfolio with slight differences in average portfolio weights but all weights close to 0.11. Figures 5(b) and 5(c) show that optimized portfolio weights are more concentrated in the long-term government bond and the low-rated corporate bond. This could partly be explained by the higher returns on these types of bonds, as indicated in Table 1. Even though the government bond with the lowest maturity generates the lowest return, it is still significantly incorporated in the optimized portfolios with an average weight above 0.10. This is mainly due to the target of duration matching and the compensation of the high duration of the bonds with high returns with one having a low duration. The MV portfolio is more concentrated in only a few bonds than the MSR portfolio, as displayed in Figure 5.



(a) MC portfolio (DNS VAR)



(b) MV portfolio (DNS diagonal)



(c) MSR portfolio (DNS diagonal)

Figure 5: The average portfolio weights in the out-of-sample period for the three types of portfolios. The GB label stands for government bond and the number for the maturity thereof. The CB stands for corporate bond and the character for its rating.

5.2.3 Effect of Sharp Interest Rate Rises

In the first six months of 2022, the interest rates on bonds increased sharply, as displayed in Figure 1. Rising interest rates often lead to negative bond returns, hence, poor portfolio performance. Thus it is relevant to investigate the performance of the portfolios in such times.

First, Figure 6 shows the evolution of the monthly excess portfolio returns in 2021 and the first six months of 2022. The excess return declines on average in the first six months of 2022 when the interest rates rise sharply. This finding is in line with expectations. For the MV and MSR portfolios, the monthly excess returns almost reach -8%. However, the decline in the excess return of the MC portfolio is less sharp, only reaching slightly below -5%. This is expected as the MC portfolio is generally more stable because it tends to diversify the portfolio to the greatest extent. Thus when unstable times with the possibility of sharply rising interest rates are expected, implementing the MC portfolio could be a solution to limit the damage to portfolio performance.

On top of this, the influence of interest rate swaps and corporate bonds on the performance

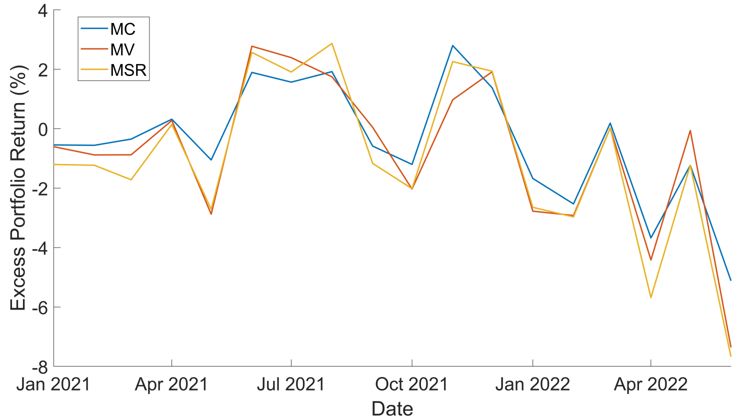


Figure 6: The evolution of the monthly excess portfolio returns for the best term structure specification for the three portfolios from January 2021 to June 2022. The first six months of 2022 indicate a period with sharp interest rate rises.

of the best portfolio (the MV DNS diagonal portfolio) in the first six months of 2022 is shown in Figure 7. Again, the decline in average portfolio performance is prevalent in the first six months of 2022. However, the fall in excess portfolio return when including the interest rate swaps and corporate bonds in the portfolio is smaller than when both are excluded. This difference is not large, the smallest difference being near zero in the first three months of 2022 to 0.5% in June 2022 to 1.2% in April 2022. This can make a huge difference to an insurer even though the differences in portfolio performance are small. For example, the total investment of the life insurance business of Nationale-Nederlanden was EUR 98,319 million in 2021, according to Nationale-Nederlanden Levensverzekering Maatschappij N.V. (2022). Thus a 1.2% increase would lead to a EUR 1180 million increase in asset value. Hence, another advantage of interest rate swaps and corporate bonds in the bond portfolio optimization is limiting the decrease of portfolio performance in times of sharply rising interest rates.

Figure 7: The evolution of the monthly excess portfolio returns for the MV DNS diagonal portfolio including and excluding interest rate swaps and corporate bonds from January 2021 to June 2022. The first six months of 2022 indicate a period with sharp interest rate rises.

