LIBOR Market Model calibration for Credit Valuation Adjustment of a portfolio of swaptions

Addressing the continuity problem: Extending a portfolio of swaptions without unexpectedly changing the exposure of the original portfolio

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Abstract

This thesis addresses the continuity problem and presents a solution to this issue. The continuity problem might occur when the expected exposure profile of a portfolio swaptions is calculated using a LIBOR Market Model (LMM) calibrated towards swaption volatilities. When such a portfolio is extended, LIBOR rate volatilities over the whole tenor structure may change, resulting in a different expected exposure profile for the swaptions in the original portfolio. Expected positive exposures (EPE) are used to calculate Credit Valuation Adjusted (CVA) charges. Unexplainable changes in CVA resulting from a portfolio extension are undesirable for practitioners. In our proposed solution, we calibrate the LMM towards Black implied caplet volatilities such that instantaneous LIBOR rate volatilities remain unaltered when extending a portfolio. We calculate a swaption-specific adjustment factor to set the simulated swaption value equal to the market observed value. EPE profiles are adjusted in a multiplicative manner with this adjustment factor. We find that our approach is a practical solution that provides reliable EPEs. Therefore, we deem our method valuable for practitioners, like banks or other financial institutions.

Keywords: *Shifted LIBOR Market Model, Interest rate swaptions, Least-Squares Monte Carlo, CVA, Expected Positive Exposure, Adjustment Factor, Calibration, Cash-settled, Physically-settled*

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Contents

1 Introduction

To provide a fair value adjustment for the valuation of financial derivatives to assess counterparty credit risk, financial institutions make use of Credit Valuation Adjustment (CVA). CVA can be defined as the difference between the true portfolio value, which includes the possibility that a counterparty may default, and the value of the risk-free portfolio where there is no risk of a counterparty default (Smith, 2015). Since CVA was introduced as a new requirement for fair value accounting in 2007/2008, adjusting the prices of financial derivatives according to credit risk has become industry practice. Therefore, it is considered that pricing this credit risk adjustment as accurately as possible is of great importance, and it is this aspect to which this thesis aims to contribute. Generally speaking, CVA can be decomposed into three parts, the first of which is the counterparty's Probability of Default (PD), which measures the likelihood of a default event. The second component is the Loss Given Default (LGD), which measures the share of an asset lost if a counterparty defaults. The third element, and also the element of interest in this thesis, is the Expected Positive Exposure (EPE), which measures the expected potential size of the obligation owed by the counterparty.

In this thesis, we will discuss simulated EPE profiles of portfolios of physically-settled, cash price-settled and collateralised cash price-settled (CCP) European swaptions within the displaced LIBOR Market Model (LMM) framework. Here LIBOR is an acronym for London Interbank Offered Rate. The LMM, with its extensions, can be used to price all types of different interest rate derivatives, e.g., swaps, caps, floors, and swaptions. In the context of swaptions, computing expected exposures can become complex as Nested Monte Carlo simulations would be needed to compute exposures at any moment in time. Since nested Monte Carlo simulations are computationally expensive and performance is highly important for derivative pricing, CVA calculation through Nested Monte Carlo schemes is not desirable. One of the ways to avoid using a Nested Monte Carlo algorithm for exposure calculation is by using the Least Squares Monte Carlo (LSMC) algorithm, introduced in Longstaff & Schwartz (2001). Herein, using regressions, a Least Squares approach to determine the continuation value of Bermudan-type options at a given moment in time is presented. Though we are not exactly interested in the continuation values of a swaption in the context of CVA, we do want to know the asset's approximated value at a later moment in time. For this, we can also use the LSMC algorithm. The value resulting from the regressions in the LSMC can be used to calculate expected exposures. Joshi & Kwon (2016) propose a method to calculate CVA motivated by a new representation of the expected positive exposure, which does not depend on the value, but solely on the sign of the regression function values. They show that their proposed method yields smaller errors in calculating CVA than the approach using the regression value directly.

To obtain expressions for the CVA of European swaptions, we first need to calibrate deterministic parameters in the LIBOR Market Model, specifically the instantaneous volatility of the LIBOR rates and the correlation between rates. For the scope of this thesis, we will only focus on the calibration of the volatilities of the forward rates using two specific types of interest rate derivatives, i.e., caplets and swaptions. The calibration of the LMM towards caplets is relatively straightforward. Under the assumption that displaced LIBOR rates are log-normally distributed and martingales within the LMM framework, we have that the price of an At-the-Money (ATM) caplet calculated by means of the Black'76 option pricing formula corresponds directly with the price of the caplet as it was calculated through the LMM. A proof of this statement will be shown later in the thesis. By setting the instantaneous LIBOR volatilities equal to the ATM Black implied caplet volatilties, we do not have to perform an optimization scheme to determine the LIBOR rate volatilities.

The calibration of the LMM towards swaptions asks for slightly more involvement. If we were to price a swaption, under specific assumptions which we will discuss later, we could calibrate the LIBOR model towards so-called co-terminal swaptions to obtain the instantaneous volatilities of the LIBOR rates. Co-terminal swaptions are swaptions with the same maturity date of the underlying swap as the swaption of interest but with a different expiry date of the option on this swap. Suppose we are interested in the exposure of a 2-year option on a 3-year swap, denoted as a 2Y3Y swaption. The set of co-terminal swaptions consists of the {1Y4Y, 2Y3Y, 3Y2Y, and 4Y1Y}-swaptions. As a result, we obtain a set of 4 specific volatilities for LIBOR rates covering the entire tenor structure of the swaption of interest. It is then possible to evaluate this swaption's price and expected exposure profiles using the LIBOR Market Model.

We could also calibrate the LMM towards all possible swaptions in the whole tenor. This implies that we will calibrate the model towards the whole swaption matrix (see for example Brigo & Mercurio (2006)). We will show that swaption volatilities in the LMM can be approximated as functions of instantaneous LIBOR volatilities. As swaption volatilities in the LMM framework do not correspond 1-on-1 to their Black implied volatilities, we need to perform an optimization scheme to estimate the swaption volatilities within the LMM.

Practitioners, like banks or other financial institutions, often have multiple trades, among which swaptions, outstanding with a counterparty. In the context of CVA, adding a new swaption to an existing portfolio of swaptions may entail an issue, which we refer to as the continuity problem. Suppose we have a portfolio of swaptions on very large notional amounts outstanding with a counterparty. Now, the counterparty wants to add trade over a longer tenor structure than the longest tenor structure already included in the portfolio. For example, the counterparty wants to add a 5Y15Y swaption to a portfolio in which the longest tenor is a 5Y10Y trade. Furthermore, suppose this additional trade is on a very small notional, making the new trade practically irrelevant. If we were to calibrate the model, under specific assumptions discussed later, towards co-terminal swaptions, we would need to select a set of co-terminals. However, the question arises which set of co-terminals needs to be selected. The set of co-terminal swaption volatilities directly impacts the volatilities and thus the dynamics of the LIBOR rates. Different LIBOR rates may yield different exposure profiles. Moreover, if we choose to calibrate the model towards a matrix of swaption volatilities, we would need to perform an optimization scheme that may result in different LIBOR rate dynamics. By adding a set of constraints to the optimization problem, we may end up with a different optimal solution. Consequently, LIBOR volatilities and ultimately expected exposures might differ by adding the practically insignificant trade. The calibration problem that arises when adding a 'longer' swaption to an existing portfolio of swaptions is what we call the continuity problem. Therefore, the question this research aims to answer is:

Can we solve the continuity problem, while still calculating reliable EPEs?

So why is the continuity a problem for practitioners like banks or other financial institutions? Expected exposure is an important element in the calculation of CVA. Modified exposures resulting from adding a new swaption to the portfolio yield a modified CVA charge. If a practitioner's exposure to a counterparty increases due to a newly added swaption, practitioners are likely to pass the CVA charge on to the counterparty. Suppose a counterparty has to pay a significant additional charge after closing a new insignificant contract. In that case, the counterparty may lose confidence in the evaluation methods of the bank or financial institution. On the other hand, if exposure significantly decreases, the counterparty will think that it had paid too much CVA charge in the past also causing discredit. Therefore, the exposure profiles of significant trades must not be modified by adding an insignificant trade. An additional reason why sudden changes in CVA are undesirable is that CVA translates directly into the profit and loss statement of a bank or financial institution. Therefore, unexplainable changes in the CVA result directly in unexplainable changes on the P&L statement. This is for various obvious reasons, among which regulatory reasons, not wanted. To the best of our knowledge, the continuity problem is not yet touched upon in the academic literature. This research aims to bring attention to the problem. Moreover, we will also suggest a practical approach to this issue, as discussed below.

We will approach the continuity problem by calibrating the model towards caplets instead of swaptions. Due to their analytical equivalence, instantaneous LIBOR volatilities can be obtained directly from caplet volatilities. Because we do not need to select a set of co-terminal swaptions or need to perform an optimization scheme, we can calibrate the model on a year-to-year basis. That is, taking a model with annually spaced LIBOR rates as an example, the instantaneous volatility of the first LIBOR rate can be obtained through the 1Y1Y caplet volatility, the volatility of the second rate using the 2Y1Y caplet volatility and so forth. Therefore, if we have a portfolio of swaptions with different tenor structures, we can simply use the set of all Black implied caplet volatilities over the entire duration to obtain specifications of the instantaneous volatilities for this tenor. Obviously, we introduce an error by calibrating the market model towards caplets in the context of swaption pricing. Market implied swaption prices will not correspond to prices obtained from the calibrated model. We will present a straightforward, practical method to decrease the error due to miscalibration in the form of a certain adjustment factor. This adjustment factor modifies the miscalibrated simulated swaption pay-offs towards the direction of analytical value. The adjustment factor ensures that at the observation date, the price of a simulated swaption will be equal to the analytical price.