5.3 Robustness Checks

First, Table 5 shows the effect of changing the target duration in the constraints on the percentages used in Section 5.2.1 to examine the importance of interest rate swaps and corporate bonds. These statistics follow from the Sharpe ratios in both tables in Appendix C. The target duration is not below ten years, as the duration matching strategy is used in practice with long-term liabilities, such that the cases below ten years are not of practical relevance. As in the case with the fixed target duration of 13.4 years, interest rate swaps and corporate bonds improve portfolio performance in terms of the Sharpe ratio. Table 5 shows that for all target durations, most of the Sharpe ratios of portfolios with interest rate swaps exceed those of portfolios excluding these. Corporate bonds never harm the Sharpe ratio of the portfolio, as at least 50% of the portfolios have a higher Sharpe ratio when including corporate bonds. The advantage of corporate bonds is more significant with interest rate swaps than without because the average increase in the Sharpe ratio when adding corporate bonds in the former is higher than in the latter for all target durations. These findings show that including interest rate swaps and corporate bonds in portfolio optimization is recommended regardless of the target duration.

Now consider the performance of the portfolios with both interest rate swaps and corporate bonds for varying target durations, as displayed in Table A1. Then the optimized portfolios generally outperform the benchmark MC portfolio in terms of the Sharpe ratio for all target durations. Hence, the conclusion that it is profitable for an investor to consider optimized portfolios using the

Table 5: The impact of changing the target duration on the number of portfolios with a higher Sharpe ratio including corporate bonds (interest rate swaps) than the Sharpe ratio when excluding them and the average increase, if there is an increase, in Sharpe ratio due to corporate bonds with and without interest rate swaps.

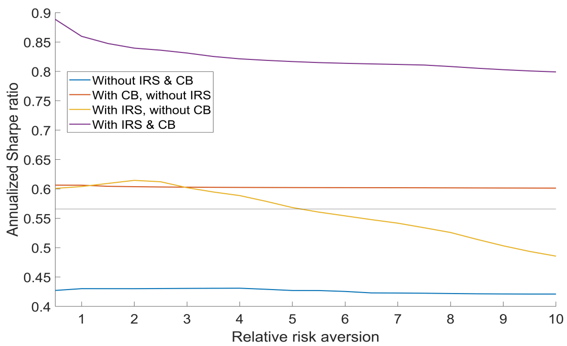
Target duration (years)	10	15	20	25	30
SR incl. CB > SR excl. CB (% of portfolios)	95.83	91.67	83.33	50.00	50.00
SR incl. IRS > SR excl. IRS (% of portfolios)	87.50	83.33	66.67	66.67	58.33
If increase, average % SR increase due to CB excl. IRS	22.48	28.88	19.66	9.84	11.54
If increase, average % SR increase due to CB incl. IRS	27.99	29.03	26.82	21.59	44.69

first two bond return moments still holds when varying the target duration. On top of that, the MV portfolio performs superior to the MSR portfolio regardless of the target duration.

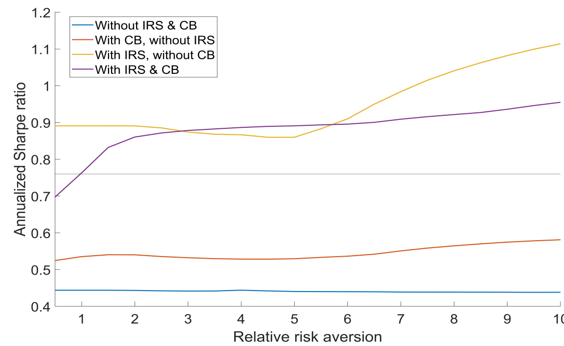
Second, Figure 8 displays the impact of varying the relative risk aversion on the portfolio performance in terms of the Sharpe ratio. This parameter only occurs in the objective function of the MV portfolio, so only the MV portfolio performance changes when the risk aversion changes. In the two uncorrelated specifications of the term structure models, the specification with interest rate swaps and corporate bonds is superior for all levels of relative risk aversion. For the correlated cases, this does not hold for all levels of relative risk aversion. The specification with interest rate swaps and corporate bonds still outperforms the specification without these two portfolio components. This superiority means interest rate swaps and corporate bonds generally improve portfolio performance in the Sharpe ratio regardless of the relative risk aversion. This improvement is the same as in Section 5.2.1.

Furthermore, Figure 8 shows that the MV portfolio outperforms the benchmark MC portfolio (the horizontal line in the figure) when interest rate swaps and corporate bonds are included. Thus optimized portfolios outperform the benchmark MC portfolio regardless of the relative risk aversion. Therefore, investors with every type of risk aversion should consider optimized portfolios over benchmark portfolios in practice.

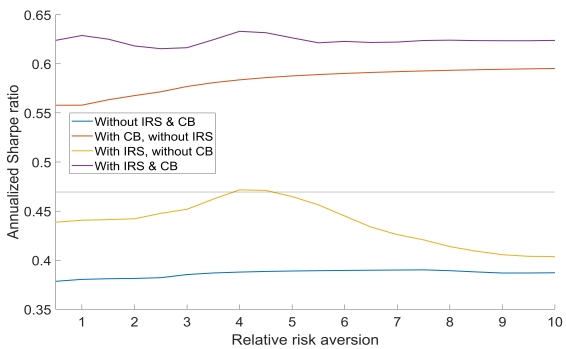
Third, Figure 4 shows that the optimized portfolio outperforms the benchmark MC portfolio in most term structure specifications for a range of liquidity risk ratios. The only specification for which this is not true is the MSR portfolio in the correlated DNS model. This superiority of the



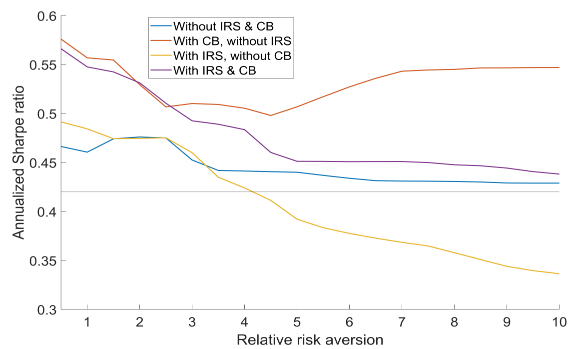
(a) DNS diagonal



(b) DNS VAR



(c) AFNS diagonal



(d) AFNS VAR

Figure 8: The impact of varying the level of relative risk aversion on the annualized Sharpe ratio of the MV portfolio. The horizontal line is the value of the annualized Sharpe ratio of the best MC portfolio with interest rate swaps and corporate bonds.

optimized models over the benchmark is in line with the findings in Section 5.2.2. The exclusion of interest rate swaps corresponds to a liquidity risk ratio of zero because then no interest rate swaps are allowed. Generally, the annualized Sharpe ratio increases when having a liquidity risk ratio unequal to zero. Thus interest rate swaps benefit an investor for any liquidity risk ratio when corporate bonds are also present. All in all, the choices of parameters generally do not affect the conclusions in this paper about the use of interest rate swaps, corporate bonds and optimized portfolios.

6 Conclusion

The main goal of this paper was to investigate the incorporation of interest rate swaps in the static bond portfolio optimization problem with a duration matching constraint. The minimum-concentration, mean-variance and maximum Sharpe ratio portfolio problems are expanded with constraints matching the duration of assets and liabilities and limiting the required collateral for interest rate swaps. Four different specifications of term structure models to estimate the bond return moments are used: the (un-)correlated dynamic Nelson-Siegel and arbitrage-free Nelson-Siegel models.

This paper finds that including interest rate swaps in the bond portfolio optimization improves portfolio performance in terms of the Sharpe ratio. This positive effect is more pronounced when corporate bonds are included in the portfolio optimization besides the government bonds. Therefore, the advice is to use government bonds, interest rate swaps and corporate bonds altogether in the static bond portfolio optimization. Using these constituents also leads to minor gains in portfolio returns when returns decline due to sharply rising interest rates. In monetary terms, the profit is significant for large insurance companies with high investments in fixed-income portfolios even though the improvements in portfolio returns are little.

Furthermore, optimized portfolios using the first two bond return moments in their objective function outperform the benchmark minimum-concentration portfolio. The minimum-concentration portfolio is the equivalent of the equally-weighted portfolio when additional constraints are present. The mean-variance portfolio generally performs the best. The mean-variance portfolio with the uncorrelated dynamic Nelson-Siegel term structure specification is the best-performing model overall. However, in periods with fast-increasing interest rates and average portfolio performance dropping, the minimum-concentration portfolio is more stable than the optimized portfolios. The minimum-concentration portfolio then limits the losses in excess portfolio returns. Therefore, it is safer to use the minimum-concentration portfolio as opposed to the optimized portfolios when expecting rapidly increasing interest rates. The minimum-concentration portfolio also has a significantly lower turnover than the optimized portfolio. This lower turnover leads to much lower transaction costs. Thus, optimized portfolios have great potential to improve portfolio performance over the benchmark model when incorporating government bonds, interest rate swaps and corporate bonds. However, investors should consider issues regarding transaction costs and the decline of portfolio performance in periods with sharply rising interest rates.