This thesis contributes to the existing literature in the following ways. We will analyse the EPE profiles of portfolios of swap-settled, cash price-settled, and collateralised cash price-settled (CCP) swaptions using a modified definition of the EPE presented in Joshi & Kwon (2016). For this method, we will present a proof showing the justification of the modification. To calculate EPE profiles, we make use of the Least Squares Monte Carlo algorithm introduced in Longstaff & Schwartz (2001). Even though EPE profiles for single physically-settled swaptions have already been investigated in Joshi & Kwon (2016), results for both single and portfolios of cash price-settled and CCP-settled swaptions are yet to be discussed. Moreover, we address the explained continuity problem and introduce a method to approach this issue. Subsequently, we will analyse the implications of this method in the context of credit valuation adjustment to provide a well-covering overview for practitioners.

The remainder of this paper is structured as follows. In Section 2 we will discuss and summarize existing literature on interest rate modelling, interest rate swaps, and CVA modelling. In Section 3 we will discuss the construction of the LIBOR Market Model. Section 4 touches upon interest rate derivative pricing and discusses the calibration of the LIBOR Model. Moreover, herein we will present our proposed solution to the continuing problem introduced above. We will explore Credit Valuation Adjustment and Least Squares Monte Carlo in Section 5. In Section 6, we will discuss our results, which we conclude and summarize in Section 7. Lastly, in Section 8 we elaborate on the limitations of this study and suggest further research.

2 Literature review

In this section, we will discuss existing literature on the key elements that play a role in this thesis. We will first touch upon the topic of interest rate modelling and more specifically, the LIBOR Market Model. Subsequently, we will discuss research covering model calibration using caplets and swaptions. Lastly, we discuss findings related to Credit Valuation Adjustment (CVA).

2.1 Literature on the LIBOR Market Model

Vasicek (1977) was one of the earliest papers to develop a model to describe the evolution of interest rates with the aim to price bonds and interest rate derivatives in the absence of arbitrage opportunities. The standard Vasicek model is a single-factor short-rate model as market risk is the only driver of interest rates, though the model can be easily extended to a multi-factor framework. The Vasicek model is a time-homogeneous equilibrium model, meaning that the drift and diffusion term parameters in the stochastic differential equation do not vary over time. The model provides a set of theoretical bond prices that usually do not precisely match the actual prices observed in the market. This issue led to the development of time-inhomogeneous interest rate models like Ho & Lee (1986) and J. Hull & White (1990). Heath et al. (1992) proposed a new framework that aims to model the evolution of the complete forward-rate curve over time instead of focusing on the short rate alone. All well-known models above have their own advantages and disadvantages, and their utility is application-specific. One of the main drawbacks of the short rate- and HJM models is that these models focus on the instantaneous interest rates, which are unobservable in the market. In the late 1990's so-called market models were developed that directly model observable market rates such as the LIBOR rate and interest swap rate. Miltersen et al. (1997), Brace et al. (1997) and Jamshidian (1997) introduced the LIBOR Market Model. The LMM models a finite number of log-normally distributed forward rates also referred to as LIBOR rates. Because of the log-normality of these rates, calibration of the LIBOR Market Model is simple, as Black's formula (Black, 1976) is consistent for the pricing of caps and floors.

Unfortunately, the LIBOR Market Model framework has its own disadvantages. Since forward rates are assumed to be log-normally distributed, the model can not cope with modelling negative interest rates. Although it was previously assumed that interest rates would never fall below zero, the current market environment shows negative rates, making this a large model pitfall. Brigo & Mercurio (2006) introduces the displaced LIBOR Market model to tackle this issue. By allowing a displacement in the simulated forward rates, one could obtain negative rates by subtracting the displacement value from the simulated rates. In this thesis, we will derive the LIBOR Market

Model along the lines of Fries (2007) and simulate interest rates accordingly.

In some situations, it is desired to find an expression for a LIBOR forward rate corresponding to dates S and T for which either date is not included in the predetermined tenor structure corresponding to the particular LMM model. Because of this, it is important to define an interpolation scheme to obtain non-standard LIBOR rates. The first of such interpolation schemes were introduced in Schlögl (2002). C. J. Beveridge & Joshi (2009) note that the proposed method in Schlögl (2002) does not exclude internal arbitrages and could lead to negative rates even in the case of a displacement of 0. Werpachowski (2010) states that the interpolation methods proposed in Schlögl (2002) and C. J. Beveridge & Joshi (2009) yield modelled interpolated forward rates with much lower volatility than the rates modelled by the regular LMM. The interpolation scheme in Werpachowski (2010) uses a log-normal, non-displaced market model and is therefore not directly applicable in our framework. Bogt (2018) extends Werpachowski (2010) in such a way that the interpolation scheme can deal with a displaced LIBOR market model. We use this method in this thesis to interpolate LIBOR rates.

Before the 2008 economic crisis, some practitioners used a single curve to discount and project future interest-rate derivative cash flows. LIBOR rates were considered risk-free, implying that discount bonds could replicate these forward rates. Before the financial crisis, LIBOR rates and risk-free OIS (Overnight Index Swaps) rates were found to be similar, and the spread between these two rates was viewed as negligible. After the crisis, however, the spread between the two curves increased significantly and has never returned to the pre-crisis level (Mercurio (2010)). This led to the conclusion that LIBOR rates were no longer considered a suitable proxy for the discount rate. Currently, OIS rates are regarded as the best available proxy for risk-neutral rates and are thus used for discounting. In Europe, the relevant OIS rate is EONIA. This is a weighted average of overnight unsecured lending rates in the European interbank market. The approach to generate future cash flows through the LIBOR Market Model and use the OIS curve for discounting is known as the multi-curve framework. This thesis will evaluate the prices and exposures of interest rate products within this framework.

2.2 Literature on LMM calibration

The pricing of interest rate derivatives is highly dependent on the interest rate process with parameters that are calibrated towards market quotes of related interest rate products. Before we can simulate interest rates according to the LIBOR Market Model, we must first appropriately set the volatilities and correlations of the LIBOR rates. This process is commonly referred to as model calibration. The (displaced) log-normal forward LIBOR model can price interest rate derivatives as caps and floors consistently the Black 1976 pricing formula. Because of the equivalence between the LMM and the Black equation for these products, we can use, e.g., At-the-Money quoted cap prices to calibrate the LIBOR Market Model. More particularly, we can use the Black implied volatilities of the caplets out of which the cap consists. Since the market quotes only caps and not caplets, we will need to obtain caplet prices and volatilities through a recursive bootstrapping method. This process is called caplet stripping and is not a trivial exercise. We find a selection of volatility stripping techniques in the literature, for example, presented in Brigo & Mercurio (2006) and P. Hagan & Konikov (2004).

Unfortunately, the equivalence between the LMM and the Black 1976 formula does not hold in the case of swaption pricing. In the LIBOR market model framework, simulated forward rates are log-normally distributed. Swap rates, a linear combination of weighted forward rates, can therefore not be log-normally distributed simultaneously. Put differently, forward and swap rates can not be both distributed log-normally under the same probability measure (Brigo & Mercurio (2006)). However, it is possible to find approximate solutions of the swap rate volatility in terms of state variables alive in the LIBOR Market Model, which thus allows model calibration to approximated swaption volatilities. The first of such was the Rebonato approximation introduced in Rebonato (1999). J. C. Hull & White (2000), Kawai (2003) and Van Appel & McWalter (2018) extended Rebonato's approximation to allow for a more sophisticated approximation of the Black consistent swaption volatilities. To find the model parameters for the complete tenor structure of a swaption of interest, we can also choose to calibrate towards co-terminal swaptions. Gatarek et al. (2007) states that the calibration of the LIBOR Market Model towards co-terminal swaptions is a widely used technique to find the model parameters. Model calibration towards either caplets or a set of swaptions with equal maturity plays a significant role in this thesis. We state that model calibration towards a complete swaption matrix or using co-terminal swaptions in the case of extending a portfolio of swaptions allows for the existence of the continuity problem. Therefore, in a setting where we aim to model the Credit Valuation Adjustment, we propose calibrating the model towards caplet volatilities, even in the context of a portfolio of swaptions.

2.3 Literature on CVA

For some time now, it has been standard practice for derivative traders to adjust the price of a derivative traded with a counterparty incorporating credit risk. This Credit Valuation Adjustment (CVA) reflects the risk of losing money because of a default by the counterparty. CVA's can be particularly complex to value (J. Hull & White (2012)). As discussed earlier, CVA can be decomposed into three different parts, i.e., the Probability of Default of the counterparty (PD), the loss given a counterparty default (LGD), and the Expected Positive Exposure on the counterparty (EPE). It does not come as a surprise that extensive literature exists on all three of these components. As we shall be focussing on the EPE only in this thesis, we will leave the literature on the PD and LGD out of this review. The expected positive exposure is defined as the average or expected positive credit exposure of a derivative. Similarly, we can define the expected negative exposure (ENE) as the average negative exposure of a derivative. The summation of the EPE and ENE forms the Expected Exposure (EE).