This paper is the first step in analysing bond portfolio optimization with interest rate swaps and a duration constraint. Extensions are possible in multiple ways. A possible extension is to expand the set of investment instruments. One can include more complicated types of bonds, such as government bonds paying coupons. Different fixed-income instruments can also be incorporated such as mortgages which are often used by insurance companies. On top of that, individual corporate bonds can be included instead of corporate bond indices as a proxy of corporate bond performance to make the analysis more realistic. However, this is likely to lead to more volatile portfolios, as individual corporate bonds have higher variations in their returns than a corporate bond index consisting of a large number of these individual bonds.

Another interesting extension is using different underlying term structure models to capture the behaviour of the yield curve more appropriately. Many possibilities for this are available as the term structure literature is vast. The multifactor Vasicek-type term structure model, the widely used Cox-Ingersoll-Ross model and the flexible Hull-White model are only a few examples. Besides the term structure models, one can extend the set of portfolios used.

A final possible extension is to include the collateral problem directly in the objective function in some way instead of including it as a constraint. This change could lead to diminishing objective values of the portfolio when the company has to take on additional liquidity due to the required collateral. In this way, the costs of the liquidity risk are included in the optimization instead of in the constraints by setting a maximum for the liquidity risk through maximally allowed collateral.

References

Alizadeh, F. and Goldfarb, D. (2003). Second-Order Cone Programming. *Mathematical Programming*, 95(1):3–51.

Andersen, E. D., Roos, C., and Terlaky, T. (2003). On implementing a Primal-Dual Interior-Point Method for Conic Quadratic Optimization. *Mathematical Programming*, 95(2):249–277.

Brandt, M. W. (2010). Portfolio Choice Problems. In *Handbook of Financial Econometrics: Tools and Techniques*, pages 269–336. Elsevier.

Caldeira, J. F., Moura, G. V., and Santos, A. A. (2016). Bond Portfolio Optimization using Dynamic Factor Models. *Journal of Empirical Finance*, 37:128–158.

Christensen, J. H., Diebold, F. X., and Rudebusch, G. D. (2011). The Affine Arbitrage-Free Class of Nelson-Siegel Term Structure Models. *Journal of Econometrics*, 164(1):4–20.

Christensen, J. H. E., Lopez, J. A., and Rudebusch, G. D. (2015). Analytical Formulas for the Second Moment in Affine Models with Stochastic Volatility.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53(2):385–407.

Deguest, R., Fabozzi, F., Martellini, L., and Milhau, V. (2018). Bond Portfolio Optimization in the Presence of Duration Constraints. *The Journal of Fixed Income*, 28(1):6–26.

Diebold, F. X. and Li, C. (2006). Forecasting the Term Structure of Government Bond Yields. *Journal of Econometrics*, 130(2):337–364.

Duffie, D. and Singleton, K. J. (1997). An Econometric Model of the Term Structure of Interest-Rate Swap Yields. *The Journal of Finance*, 52(4):1287–1321.

EIOPA (2019). Report on Insurers' Asset and Liability Management in relation to the Illiquidity of their Liabilities. Technical Report EIOPA-BoS-19-593, European Insurance and Occupational Pensions Authority.

ESMA (2014). Draft Technical Standards on the Clearing Obligation – Interest Rate OTC Derivatives. Technical report, European Securities and Markets Authorities.

Flockermann, P., Garston, G., and de Albuquerque, I. (2020). Sizing Liquidity Buffers for Margined Derivatives. Technical report, Deloitte AG: Financial Risk.

Gandelman, N. and Hernández-Murillo, R. (2015). Risk Aversion at the Country Level. Technical Report 2014-005A, Federal Reserve Bank of St. Louis.

Grinblatt, M. (2001). An Analytic Solution for Interest Rate Swap Spreads. *International Review* *of Finance*, 2(3):113–149.

Hull, J. and White, A. (1990). Pricing Interest-Rate-Derivative Securities. *The Review of Financial Studies*, 3(4):573–592.

Jagannathan, R. and Ma, T. (2003). Risk Reduction in Large Portfolios: Why imposing the Wrong Constraints Helps. *The Journal of Finance*, 58(4):1651–1683.

Johannes, M. and Sundaresan, S. (2007). The Impact of Collateralization on Swap Rates. *The Journal of Finance*, 62(1):383–410.

Joslin, S., Singleton, K. J., and Zhu, H. (2011). A New Perspective on Gaussian Dynamic Term Structure Models. *The Review of Financial Studies*, 24(3):926–970.

Korn, O. and Koziol, C. (2006). Bond Portfolio Optimization: A Risk-Return Approach. *The Journal of Fixed Income*, 15(4):48–60.

Lan, C. (2015). An Out-of-Sample Evaluation of Dynamic Portfolio Strategies. *Review of Finance*, 19(6):2359–2399.

Landsman, Z. (2008). Minimization of the Root of a Quadratic Functional under an Affine Equality Constraint. *Journal of Computational and Applied Mathematics*, 216(2):319–327.

Ledoit, O. and Wolf, M. (2008). Robust Performance Hypothesis Testing with the Sharpe Ratio. *Journal of Empirical Finance*, 15(5):850–859.

Markowitz, H. (1952). Portfolio Selection. *The Journal of Finance*, 7(1):77–91.

Mehrotra, S. (1992). On the Implementation of a Primal-Dual Interior Point Method. *SIAM Journal on Optimization*, 2(4):575–601.

Michaud, R. O. (1989). The Markowitz Optimization Enigma: Is 'Optimized' Optimal? *Financial Analysts Journal*, 45(1):31–42.

Nationale-Nederlanden Levensverzekering Maatschappij N.V. (2022). 2021 Annual Report. `https://www.nn.nl/Over-NationaleNederlanden/Wie-zijn-wij/Jaarverslagen.htm`.

Pearson, J. W. and Gondzio, J. (2017). Fast Interior Point Solution of Quadratic Programming Problems arising from PDE-Constrained Optimization. *Numerische Mathematik*, 137(4):959–999.

Puhle, M. (2008). *Bond Portfolio Optimization*, volume 605. Springer Science & Business Media.

Redington, F. M. (1952). Review of the Principles of Life-Office Valuations. *Journal of the Institute of Actuaries*, 78(3):286–340.

Schaible, S. (1974). Parameter-Free Convex Equivalent and Dual Programs of Fractional Programming Problems. *Zeitschrift für Operations Research*, 18(5):187–196.

Skinner, F. (2004). *Pricing and Hedging Interest and Credit Risk Sensitive Instruments*, chapter 7+8, pages 159–226. Elsevier.

Smith, D. J. (2014). *Bond Math: The Theory Behind the Formulas*, chapter 8, pages 173–203. John Wiley & Sons.

van Bragt, D., Steehouwer, H., and Waalwijk, B. (2010). Market Consistent ALM for Life Insurers—Steps toward Solvency II. *The Geneva Papers on Risk and Insurance-Issues and Practice*, 35(1):92–109.

Widyatantri, F. and Husodo, Z. A. (2020). Asian Emerging Market Government Bond Portfolio Optimization using Mean-Variance Analysis in the Presence of Duration Constraint. *Advances in Economics, Business and Management Research*, 160.

Appendix A Kalman Filter

In this appendix, the implementation of the estimation of the model parameters using the Kalman filter for the the DNS and the AFNS model is described. The approach by Christensen et al. (2011) is followed.

A.1 Dynamic Nelson-Siegel Model

The DNS model in state space form is

$$y_t = B(\tau)X_t + \varepsilon_t, \tag{48}$$

$$X_t = (I - \Phi)\mu + \Phi X_{t-1} + \eta_t. \tag{49}$$

The initial values for the Kalman filter are the unconditional mean and variance under the P-measure, following Christensen et al. (2011). Thus $X_{1|0} = \mu$ and $P_{1|0} = V$ where V is the solution of $V = \Phi V \Phi' + Q$. To ensure stationarity such that the solution exists, the eigenvalues of Φ are restricted to be less than one. Then the solution is $V = (I - \Phi \otimes \Phi)^{-1} \operatorname{vec}(Q)$.
The first step of the Kalman filter is the prediction step which is

$$\hat{X}_{t+1|t} = (I - \Phi)\mu + \Phi \hat{X}_{t|t}, \tag{50}$$

$$P_{t+1|t} = \Phi P_{t|t} \Phi' + Q. \tag{51}$$

Then the update step is

$$\hat{X}_{t+1|t+1} = \hat{X}_{t+1|t} + P_{t+1|t}B(\tau)'(B(\tau)P_{t+1|t}B(\tau)' + H)^{-1}(y_{t+1} - B(\tau)\hat{X}_{t+1|t}), \tag{52}$$

$$P_{t+1|t+1} = P_{t+1|t} - P_{t+1|t}B(\tau)'(B(\tau)P_{t+1|t}B(\tau)' + H)^{-1}B(\tau)P_{t+1|t}. \tag{53}$$