A great part of the thesis consists of comparing EPE profiles for swaptions and portfolios of swaptions within a LIBOR Market Model framework. Obtaining expressions for the EPE shows great resemblance with obtaining expressions of the continuation value of Bermudan or American type options. Therefore, before diving into the topic of CVA calculation, we will first provide a summary of the valuation of these derivatives. For a long time, pricing American- and Bermudan-style options has been a complex exercise due to the absence of closed-form solutions for these products. An approach by constructing a binomial tree Cox (1996) and multidimensional generalizations Boyle (1988) have been proposed, though these methods only work well in a low-dimensional setting. As the lattice size grows exponentially, the computational time increases significantly, and accurate valuation becomes more complex. To solve this problem, also known as the curse of dimensionality, several methods have been introduced to estimate the continuation values, among which the randomized tree method (Broadie & Glasserman, 1997), the stochastic mesh method (Broadie et al., 2004) and regression-Based methods (Carriere (1996), and Longstaff & Schwartz (2001)). Longstaff & Schwartz (2001) introduce the Least Squares Monte Carlo (LSMC) method to estimate the continuation value of a financial product or portfolio of products by regressing the ex-post realized pay-off from continuation on (a set of) polynomial functions of state variables. For the case of swaptions, this set of state variables contains, for example, the set of simulated LIBOR rates. In the context of CVA, we are not interested in the continuation value of a derivative but in its expected value at a future moment in time. This resembles then the expected exposure we have to a counterparty. In order to estimate the expected exposure of a derivative at a specific moment in time, we need to choose a set of orthogonal basis functions and project the realized simulated discounted pay-off on the space spanned by these functions. This thesis will use the LSMC algorithm to find the expected exposures.

In the literature, we find many examples of extensions of and further research into the Least Squares Monte Carlo algorithm, not all necessarily related to CVA. Areal et al. (2008) conduct an extensive sensitivity analysis to evaluate the impact of the choice and number of polynomial basis functions and the number of paths on the algorithm's accuracy. Though not much difference in accuracy could be found for plain vanilla American options using a variety of basis functions, the weighted Laguerre function did have a slight advantage over other functions, especially in the context of portfolios of options. Moreover, Areal et al. (2008) tried different Least Squares algorithms, among which Singular Value Decomposition (SVD) on the matrix of state variables, which improved estimation accuracy. Moreno & Navas (2003) proposed a variance reduction technique using importance sampling centred on Girsanov's theorem. By sampling more In-the-Money paths in the Monte Carlo simulation, a decrease in the variance of the price of an American option can be achieved. Another method to reduce the variance of the estimated continuation value is antithetic sampling, already introduced in Longstaff & Schwartz (2001). Here, by the nature of the increments of the Brownian motion, we can increase the number of sampled paths by considering not only the simulated paths but also their exact opposite pairs. Therefore, we obtain more simulated paths without the burden of simulating more individual paths.

Though the LSMC algorithm is still widely applied to price financial derivatives. Interestingly, Joshi & Kwon (2016) states that the regression functions used in the method do not surely provide correct regressed option values over all simulated paths, which could lead to a bias in both pricing and CVA. Joshi & Kwon (2016) proposes a method to calculate CVA motivated by a new representation of the expected positive exposure, which does not depend on the value, but solely on the sign of the outcome of the regression functions. They show that a comparison of CVA for Bermudan swaptions suggests that the newly introduced method leads to smaller errors than the standard approach, which uses the regressed value directly to calculate CVA. Considering the simplicity and improved accuracy of the JKLSM compared to other methods to calculate CVA using Least Squares Monte Carlo, we will use this modified presentation of the EPE to evaluate exposure profiles of portfolios of swaptions.

3 The LIBOR Market Model

3.1 Interest rate modelling

In this section, we will construct and interpret the displaced LIBOR market model. The mathematical framework used to construct the LMM and which we further use in this thesis can be found in Appendix 9.1. For the sake of clarity, only the main results are stated in this section. We provided an extensive derivation of the LMM in the Appendix. This derivation follows very closely Fries (2007).

3.1.1 Model set-up

We assume a tenor structure given by $0 = T_0 < T_1 < ... < T_N < \infty$. The accrual factor between dates T_i and T_{i+1} is given by $T_{i+1} - T_i = \tau_i$. The tenor dates correspond to LIBOR forward rates with maturity $T_0, .., T_{N-1}$. The subscript $T_{(\cdot)}$ implies the relation between the forward rate and a zero-coupon bond maturing at time $T_{(\cdot)}$, which we will refer to back later. The goal of this section is to define a model that generates forward rates, which are commonly refered to as forward LIBOR rates. The forward LIBOR rate fixed at T_i over the period $[T_i, T_{i+1}]$ is defined as $L(t, T_i, T_{i+1})$. This notation implies that we observe at time t the LIBOR rate over the period $[T_i, T_i + \tau_i]$. To make notation neat, we will represent this rate by $L(t, T_i)$. We will start by formalising the displaced LIBOR market model with correlated Brownian motions with probability measures corresponding to one specific LIBOR rate. Then we will proceed by constructing a model under the Terminal measure. Finally, the model using the risk-neutral Spot measure is derived. The latter model will be used in this thesis to generate the forward rates to obtain expressions for the EPE of swaptions.

First, we define integers $q(t)$ for $t \in [T_0, T_{N-1}]$ by $T_{q(t)-1} \leq t < T_{q(t)}$. We assume a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[T_0,T_N]}, \mathbb{P}^{T_i})$, where $(\Omega, \mathcal{F})$ denotes the sample space, $\mathcal{F}_t$ represents a filtration and $\mathbb{P}^{T_i}$ the T_i forward measure. In Appendix 9.1 we elaborate on sample spaces, filtration and probability measures. Moreover, let $\tilde{W}^{T_i}(t)$ for now be a 1-dimensional Wiener process under this T_i probability measure, as defined in Appendix Definition 9.8. Later we will touch upon the concept d-dimensional Wiener processes. Let $P(t, T)$ be an arbitrage free zero-coupon bond that pays out 1 at maturity T, for which holds that the bond prices $P(\cdot, T_i)$ are priced under the risk-neutral measure $\mathbb{Q}$. Pricing under the risk-neutral measure implies that the price of an asset is exactly equal to the value of its discounted expectation under this measure. We elaborate more on the risk-neutral pricing and the risk-neutral measure in the Mathematical Framework in Appendix 9.1.

bability measure such that each share price is exactly equal to the discounted expectation of the share price under this measure.

Definition 3.1 (**Simply-compounded forward rate**). *The simple compounded forward rate $L(t, T_i)$, valued at time t with fixing date T_i and payment date $T_i + \tau_i = T_{i+1}$ is defined as*

$$L(t, T_i) = \frac{1}{\tau_i}\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right)$$
$$= \frac{1}{\tau_i}\left(\frac{P(t, T_i) - P(t, T_{i+1})}{P(t, T_{i+1})}\right). \tag{3.1}$$

Definition 3.1 shows that forward rates can be expressed in discounted zero-coupon bonds. Moreover, Equation 3.1 implies that

$$\frac{P(t, T_{i+1})}{P(t, T_i)} = \frac{1}{1 + \tau_i L(t, T_i)} \tag{3.2}$$

which entails that

$$\frac{P(t, T_1)}{P(t, T_0)} \cdot \frac{P(t, T_2)}{P(t, T_1)} \cdots \frac{P(t, T_N)}{P(t, T_{N-1})} = \frac{P(t, T_N)}{P(t, T_0)} = \prod_{i=0}^{N-1} \frac{1}{1 + \tau_i L(t, T_i)}. \tag{3.3}$$

Evaluating this expression at $t = T_0$ gives

$$P(T_0, T_N) = \prod_{i=0}^{N-1} \frac{1}{1 + \tau_{i-1} L(T_0, T_i)}. \tag{3.4}$$

At any day $t \in [T_0, T_N]$, information of the LIBOR rates is not sufficient to uniquely determine the discount bonds for the entire tenor period. Suppose $T_j < t < T_{j+1}$, $j \in \{0, ..., N-1\}$ and we want to know $P(t, T_i)$ for $i > j + 1$. Let us take the last day included in the tenor structure, i.e. T_N, so that we can find

$$P(T_{j+1}, T_N) = \prod_{i=j+1}^{N-1} \frac{1}{1 + \tau_i L(t, T_i)}. \tag{3.5}$$

However, this bond only discounts payments from T_N until T_{j+1} and we have defined $t < T_{j+1}$. We thus need another discount factor to discount from T_{j+1} to t. Here our definition of $q(t)$ comes into use, since $q(t)$ is the index of the first payment date after t. Therefore, we can define

$$P(t, T_N) = P(t, T_{q(t)}) \prod_{i=j+1}^{N-1} \frac{1}{1 + \tau_i L(t, T_i)}, \tag{3.6}$$

and more generally for $i \geq q(t)$

$$P(t, T_i) = P(t, T_{q(t)}) \prod_{j=q(t)}^{i-1} \frac{1}{1 + \tau_j L(t, T_j)}. \tag{3.7}$$

Thus, we can only determine discount bond prices at time t not part of the tenor times in case we also specify the discount bond $P(t, T_{q(t)})$. Glasserman (2004) refers to this discount bond as the current price of the shortest maturity bond.

Since $P(t, T_i)$ and $P(t, T_{i+1})$ are both tradable assets and priced under the risk-neutral measure, the difference between these assets is also a tradable asset and valued under the risk-neutral measure. Therefore, discounting this value by its numéraire, $P(t, T_{i+1})$, yields that the LIBOR rate $L(t, T_i)$ as in Equation 3.1 is a martingale under the corresponding T_{i+1}-forward measure (see for example C. Beveridge et al. (2008)). We can then construct the dynamics of the forward rates in the LIBOR market model, associated with their specific forward measures, given by

$$dL(t, T_i) = L(t, T_i)\sigma(t, T_i)d\tilde{W}^{T_{i+1}}(t), \quad t \in [0, T_{i-1}], \ \ i \in 1, ..., N. \tag{3.8}$$

Here, $\sigma(t, T_i)$ denotes the instantaneous volatility of the forward rate $L(t, T_i)$ and is assumed to be deterministic. Note that since the LIBOR rate $L(t, T_i)$ under the T_{i+1}-forward measure is a T_{i+1}-martingale, the rate follows a drift-less process. Thus, in the above declared LIBOR Market Model, the dynamics of the forward rate are modelled as drift-less Geometric Brownian Motions.