As maximum likelihood function to estimate the unknown parameters, the prediction error decomposition is used. This is because the consecutive observations of the yields are not independent as the underlying factors have a dynamic specification. Following the specification by Christensen et al. (2011) the prediction error decomposition is

$$\log l(y_1, \ldots, y_T; \psi) = \sum_{t=1}^{T} \Bigg(- \frac{N}{2}\log(2\pi) - \frac{1}{2}\log|B(\tau)P_{t+1|t}B(\tau)' + H| - \frac{1}{2}(y_{t+1} - B(\tau)\hat{X}_{t+1|t})'(B(\tau)P_{t+1|t}B(\tau)' + H)^{-1}(y_{t+1} - B(\tau)\hat{X}_{t+1|t})\Bigg), \tag{54}$$

where $\psi = (\mu, \Phi, H, Q, \lambda)$ is the parameter vector. For the optimization the Quasi-Newton algorithm using the BFGS Quasi-Newton method with a cubic line search procedure is used, as this is the default method for the Matlab function *fminunc*.

A.2 Arbitrage-Free Nelson-Siegel Model

In the Arbitrage-Free Nelson-Siegel model, the stochastic differential equation under the P-measure for the state variables is

$$\mathrm{d}X_t = K[\theta - X_t]\mathrm{d}t + \Sigma \mathrm{d}W_t, \tag{55}$$

according to Christensen et al. (2011). Then the state space form of the AFNS model is

$$y_t = -\frac{A(\tau)}{\tau} + B(\tau)X_t + \varepsilon_t, \tag{56}$$

$$X_t = (I - \exp(-K\Delta t))\theta + \exp(-K\Delta t)X_{t-1} + \eta_t. \tag{57}$$

The initial values for the Kalman filter are the unconditional mean and variance under the P-measure, following Christensen et al. (2011), so $X_{1|0} = \theta$ and $P_{1|0} = \int_0^\infty e^{-Ks}\Sigma\Sigma' e^{-K's} ds$.
The first step of the Kalman filter is the prediction step which is

$$\hat{X}_{t+1|t} = (I - \exp(-K\Delta t))\theta + \exp(-K\Delta t)\hat{X}_{t|t}, \tag{58}$$

$$P_{t+1|t} = \exp(-K\Delta t)P_{t|t}\exp(-K\Delta t)' + Q_t, \tag{59}$$

where $Q_t = \int_0^{\Delta t} e^{-Ks}\Sigma\Sigma' e^{-K's} ds$ with $\Delta t = \frac{1}{12}$ being the time between observations.
Then the update step is

$$\hat{X}_{t+1|t+1} = \hat{X}_{t+1|t} + P_{t+1|t}B(\tau)'(B(\tau)P_{t+1|t}B(\tau)' + H)^{-1}(y_{t+1} + \frac{A(\tau)}{\tau} - B(\tau)\hat{X}_{t+1|t}), \tag{60}$$

$$P_{t+1|t+1} = P_{t+1|t} - P_{t+1|t}B(\tau)'(B(\tau)P_{t+1|t}B(\tau)' + H)^{-1}B(\tau)P_{t+1|t}. \tag{61}$$

As maximum likelihood function to estimate the unknown parameters, the prediction error decomposition is used. This is because the consecutive observations of the yields are not independent as the underlying factors have a dynamic specification. Following the specification by Christensen et al. (2011) the prediction error decomposition is

$$\log l(y_1, \ldots, y_T; \psi) = \sum_{t=1}^{T} \Bigg(-\frac{N}{2}\log(2\pi) - \frac{1}{2}\log|B(\tau)P_{t+1|t}B(\tau)' + H| - \frac{1}{2}(y_{t+1} + \frac{A(\tau)}{\tau} - B(\tau)\hat{X}_{t+1|t})'(B(\tau)P_{t+1|t}B(\tau)' + H)^{-1}(y_{t+1} + \frac{A(\tau)}{\tau} - B(\tau)\hat{X}_{t+1|t})\Bigg), \tag{62}$$

where $\psi = (\theta, K, H, \Sigma, \lambda)$ is the parameter vector. For the optimization the Quasi-Newton algorithm using the BFGS Quasi-Newton method with a cubic line search procedure is used, as this is the default method for the Matlab function *fminunc*.

Appendix B HAC Inference Test

The HAC inference test by Ledoit and Wolf (2008) tests the statistical significance of the difference in the Sharpe ratio of two portfolios. The excess returns are $R_{p1,t}$ and $R_{p2,t}$ for $t = 1, \ldots, T$ for portfolios $p1$ and $p2$, respectively. Both excess return series are assumed to be stationary processes with estimated mean and covariance matrix as

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_{p1} \\ \hat{\mu}_{p2} \end{pmatrix} \quad \text{and} \quad \hat{\Sigma} = \begin{pmatrix} \hat{\sigma}^2_{p1} & \hat{\sigma}_{p1,p2} \\ \hat{\sigma}_{p1,p2} & \hat{\sigma}^2_{p2} \end{pmatrix}. \tag{63}$$

The Sharpe ratios of the two portfolios differ by

$$\hat{\Delta}_{Sharpe} = \hat{SR}_{p1} - \hat{SR}_{p2} = \frac{\hat{\mu}_{p1}}{\hat{\sigma}_{p1}} - \frac{\hat{\mu}_{p2}}{\hat{\sigma}_{p2}}. \tag{64}$$

Define $\mathrm{E}[R_{p1,1}] = \alpha_{p1}$ and $\mathrm{E}[R_{p2,1}] = \alpha_{p2}$ with estimates $\hat{\alpha}_{p1}$ and $\hat{\alpha}_{p2}$. Also define $v = (\mu_{p1}, \mu_{p2}, \alpha_{p1}, \alpha_{p2})$ and $\hat{v} = (\hat{\mu}_{p1}, \hat{\mu}_{p2}, \hat{\alpha}_{p1}, \hat{\alpha}_{p2})$ such that $\hat{\Delta}_{Sharpe} = f(\hat{v})$ where

$$f(\hat{\mu}_{p1}, \hat{\mu}_{p2}, \hat{\alpha}_{p1}, \hat{\alpha}_{p2}) = \frac{\hat{\mu}_{p1}}{\sqrt{\hat{\alpha}_{p1} - \hat{\mu}^2_{p1}}} - \frac{\hat{\mu}_{p2}}{\sqrt{\hat{\alpha}_{p2} - \hat{\mu}^2_{p2}}}. \tag{65}$$

The assumption that $\sqrt{T}(\hat{v} - v) \xrightarrow{d} N(0, \Psi)$ holds under mild regularity conditions where Ψ is an unknown symmetric positive semi-definite matrix. Using the Delta method, one can find $\sqrt{T}(\hat{\Delta}_{Sharpe} - \Delta_{Sharpe})) \xrightarrow{d} N(0, \nabla' f(v) \Psi \nabla f(v))$. The standard error for $\hat{\Delta}_{Sharpe}$ is then given by

$$s(\hat{\Delta}_{Sharpe}) = \sqrt{\frac{\nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}{T}}, \tag{66}$$

when $\hat{\Psi}$ is a consistent estimator. To test the null hypothesis $H_0 : \Delta_{Sharpe} = 0$, the p-value of the two-sided test is

$$\hat{p} = 2\Phi\left(-\frac{|\hat{\Delta}_{Sharpe}|}{s(\hat{\Delta}_{Sharpe})}\right), \tag{67}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. As consistent estimator of $\hat{\Psi}$, Ledoit and Wolf (2008) use a group of estimators called the heteroskedasticity and autocorrelation robust (HAC) kernel estimators. The kernel incorporated in this paper is one of the kernels in Ledoit and Wolf (2008), that is the commonly used Parzen-Gallant kernel. The code for the HAC inference test is available on Michael Wolf's website.[5]

[5] `https://www.econ.uzh.ch/en/people/faculty/wolf/publications.html#Programming_Code`

Appendix C Effect of Varying Target Duration on Sharpe Ratio

Table A1: Annualized Sharpe ratio of the bond portfolios including interest rate swaps for the out-of-sample period for varying target durations.