3.1.2 Displaced LIBOR rate

The LIBOR Market Model, in its log-normal form, does not allow for negative interest rates. For a very long time, this characteristic has been one of the advantages of the model. However, for the past few years, negative interest rate have become more common and thus the model's non-negativity constraint on the forward rates has become a disadvantage. We can solve this problem by shifting the forward rates by a constant. This adjusted model has become known as the displaced LIBOR Market Model. The displaced forward rates are defined as follows, for some $\beta \in \mathbb{R}$

$$\begin{aligned} \tilde{L}(t, T_i) &= L(t, T_i) + \beta, \\ &= \frac{1}{\tau_i}\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right) + \beta. \end{aligned} \tag{3.9}$$

The deterministic and constant parameter β is added to make sure that the initial displaced forward rates are positive. Thus, we can assume that the displaced forward rates have a log-normal distribution. Moreover, it is assumed that $\tilde{L}(t, T_i) := L(t, T_i) + \beta > 0$ for all $i \in \{0, ..., N-1\}$, with $\tau_i\beta < 1$. In this thesis, we simulate the displaced forward rates, which are strictly positive and by subtracting a positive value of β, we can obtain forward rates that are negative.

The assumption stated above is effectively a restriction on the price of the zero-coupon bonds, as the restriction boils down to $P(0, T_i) > (1 - \beta\tau_i)\, P\,(0, T_{i+1})$, using Equation 3.1 and herein setting $t = 0$. This relation indicates that the prices of zero-coupon bonds can be slightly increasing or decreasing by some amount. For $\beta = 0$, we find that the prices are decreasing for increasing values of i.

3.1.3 Creating a correlated d-dimensional Wiener process

Previously, we have stated that the forward rates are modelled using a single Brownian motion, $\tilde{W}^{T_i}(t)$, for $i \in [1, ..., N]$ under the T_{i+1} forward measure as described in Appendix 9.1 Definition 9.9. This framework can be extended by allowing the 1-dimensional Brownian motion to be a d-dimensional Wiener process, consisting of either independent or correlated Brownian motions. We first define the d-dimensional Brownian motion under the T_{i+1} forward measure as $\tilde{W}^{T_{i+1}}(t) = \left(\tilde{W}_1^{T_{i+1}}(t), ..., \tilde{W}_d^{T_{i+1}}(t)\right)$. To obtain an expression for the correlation between two d-dimensional Brownian motions $W^{T_{i+1}}$ and $W^{T_{i+1}}$ for $i, j \in \{0, N-1\}$, we first introduce the concept of quadratic variation of a single process and the quadratic covariation between two processes X and Y.

Definition 3.2 (**Quadratic variation of a process**). *The quadratic variation of stochastic process $X(t)$ is a path-wise measurement of its the variance and can be defined as*

$$\langle X \rangle_t = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2, \tag{3.10}$$

where Π is considered a partition of the interval $[0, t]$ and the mesh of the partition $||\Pi|| = \max_{1 \le i \le n-1} \{t_{i+1} - t_i\}$, is the length of the longest of these subintervals.

Definition 3.3 (**Quadratic covariation of processes X and Y**). *The quadratic covariation of processes X and Y can be defined as*

$$\langle X, Y \rangle_t = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

Moreover, if $\tilde{W}^{(\cdot)}(t)$ is a Wiener process under a probability measure, then it can be shown that $\langle \tilde{W}^{(\cdot)}(t) \rangle = t$ and in differential form

$$d\langle \tilde{W}^{(\cdot)}(t) \rangle = (d\tilde{W}^{(\cdot)}(t))^2 = dt. \tag{3.11}$$

Now let $\tilde{W}^{T_{i+1}} = (\tilde{W}_1^{T_{i+1}}, \tilde{W}_2^{T_{i+1}}, ..., \tilde{W}_d^{T_{i+1}})$ be a d-dimensional Wiener process under the forward measure T_{i+1} and $d\langle \tilde{W}^{T_{i+1}}(t), \tilde{W}^{T_{j+1}}(t) \rangle = dt$ if $i = j$ and 0 otherwise. We want to construct a d-dimensional Wiener process $W^{T_{i+1}}(t)$ based on $\tilde{W}^{T_{i+1}(t)}$ that has correlation $\rho_{ij}(t)$ with $W^{T_{i+1}}(t)$ given by

$$d\langle W^{T_i}(t), W^{T_j}(t) \rangle = \rho_{ij}(t) dt. \tag{3.12}$$

To construct such a Brownian motion, we make use of a decomposition of the correlation matrix $\mathrm{P}(t)$. Suppose $\mathrm{P}(t) = C(t)C(t)^T$, with $C(t)$ being a lower triangular matrix. We can now construct the dynamics of the Wiener process $W^{T_{i+1}}(t))$ follows

$$dW^{T_{i+1}}(t) = C_t d\tilde{W}^{T_{i+1}}(t). \tag{3.13}$$

Moreover, suppose $C_t = (c_{ij}(t))_{n\times d}$, then we can rewrite Equation 3.13 to

$$dW^{T_{i+1}}(t) = \sum_{k=1}^{d} c_{ik}(t) d\tilde{W}_k^{T_{i+1}}(t). \tag{3.14}$$

Now, we can find an expression for the quadratic variation in differential form given by

$$\begin{aligned} d\langle W^{T_{i+1}}(t), W^{T_{i+1}}(t)\rangle &= d\langle \sum_{k=1}^{d} c_{ik}(t) d\tilde{W}_k^{T_{i+1}}(t), \sum_{l=1}^{d} c_{jl}(t) d\tilde{W}_l^{T_{i+1}}(t)\rangle, \\ &= \sum_{k=1}^{d} c_{ik}(t) c_{jk}(t) dt, \\ &= \rho_{ij}(t) dt. \end{aligned} \tag{3.15}$$

We have thus shown that $W^{T_{i+1}}(t)$ and $W^{T_{j+1}}(t)$ are correlated. Note that this holds for $i, j \in \{0, ..., N-1\}$. Finally, taking account of the displaced LIBOR rates and the correlated d-dimensional Brownian motion, we postulate that

$$d\tilde{L}(t, T_i) = \tilde{L}(t, T_i)\sigma(t, T_i) dW^{T_{i+1}}(t), \tag{3.16}$$

with initial condition

$$\tilde{L}(0, T_i) = \frac{1}{\tau_i}\left(\frac{P(0, T_i)}{P(0, T_{i+1})} - 1\right) + \beta. \tag{3.17}$$

Our research focusses on calibration of the LIBOR Market Model with respect to the instantaneous volatilities of the LIBOR rates. Instead of using a d-dimensional correlated Brownian motion, we wil simulate forward LIBOR rates using a 1-dimensional Brownian motion. This implies that $\rho_{i,j}(t) = 1$.

3.1.4 The LIBOR market model under the spot measure

As it is inconvenient to use a different simulation measure and a different Brownian motion for every simulated forward rate, it is preferred to use a single simulation measure. One of the measures one could choose is the spot measure, i.e., $\mathbb{Q}^S$, characterised by the discretely compounded money market account as numéraire. The LIBOR Market Model under the dynamics of the spot measure was introduced in Jamshidian (1997). Generally, when we set a specific numéraire, only one LIBOR rate can be a martingale, while all the other LIBOR rates are not. However, as we want to simulate LIBOR rates in an arbitrage-free framework, implying all LIBOR rates should adhere to the martingale property, a drift function will be introduced by which all LIBOR rates will be adjusted. Therefore the LIBOR rates will be modelled according to a Geometric Brownian motion with drift generally given by

$$d\tilde{L}(t, T_i) = \mu^S(t, T_i)\tilde{L}(t, T_i) dt + \sigma(t, T_i)\tilde{L}(t, T_i) dW^S. \tag{3.18}$$

Under the spot measure $\mathbb{Q}^S$ the arbitrage-free data generating process of the LIBOR rates is as follows

$$d\tilde{L}(t,T_i) = \tilde{L}(t,T_i)\left(\sum_{j=q(t)+1}^{i} \frac{\tau_j \tilde{L}(t,T_j)\rho_{i,j}(t)\sigma(t,T_i)\sigma(t,T_j)}{1+\tau_j L(t,T_j)}\right)dt$$
$$+\,\sigma(t,T_i)\tilde{L}(t,T_i)dW^S(t), \qquad t \in [T_{i-1}], \tag{3.19}$$

A full derivation of the dynamics of the LIBOR rates under the $\mathbb{Q}^S$ measure is given in Appendix 9.3, where we closely follow Fries (2007). The above-stated formulas describing the dynamics of the LMM are still quite generic. Thus, in order to simulate LIBOR rates, we need to specify the parameters instantaneous volatilities of the LIBOR rates $\sigma(t,T_i)$.