	Target Duration (years)	10	15	20	25	30
Excluding Corporate Bonds						
MC	DNS diagonal	0.6186	0.4059	0.1939	-0.0049	-0.1634
MC	DNS VAR	0.6202	0.5070	0.4704	0.3300	0.2031
MC	AFNS diagonal	0.6220	0.3491	0.1463	0.0659	0.0441
MC	AFNS VAR	0.6206	0.3236	0.1068	0.0023	-0.0306
MV	DNS diagonal	0.8038	0.5388	0.4172	0.3393	0.2566
MV	DNS VAR	0.9675	0.8935	0.8120	0.7252	0.6699
MV	AFNS diagonal	0.5937	0.3854	0.2584	0.1802	0.1356
MV	AFNS VAR	0.7153	0.4040	0.2373	0.1226	0.0486
MSR	DNS diagonal	0.5850	0.3112	0.1815	0.1030	-0.0141
MSR	DNS VAR	0.7065	0.4534	0.3166	0.2181	0.1918
MSR	AFNS diagonal	0.6744	0.3500	0.2091	0.1546	0.1037
MSR	AFNS VAR	0.7266	0.3456	0.2069	0.0974	0.0735
Including Corporate Bonds						
MC	DNS diagonal	0.7660	0.4720	0.1386	-0.0785	-0.2565
MC	DNS VAR	0.7877	0.7496	0.6552	0.3553	0.2369
MC	AFNS diagonal	0.7889	0.3403	0.1604	0.0752	0.0451
MC	AFNS VAR	0.7787	0.2997	0.0890	-0.0152	-0.0508
MV	DNS diagonal	1.2032	0.7382	0.4681	0.3020	0.1770
MV	DNS VAR	0.8090	0.9015	0.9132	0.7390	0.6018
MV	AFNS diagonal	0.8308	0.5375	0.3185	0.1926	0.1198
MV	AFNS VAR	0.8603	0.4487	0.2291	0.0843	0.0175
MSR	DNS diagonal	0.9193	0.4998	0.2787	0.1550	0.0078
MSR	DNS VAR	0.7939	0.4815	0.3009	0.2002	0.1850
MSR	AFNS diagonal	0.7629	0.4449	0.3049	0.1812	0.1084
MSR	AFNS VAR	0.8150	0.4971	0.2449	0.1489	0.0000

Table A2: Annualized Sharpe ratio of the bond portfolios excluding interest rate swaps for the out-of-sample period for varying target durations.

	Target Duration (years)	10	15	20	25	30
Excluding Corporate Bonds						
MC	DNS diagonal	0.6183	0.3285	0.1892	0.1142	0.0986
MC	DNS VAR	0.6202	0.3303	0.1911	0.1177	0.1026
MC	AFNS diagonal	0.6220	0.3319	0.1926	0.1190	0.1045
MC	AFNS VAR	0.6206	0.3307	0.1915	0.1180	0.1029
MV	DNS diagonal	0.6590	0.3571	0.2058	0.1444	0.0993
MV	DNS VAR	0.6164	0.3889	0.2729	0.1697	0.1026
MV	AFNS diagonal	0.5470	0.3283	0.2193	0.1505	0.1044
MV	AFNS VAR	0.6454	0.4029	0.2818	0.1769	0.1028
MSR	DNS diagonal	0.5545	0.2945	0.1207	0.0675	0.0205
MSR	DNS VAR	0.6296	0.3669	0.1515	0.0582	0.0295
MSR	AFNS diagonal	0.6529	0.3366	0.2193	0.1473	0.0543
MSR	AFNS VAR	0.6569	0.1596	0.1339	0.0889	0.0513
Including Corporate Bonds						
MC	DNS diagonal	0.6723	0.3461	0.1932	0.1143	0.0990
MC	DNS VAR	0.6749	0.3489	0.1974	0.1177	0.1028
MC	AFNS diagonal	0.6765	0.3504	0.1987	0.1190	0.1045
MC	AFNS VAR	0.6752	0.3492	0.1977	0.1180	0.1031
MV	DNS diagonal	0.9858	0.4865	0.2421	0.1572	0.0997
MV	DNS VAR	0.7716	0.4550	0.2807	0.1727	0.1027
MV	AFNS diagonal	0.8040	0.4723	0.2924	0.1788	0.1047
MV	AFNS VAR	0.7979	0.4776	0.2973	0.1676	0.1033
MSR	DNS diagonal	0.8746	0.3221	0.1576	-0.0021	-0.0292
MSR	DNS VAR	0.6936	0.3973	0.2304	0.0362	-0.0163
MSR	AFNS diagonal	0.7529	0.4433	0.2793	0.1763	0.1033
MSR	AFNS VAR	0.6970	0.4137	0.2072	0.0584	0.0253