4 Calibration of the LIBOR Market Model

Before we can use the LIBOR Market Model as derived in Section 3, it is necessary to determine values for the parameters incorporated in the model. The aim is to estimate the parameters of the LIBOR Market model such that these are consistent with market information as implied by the quoted prices of financial instruments. This process, also known as model calibration, is essential to obtain correctly simulated forward rates. The dynamics of the forward rates, as given in Equation 3.19, depend on two parameter types, i.e., the instantaneous volatility between rates and the correlation between forward rates. With calibration, we want to estimate the parameters $\sigma^2(t, T_i)$ and $\rho_{i,j}$, such that the market price of the financial asset in scope corresponds to the price obtained from the model. In this thesis, we will only focus on calibrating the LIBOR Market Model towards the former, i.e., the instantaneous volatility of the LIBOR rate. We consider model calibration using ATM caplet and swaption prices. First, we summarize methods in the existing literature on pricing interest rate caplets and swaptions in the LMM framework in relation to Black'76. Hereafter, we will elaborate on how these instruments can be used to calibrate the LMM.

4.1 Pricing caplets in the LMM

Within the framework of the LIBOR Market Model, caps and floors may be priced using the Black'76 formula, as defined in Appendix 9.1 Theorem 9.1. A buyer of an interest rate cap buys protection for rising floating interest rates. The product essentially consists of a series of sequentially maturing European call options on the interest rate. These call options are commonly referred to as caplets. A cap is a bilateral contract that provides a maximum (or cap) on the floating rate payments. An interest rate cap holder receives payments at the end of each period when the floating interest rate exceeds the fixed strike rate incorporated in the underlying caplet. The pay-off of a caplet at time T_{i+1} on the LIBOR rate is defined as

$$CF_{Caplet}(T_{i+1}) = \tau_i \left(L(T_i, T_i) - K \right)^+ . \tag{4.1}$$

Above, K denotes the strike price of the caplet. The pay-off of a floorlet, i.e., a European put option on the interest rate, resembles much the pay-off of a caplet. Due to the similarity in pay-offs, we will only discuss the pricing of caplets in this section. If we assume that LIBOR rates are log-normally distributed and for now have a constant volatility $\sigma(\cdot, T_i) = \sigma_i > 0$, we can calculate the price of a caplet on time period $[T_{i-1}, T_i]$ evaluated at time $t = 0$, denoted as $V_{Capl}^{BLACK}(0, T_i, K, \sigma_i)$, by means of the well-known Black 1976 formula (Black (1976)). In a setting where we simulate displaced

LIBOR rates, we also have to displace the strike rate. Therefore, we have that

$$V_{Capl}^{BLACK}(0,T_i,K,\sigma_i) = \tau_i P(0,T_{i+1})(\tilde{L}(0,T_i)\Phi(d_1) - \tilde{K}\Phi(d_2)]),$$
$$d_1 = \frac{ln\left(\frac{\tilde{L}(0,T_i)}{\tilde{K}}\right) + \frac{1}{2}\sigma_i^2 T_i}{\sigma_i\sqrt{T_i}},$$
$$d_2 = d_1 - \sigma_i\sqrt{T_i}. \tag{4.2}$$

The proof of the claim in Equation 4.2 is stated in Appendix 9.4. In the expression above $\tilde{K}$ is denoted as $K+\beta$. Here $\Phi(\cdot)$ denotes the standard normal distribution function and σ_i denotes the constant volatility of the caplet expiring at time T_i and paying at T_{i+1}. Due to its popularity, market prices of caplets are quoted in terms of Black implied volatilities, σ_i^{Black}. The Black implied volatility is the volatility in the Black'76 equation that corresponds with a given caplet price. In practice, caplets are not traded on their own. The market only quotes prices and Black implied volatilities of caps. The caplet volatilities can be derived from the cap volatilities by means of a bootstrapping algorithm. This process is commonly referred to as caplet stripping.

The LIBOR Market Model is constructed such that the LIBOR forward rates are log-normally distributed. As a consequence of this log-normality, it holds that the value of a caplet priced through the LMM is exactly the same as the price calculated with the Black'76 formula. Moreover, it may be expected that the Black implied volatility for the i-th caplet, σ_i^{Black}, is some average of the time-dependent instantaneous volatilities of the LIBOR rates $\sigma(\cdot,T_i)$, discussed in Section 3. We will show that this assumption is justified. To price the i-th caplet in the LMM framework, we will use the T_{i+1}-th forward measure. As we have seen earlier in Equation 3.16, under the forward measure $\mathbb{Q}^{T_{i+1}}$, the log-normally distributed shifted LIBOR rate $\tilde{L}(0,T_i)$ is a martingale. The solution of the stochastic differential equation in Equation 3.16 can be found through Itô's formula and is given as

$$\tilde{L}(t,T_i) = \tilde{L}(0,T_i)e^{-\frac{1}{2}\int_0^t\|\sigma(s,T_i)\|^2ds+\int_0^t\sigma(s,T_i)dW^{T_{i+1}}(s)}, \quad 0\le t\le T_i. \tag{4.3}$$

Here $\sigma(\cdot,T_i)$ denotes the time-dependent instantaneous volatility of the LIBOR rate $\tilde{L}(t,T_i)$. The price of our caplet of interest within the LIBOR Market Model framework can now be computed using the fundamental theorem of asset pricing as given in Appendix 9.1 Equation 9.3

$$
\begin{aligned}
V_{Capl}^{LMM}(0, T_i, K, \sigma(\cdot, T_i)) &= \tau_i P(0, T_{i+1}) \mathbb{E}^{\mathbb{Q}_{T_{i+1}}} \left[\frac{(L(T_i, T_i) - K)^+}{P(T_{i+1}, T_{i+1})} \right], \\
&= \tau_i P(0, T_{i+1}) \mathbb{E}^{\mathbb{Q}_{T_{i+1}}} \left[\frac{((L(T_i, T_i) + \beta) - (K + \beta))^+}{P(T_{i+1}, T_{i+1})} \right], \\
&= \tau_i P(0, T_{i+1}) \mathbb{E}^{\mathbb{Q}_{T_{i+1}}} \left[\frac{(\tilde{L}(T_i, T_i) - \tilde{K})^+}{P(T_{i+1}, T_{i+1})} \right], \\
&= \tau_i P(0, T_{i+1}) \mathbb{E}^{\mathbb{Q}_{T_{i+1}}} \left[(\tilde{L}(T_i, T_i) - \tilde{K})^+ \right].
\end{aligned} \tag{4.4}
$$

Again, due to the log-normality of the LIBOR rate, we can analytically derive the expectation in Equation 4.4. We find that

$$
\begin{aligned}
V_{Capl}^{LMM}(0, T_i, K, \sigma(\cdot, T_i)) &= \tau_i P(0, T_{i+1})(\tilde{L}(0, T_i)\Phi(d_1) - \tilde{K}\Phi(d_2)]), \\
d_1 &= \frac{\ln\left(\frac{\tilde{L}(0,T_i)}{\tilde{K}}\right) + \frac{1}{2}\int_0^{T_i} \sigma(s, T_i)^2 ds}{\sqrt{\int_0^{T_i} \sigma(s, T_i)^2 ds}}, \\
d_2 &= d_1 - \sqrt{\int_0^{T_i} \sigma(s, T_i)^2 ds}.
\end{aligned} \tag{4.5}
$$

The price of a caplet using the LIBOR Market Model can also be quoted in terms of its Black implied volatility. Comparing Equation 4.2 and Equation 4.5, we can define the constant Black implied volatility of a caplet within the framework of the LIBOR Market Model by

$$
\sigma_i^{Black} = \sqrt{\frac{1}{T_i} \int_0^{T_i} \sigma(s, T_i)^2 ds}, \tag{4.6}
$$

where $\sigma(\cdot, T_i)$ denotes the instantaneous volatility of the i-th LIBOR rate. Given an observed implied volatility σ_i^{Black}, there are various functions of $\sigma(t, T_i)$ that could be integrated to σ_i^{Black}. In this thesis, we choose a specific instantaneous volatility of the i-th LIBOR rate. Namely, we set $\sigma(t, T_i) = \sigma_i^{Black}$, such that the equality in Equation 4.6 holds. By setting $\sigma(t, T_i) = \sigma_i^{Black}$, we assume that each LIBOR rate has constant instantaneous volatility regardless of t. Concluding, we choose

$$
\sigma(t, T_i) = \sigma_i^{Black}, \quad t \in [0, T_i), \quad i \in \{1, ..., N\}, \tag{4.7}
$$

where $i \in \{1, ..., N\}$ denotes the $i-$th caplet in the set of N caplets considered to determine the instantaneous LIBOR volatilities over the period $[T_0, T_N]$. It is important to note that the Black implied volatility does not depend on the strike of a caplet. Thus, the model can calibrate to a single caplet per maturity only. Therefore the calibrated LMM model can not render a caplet smile. We can construct the Table 1 that depicts the term structure of the LIBOR rates volatilities, emphasizing the constant instantaneous volatilities per LIBOR rate.

Vol of \t ∈	$[T_0,T_1)$	$[T_1,T_2)$	$[T_2,T_3)$	$[T_3,T_4)$	$\cdots$	$[T_{N-1},T_N)$
$\tilde{L}(t,T_1)$	σ_1^{Black}	0	0	0	0	0
$\tilde{L}(t,T_2)$	σ_2^{Black}	σ_2^{Black}	0	0	0	0
$\tilde{L}(t,T_3)$	σ_3^{Black}	σ_3^{Black}	σ_3^{Black}	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\tilde{L}(t,T_N)$	σ_{N-1}^{Black}	σ_{N-1}^{Black}	σ_{N-1}^{Black}	σ_{N-1}^{Black}	$\cdots$	σ_{N-1}^{Black}

Table 1: Term structure depicting the constant instantaneous volatilities of the LIBOR rates.

It is not the first time that the suggestion of constant LIBOR rate volatilities has been made. For example, Brigo & Mercurio (2006) considers constant LIBOR volatilities as one of the proposed methods to calibrate the LMM, and Pelsser & Pietersz (2003) suggests that in the case constant LIBOR volatilities are considered to specify the term structure, the estimation of the sensitivity of underlying swaps included in Bermudan swaptions towards implied volatility is less uncertain.

On the other hand, we could have also chosen a parametric approach to determine a function to specify the instantaneous volatility $\sigma(t,T_i)$ that integrates to σ_i^{Black}. Rebonato (2012) states that such a parametric function may be given as

$$\sigma(t,T_i) = (a + b(T_i - t)) \cdot e^{-c(T_i - t)} + d, \tag{4.8}$$

though other functions also exist. To specify the instantaneous volatility of the LIBOR rates, we need to assign values to the parameters a, b, c and d in Equation 4.8. To do so, we minimize the following problem, with respect to a, b, c and d

$$\min_{\{a,b,c,d\}} \sum_{i=1}^{N} \left(\sigma_i^{Black} - \sqrt{\frac{1}{T_i} \int_0^{T_i} [(a + b(T_i - s)) \cdot e^{-c(T_i - s)} + d]^2 ds} \right)^2, \tag{4.9}$$

where N is the number of caplets used for the calibration process. After optimization procedure, we can thus obtain the instantaneous volatilities of the LIBOR rates $\sigma(t,T_i)$ by filling in the optimized parameters in Equation 4.8. We will refer back to the parametric approach and more specifically to Equation 4.8 when we discuss LMM calibration towards swaption volatilities.

The calibration process stated above considers a LIBOR Market Model with no volatility smile modelling. In this research, we will calibrate the model to ATM caplet volatilities. Clearly, from a theoretical point of view, this is a limitation of this research. However, for the scope of this thesis, we are specifically interested in presenting an approach to calibrate caplets in the context of calculating EPE profiles for a portfolio of swaptions. This thesis aims to bring attention to this approach and not to present an exhaustive overview of how to calibrate the LMM as accurately as possible.

4.2 Interest rate swaps and the swap rate

Before we dive into the pricing of options on interest rate swaps, it may be beneficial to discuss first the fundamentals of these types of swaps and introduce notation. An interest rate swap, from here onwards simply 'swap', is a bilateral contract to exchange future interest payment streams based on an agreed-upon principal amount or notional. These cash flow streams are referred to as legs. A swap contract contains a fixed leg and a floating leg. The fixed leg pays a fixed interest rate specified in the contract. The floating leg payments depend on, for example, LIBOR rates, but other reference rates like EURIBOR or EONIA can also be used. In this thesis, a swap is characterised by a specific tenor structure on which both fixed and floating leg payments occur. In reality, this is not necessarily the case, as payment dates may differ among legs. Let the tenor structure be given as

$$0 \leq T_\alpha < T_{\alpha+1} < ... < T_\beta, \quad \tau_i = T_{i+1} - T_i, \quad i \in \{\alpha, ..., \beta - 1\}.$$

Within this structure, T_α and T_β denote the start and end of the swap, respectively. In case $T_0 = T_\alpha$, the swap is referred to as spot-starting, while if $T_0 < T_\alpha$ it is a forward-starting swap. The LIBOR rates are observed at the beginning of each period $[T_i, T_{i+1}]$; however, payments occur at the end of each period. Thus, the fixed side of the swap pays $\tau_i \cdot K$, where K again denotes the fixed rate. The floating side pays $\tau_i \cdot \tilde{L}(T_i, T_i)$.

Equation 3.16 suggests that the displaced forward LIBOR rate for the period $[T_i, T_{i+1}]$ is a martingales under the $\mathbb{Q}^{T_{i+1}}$ probability measure. Logically, the non-displaced forward LIBOR rate is also a martingale, since the displacement is a constant value that can be taken out of the expectation. Due to the martingale property, we have that the present value of the floating leg of a swap can be determined according to

$$\begin{aligned} V_{float}(t) &= \sum_{i=\alpha}^{\beta-1} \tau_i P(t, T_{i+1}) \mathbb{E}^{\mathbb{Q}^{T_{i+1}}} \left[L(T_i, T_i) \middle| \mathcal{F}_t \right] \\ &= \sum_{i=\alpha}^{\beta-1} \tau_i P(t, T_{i+1}) L(t, T_i). \end{aligned} \tag{4.10}$$

The present value of the fixed leg of the swap is straightforward since

$$V_{fixed}(t) = \sum_{i=\alpha}^{\beta-1} \tau_i P(t, T_{i+1}) \cdot K \tag{4.11}$$

$$= A(t, T_\alpha, T_\beta) \cdot K, \tag{4.12}$$

where $A(t, T_\alpha, T_\beta)$ implicitly implies the present value of a basis point (PVBP). We can compute the value of a interest rate swap as follows

$$V_{swap}(t) = \omega \left(V_{float}(t) - V_{fixed}(t) \right), \tag{4.13}$$

where $\omega = 1$ from a perspective of the agent who pays the fixed rate and $\omega = -1$ if the agent pays the floating rate. It is often the case that the value of an interest rate swap at initiation is zero or close to zero. Thus, we can find the value of the fixed rate which gives the swap a zero present value at initiation, also defined as the swap rate. Since the present value is 0, we should have that $V_{float}(t) = V_{fixed}(t)$. Therefore, the swap rate of a swap starting in T_α and maturing in T_β, denoted as $S_{\alpha,\beta}(t)$, can be found through

$$\begin{aligned}\sum_{i=\alpha}^{\beta-1} \tau_i P(t,T_{i+1}) L(t,T_i) &= \sum_{i=\alpha}^{\beta-1} \tau_i P(t,T_{i+1}) S_{\alpha,\beta}(t) \\ \Leftrightarrow S_{\alpha,\beta}(t) &= \frac{\sum_{i=\alpha}^{\beta-1} \tau_i P(t,T_{i+1}) L(t,T_i)}{\sum_{i=\alpha}^{\beta-1} \tau_i P(t,T_{i+1})} \\ \Leftrightarrow S_{\alpha,\beta}(t) &= \frac{\sum_{i=\alpha}^{\beta-1} \tau_i P(t,T_{i+1}) L(t,T_i)}{A(t,T_\alpha,T_\beta)}. \end{aligned} \tag{4.14}$$

Rearranging terms in Equation 4.14 yields

$$S_{\alpha,\beta}(t) \cdot A(t,T_\alpha,T_\beta) = \sum_{i=\alpha}^{\beta-1} \tau_i P(t,T_{i+1}) L(t,T_i) = V_{float}(t). \tag{4.15}$$

Therefore, using Equation 4.13, we can define the present value of an interest rate swap as

$$\begin{aligned} V_{swap}(t) &= \omega \left(S_{\alpha,\beta}(t) \cdot A(t,T_\alpha,T_\beta) - A(t,T_\alpha,T_\beta) \cdot K \right), \\ &= \omega \cdot A(t,T_\alpha,T_\beta) \cdot \left(S_{\alpha,\beta}(t) - K \right). \end{aligned} \tag{4.16}$$

Now, if we define weights $w_i(t)$ as

$$w_i(t) \stackrel{\text{def}}{=} \frac{\tau_i P(t,T_{i+1})}{\sum_{k=\alpha}^{\beta-1} \tau_k P(t,T_{k+1})}, \tag{4.17}$$

the swap rate as in Equation 4.15 can be expressed as a weighted average of LIBOR rates

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha}^{\beta-1} w_i(t) L(t,T_i). \tag{4.18}$$

Equation 3.7 shows the relation between the discount bonds in terms of LIBOR rates. Inserting this definition into Equation 4.17, yields an expression of the weights in terms of LIBOR rates

$$w_i(t) = \frac{\tau_i \prod_{j=q(t)}^{i} \frac{1}{1+\tau_j L(t,T_j)}}{\sum_{k=\alpha}^{\beta-1} \tau_k \prod_{j=q(t)}^{k} \frac{1}{1+\tau_j L(t,T_j)}}. \tag{4.19}$$

We can thus conclude that the swap rate is a linear function purely in terms of LIBOR rates denoted by

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha}^{\beta-1} w_i(t) L(t,T_i). \tag{4.20}$$

4.3 Pricing swaptions in the Swap Rate Model

The main interest rate derivative we are concerned with in this thesis is the option on an interest rate swap, also called a swaption. The holder of a swaption has the right but not the obligation to enter an interest rate swap somewhere in the future at an agreed-upon fixed rate. We are only considering European swaptions, i.e., swaptions for which the expiry date is predetermined and early exercise is not possible. Recall that the position in the swap could be such that one would pay or receive the fixed rate. Similarly, we distinguish between payer and receiver swaptions. Therefore, the holder of a payer swaption has the right but not the obligation to enter a swap in which the holder of the option will pay the predetermined fixed rate and receives floating LIBOR rates. Likewise, we can define a receiver swaption. Swaptions can be physically- or cash-settled. In the former, if the swaption expires and the swaption is In-the-Money (thus for a payer swaption in case the swap rate is higher than the fixed-rate and for a receiver swaption vice versa), the swap is initiated. In the case of a cash-settled swaption, at expiry and again considering that the swaption is ITM, the holder of the swaption will receive a cash payment equal to the present value of the underlying swap at expiry.

This research assumes that the swaption's expiry date coincides with the underlying swap's start date. As previously indicated, the tenor of the swap is $T_\beta - T_\alpha$. If the swap's tenor would be reduced to a single period, i.e., $\beta - \alpha = 1$, a payer swaption would fundamentally be the same as a caplet. In the same way, a receiver swaption with an underlying swap tenor of one period is equivalent to a floorlet.

Will we now construct the value of the pay-off of a physically-settled swaption. After expiry, if exercised, the pay-off of a physically-settled swaption is equal to the pay-off of the underlying swap. Let us formally define ω such that

$$\omega = \begin{cases} +1 & \text{“option holder pays fixed rate”} \\ -1 & \text{“option holder receives fixed rate”}. \end{cases}$$

It is clear that the holder of an option paying the fixed interest rate K only wants to exercise the option if this rate is lower than the swap rate implied by the market at the date of expiry. Thus, only if $S_{\alpha,\beta}(T_\alpha) > K$. For the holder of a receiver swaption the opposite holds. Here we have assumed that the fixed rate specified in the contract remains the same during the term of the contract. The value of the pay-off of a physically-settled swaption evaluated at T_α is given by

$$\mathrm{V}_{PhysicalSwaption}(T_\alpha) = A(T_\alpha, T_\alpha, T_\beta) \cdot \omega \cdot (S_{\alpha,\beta}(T_\alpha) - K)^+, \tag{4.21}$$

We can evaluate the value of a swaption using the fundamental theorem of asset pricing as stated in Appendix 9.1 Equation 9.3. To evaluate the price of a swaption at time $t < T_\alpha$, we introduce the probability measure $\mathbb{Q}^{\alpha,\beta}$ for which the swap rate $S_{\alpha,\beta}$ is a martingale. We can define the value of a swaption at time t through

$$V_{PhysicalSwaption}(t) = A(t,T_\alpha,T_\beta)\cdot \mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\left[\left.\frac{A(T_\alpha,T_\alpha,T_\beta)\cdot\omega\cdot(S_{\alpha,\beta}(T_\alpha)-K)^+}{A(T_\alpha,T_\alpha,T_\beta)}\right|\mathcal{F}_t\right], \tag{4.22}$$
$$= A(t,T_\alpha,T_\beta)\cdot\omega\cdot\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\middle|\mathcal{F}_t\right].$$

For now, assume that $S_{\alpha,\beta}(t)$ is also displaced log-normally distributed under $\mathbb{Q}^{\alpha,\beta}$. Since $S_{\alpha,\beta}(t)$ is log-normally distributed and a martingale we can find the value of a swaption at time t through the shifted Black'76 formula. For this, we assume that the instantaneous volatility of the swap rate $\sigma_{\alpha,\beta}$ is constant. The proof for this statement is very similar to the proof presented in Appendix 9.4, so we choose to omit it. The value of a swaption at time T_i by means of Black'76 is

$$V^{BLACK}_{Swaption}(0,T_i,K,\sigma_{\alpha,\beta}) = A(0,T_\alpha,T_\beta)\cdot\omega\cdot(\tilde{S}_{\alpha,\beta}(0)\Phi(d_1)-\tilde{K}\Phi(d_2)]),$$
$$d_1 = \frac{ln\left(\frac{\tilde{S}_{\alpha,\beta}(0)}{\tilde{K}}\right)+\frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}\sqrt{T_\alpha}},$$
$$d_2 = d_1 - \sigma_{\alpha,\beta}\sqrt{T_\alpha}. \tag{4.23}$$

The market standard is to quote the prices of swaptions in terms of Black implied volatilities. Therefore, $\sigma_{\alpha,\beta} = \sigma^{Black}_{\alpha,\beta}$ and can be derived from market quotes. Also for time dependent instantaneous volatilities, $\sigma_{\alpha,\beta}(t)$, we can derive that Black'76 model. Again the derivation is very similar to the derivation in Appendix 9.4, so we choose to omit it. Finally, we can denote that Black implied volatility in terms of the time dependent instantaneous volatilities of the swap rate

$$\sigma^{Black}_{\alpha,\beta} = \sqrt{\frac{1}{T_\alpha}\int_0^{T_\alpha}\sigma^2_{\alpha,\beta}(t)dt}. \tag{4.24}$$

Where we consider the instantaneous volatility of the swap rate $\sigma_{\alpha,\beta}(t)$ at time t, $0\le t\le T$, $\sigma_{\alpha,\beta}$: $[0,T_i]\to[0,\infty)$.

4.4 Volatility approximation methods

Unfortunately, under the LIBOR Market Model, the swap rate $S_{\alpha,\beta}(t)$ is not log-normally distributed as it is a weighted sum of LIBOR rates which are log-normally distributed themselves. Put differently, if the LIBOR rates are log-normally distributed and martingales under their own probability measure, the swap rates can not be log-normal under the $\mathbb{Q}^{\alpha,\beta}$, nor can they be martingales under any of the LMM probability measures. Nevertheless, if we can find a manner to express the average percentage volatility $\sigma^{Black}_{\alpha,\beta}$ as a function of the volatility and correlation of the forward rates that drive the LIBOR Market Model, we can approximately price swaptions using the Black'76 formula.

4.4.1 Rebonato approximation

The first approximation of the average percentage variance was shown in Rebonato (1999) and further elaborated upon in Jackel & Rebonato (2003). Rebonato's method assumes the weights and LIBOR rates in Equation 4.20 are independent and that the weights stay steady over time. This claim is often supported by arguing that the volatility in the weights in Equation 4.20 is negligible compared to the volatility of the LIBOR rates. In this light, the weights in Equation 4.20 are frozen at time $t = 0$, such that we can approximate the swap rate through

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha}^{\beta-1} w_i(0) L(t, T_i). \tag{4.25}$$

With that, we can differentiate both sides of Equation 4.25 and filling in the expression of the LIBOR rates as derived in Section 3.1.4 and the definition of the dynamics of the LIBOR rates as in Equation 3.16, we find

$$\begin{aligned} dS_{\alpha,\beta}(t) &\approx \sum_{i=\alpha}^{\beta-1} w_i(0) dL(t, T_i), \qquad (4.26) \\ &= \sum_{i=\alpha}^{\beta-1} w_i(0)[\mu(t)\tilde{L}(t, T_i)dt + \sigma_i(t)\tilde{L}(t, T_i)dW^{T_{i+1}}(t)]. \end{aligned}$$

We can now find an expression for the differential of the quadratic variation of the swap rate by

$$\begin{aligned} d\langle S_{\alpha,\beta}(t), S_{\alpha,\beta}(t)\rangle &\approx \sum_{i=\alpha}^{\beta-1}\sum_{j=\alpha}^{\beta-1} w_i(0)w_j(0)d\tilde{L}(t, T_i)d\tilde{L}(t, T_j), \qquad (4.27) \\ &= \sum_{i,j=\alpha}^{\beta-1} w_i(0)w_j(0)\tilde{L}(t, T_i)\tilde{L}(t, T_j)\sigma_i(t)\sigma_j(t)\rho_{i,j}(t)dt \end{aligned}$$

where for $\rho_{i,j}(t)$ it holds that $dW_i(t)dW_j(t) = \rho_{i,j}(t)dt$. If the swap rate follows an Ito diffusion process it holds that $d\langle S_{\alpha,\beta}(t), S_{\alpha,\beta}(t)\rangle = \delta^2(t)dt$, with $\delta = \tilde{S}_{\alpha,\beta}(t)\sigma_{\alpha,\beta}(t)$. Together with Equation 4.27 we can find an expression for the approximation of $\sigma^2_{\alpha,\beta}(t)$ in terms of variables of the LIBOR Market Model, through

$$d\langle S_{\alpha,\beta}(t), tildeS_{\alpha,\beta}(t)\rangle = S^2_{\alpha,\beta}(t)\sigma^2_{\alpha,\beta}(t)dt \approx \sum_{i,j=\alpha}^{\beta-1} w_i(0)w_j(0)\tilde{L}(t, T_i)\tilde{L}(t, T_j)\sigma_i(t)\sigma_j(t)\rho_{i,j}(t)dt, \tag{4.28}$$

$$\Leftrightarrow \quad \sigma^2_{\alpha,\beta}(t) \approx \frac{1}{\tilde{S}^2_{\alpha,\beta}(t)} \sum_{i,j=\alpha}^{\beta-1} w_i(0)w_j(0)\tilde{L}(t, T_i)\tilde{L}(t, T_j)\sigma_i(t)\sigma_j(t)\rho_{i,j}$$

Note that forward LIBOR rates are only frozen to obtain an approximate expression of the instantaneous swaption volatility in terms of instantaneous forward LIBOR rate volatility and clearly not for simulation purposes. Apart from freezing the weights in Equation 4.25, Rebonato (1999) additionally assumes that all LIBOR rates $L(t, T_i)$ are frozen at the very first moment as well $\forall t$, such that

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) L(0, T_i) = S_{\alpha,\beta}(0). \tag{4.29}$$

Replacing the instantaneous volatility of the swap rate in Equation 4.24 with the expressions for $\sigma^2_{\alpha,\beta}(t)$ above and taking into account the freezing of the LIBOR rates, we find the approximated Black volatilities according to Rebonato (1999) by

$$\sigma^{Rebonato}_{S_{\alpha,\beta}}(T_\alpha)^2 = \frac{1}{T_\alpha} \sum_{i,j=\alpha}^{\beta-1} \frac{w_i(0) w_j(0) \tilde{L}(0, T_i) \tilde{L}(0, T_j)}{\tilde{S}^2_{\alpha,\beta}(0)} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) \rho_{i,j}(t) dt. \tag{4.30}$$

The volatility obtained through Equation 4.30 can be used as a proxy for the Black volatility of the swap rate, $\sigma^{Black}_{\alpha,\beta}$. If we replace instantaneous volatilities $\sigma_i(t)$ and $\sigma_j(t)$ with parametrization as given in Equation 4.8, we can find the set of parameters a, b, c and d by minimizing the following problem (see Rebonato (2012))

$$\min_{\{a,b,c,d\}} \sum_{m=1}^{M} \left(\sigma^{Black}_{S_{\alpha,\beta},m} - \sigma^{Rebonato}_{S_{\alpha,\beta},m}(T_\alpha) \right)^2, \tag{4.31}$$

where M is the number of swaptions take into consideration for the calibration process, $\sigma^{Black}_{\alpha,\beta,m}$ is the ATM Black implied swaption volatility for the m-th swaption and $\sigma^{Rebonato}_{S_{\alpha,\beta}}(T_\alpha)^2$ is the Rebonato approximation of the m-th swaption with expiry date T_α and maturity date T_β. The goal is thus to find the parameters a, b, c and d such that the table filled with swaption volatilities approaches as much as possible the table filled with Black implied volatilities. The optimal values a^*, b^*, c^*, d^* are then used to construct instantaneous volatilities of the LIBOR rates according to Equation 4.8. However, as the number of swaption volatilities to be estimated is often much larger than the numbers of free parameters to estimate these volatilities, the estimated swaption volatilities can differ from the Black implied volatilities.

4.5 Impracticalities of calibration methods

From a practical point of view, estimating a number of swaption volatilities greater than the number of free parameters brings another possible complication when we would extend an existing portfolio of swaptions. Suppose the swaption we want to include in the portfolio has a larger tenor than the largest tenor included in the portfolio. Therefore, the swaption volatility corresponding to the new swaption is not included yet in the matrix of estimated swaption volatilities. We would need to minimize Equation 4.31 leading to potentially different values of a, b, c and d and thus directly to different values of the instantaneous volatilities of the LIBOR rates. Therefore, the simulated values of exposures for all swaptions may differ after including the new swaption in the portfolio. This effect is undesirable. What if we would have that this newly added swaption with a long tenor is a trade over a very small notional and is relatively insignificant compared to other existing trades in the portfolio. By adding the insignificant trade, exposure profiles of relevant trades will alter. A practitioner could consider a very large swaption volatility matrix so that every potentially added swaption is already covered. This, however, brings two difficulties. First of all, what would we consider to be large enough? This question may possibly be answered by the trading mandate of the respective financial institution. More importantly, if we would increase the size of the matrix of estimated swaption volatilities, we would also increase the error in the estimation of the swaption volatilities. This is simply a result of increasing the number of estimated volatilities while keeping the number of free parameters the same.

We could impose assumptions that allow optimization-free calibration using a set of co-terminal swaptions. Let $\sigma_{S_{\alpha,\beta},m}^{Rebonato}(T_\alpha) = \sigma_{S_{\alpha,\beta},m}^{Black}$ and additionally assume that the instantaneous LIBOR volatilities in Equation 4.30 are independent of time t. We then have that

$$\sigma_{S_{\alpha,\beta},m}^{Black} = \sqrt{\frac{1}{T_{\alpha_m}} \sum_{i,j=\alpha_m}^{\beta_m - 1} \frac{w_i(0)w_j(0)L(0,T_i)L(0,T_j)\sigma_i\sigma_j}{S_{\alpha_m,\beta_m}^2(0)} \int_0^{T_{\alpha_m}} \sigma_i\sigma_j\rho_{i,j}(t)dt.} \tag{4.32}$$

Here α_m and β_m denote that exercise and maturity date of the m-th swaption. To find a solution for $\sigma_i, \quad i \in \{1, ..., N\}$, we can use the set of co-terminal swaptions. The benefit of this is that we do not have to perform an optimization scheme to determine the instantaneous volatilities of the LIBOR rates. 1 depicts the situation in case we want to specify the LIBOR dynamics to price a 2Y4Y swaption. For this, we need to specify the LIBOR volatilities $\{\sigma_1, ..., \sigma_5\}$. Using the swaption volatilities of the co-terminal swaptions, we can iteratively solve corresponding instantaneous LIBOR volatilities. With the 5 known Black implied swaption volatilities, we can solve exact for the 5 LIBOR volatilities, in case an exact solution exists. More generally, we can specify α.

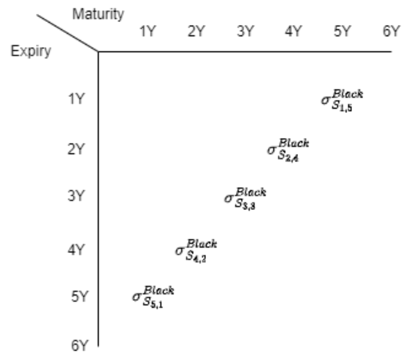


Figure 1: This table shows the set of co-terminal swaptions volatilities used to calibrate the LIBOR Market Model to price a 2Y4Y swaption. Since we have 5 known swaption volatilities and 5 unknown instanteneous LIBOR volatilities to estimate, we can find an exact solution to the system of equations.

The above approach, however, does not provide a solution for the continuity problem. If we extended the existing portfolio with a swaption over a longer tenor, we would need to find a suitable set of co-terminal swaptions. If we would choose the set of co-terminal swaptions of the newly added insignificant swaption, we will find different values for σ_i then if we would use the co-terminal set of swaptions corresponding to the previous longest tenor in the portfolio. Therefore, the continuity problem is still present.

We propose a straightforward approach to tackle the above-discussed problems while calibrating the LMM. Since we can use analytical ATM caplet volatilities to obtain instantaneous LIBOR rate volatilities, we can safely add a swaption with a longer tenor structure to an existing portfolio over a shorter structure. The LIBOR rate volatilities of the rates already included in the shorter structure will not change after adding the new trade over a longer time. Therefore the LIBOR rate dynamics over the shorter structure will not change and exposures of trades within the shorter structure will not be modified. This concept is graphically presented in Figure 2.

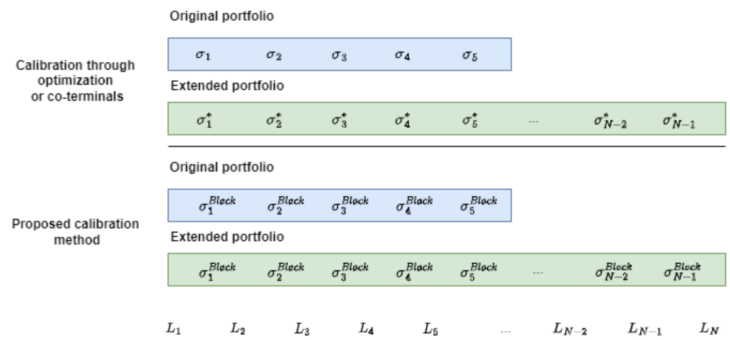


Figure 2: The LIBOR volatilities of an extended portfolio using either an optimization method or calibration to co-terminals are not equal to the volatilities of the original portfolio. Ultimately exposure profiles of the shorter portfolio may change. Using Black implied caplet volatilities as instantaneous LIBOR volatilities yields the desired result of non-changed exposure profiles when adding a swaption to the portfolio.

Since we are using caplet volatilities to obtain exposure profiles for swaptions, we will propose an adjustment factor correcting the resulting offset to some extent. We elaborate on this in Section 4.6. Ultimately, we will use expected exposure to calculate CVA. Therefore, we will touch upon the topic of CVA and its calculation in Section 5.

4.6 Adjustment factor

As previously discussed, calibrating the LMM towards swaptions allows for the existence of the continuity problem. One way to face this problem is by calibrating the LIBOR volatilities towards caplet volatilities. Using Black implied ATM caplet volatilities directly as instantaneous LIBOR rate volatilities allows us to both avoid an optimization scheme to determine LIBOR volatilities and to face the dilemma of finding a suitable set of co-terminal swaptions. Therefore, when adding a trade to an existing portfolio of swaptions, the instantaneous LIBOR volatilities of the first part of the tenor, already include in the portfolio, will not change. Consequently, expected exposures will remain the same. Instantaneous LIBOR volatilities for the part of the tenor that is not covered yet in the portfolio can simply be found by selecting the corresponding Black implied ATM caplet volatilities. We can thus consistently calibrate the model to all current and potential future trades.

A clear drawback of this method is that we are calibrating a model towards caplet volatilities to value swaptions. Therefore, the model implied forward rate volatility leads to a simulated swaption value, denoted as V_{sim}, that we do not observe in the market. We will try to tackle this misalignment by introducing a swaption-specific adjustment factor that modifies the pay-offs of the simulated swaption price calibrated in some manner towards their analytical or market-implied value. Let C

be the adjustment factor denoted as

$$C = \frac{V_{analytical}}{V_{sim}}, \tag{4.33}$$

where $V_{analytical}$ represents the analytical value of the swaption calculated with the Black'76 formula and V_{sim} denotes the simulated value of the swaption. To calculate the analytical value, we need swaption volatilities that can capture the volatility smile well. For this a stochastic volatility model can be used. The stochastic model we use in this thesis is the SABR model introduced in P. S. Hagan et al. (2002). Since the analytical calculation of swaption prices is not the focus of this thesis, we leave out the details on how appropriate SABR swaption volatilities are obtained. The factor C shows how much our simulated value differs from the analytical value of the swaption. This deviation is probably not only because of how we specify the dynamics of our model, but also because we are comparing analytical values with simulated values. Still, we can use C to adjust the simulated swaption pay-offs such that the simulated swaption value corresponds exactly with the implied or analytical value on the observation date. After obtaining the adjustment factor, we can modify the pay-offs resulting from the simulation procedure, considering the adjustment factor. The simulated cash flow of a payer swaption with expiry date T_α and maturity date of the underlying T_β, with a notional amount of 1, becomes

$$CF^*(T_{i+1}) = C \cdot \tau_i \cdot (L(t, T_i) - K)^+. \tag{4.34}$$

Clearly, modifying the pay-off function with an adjustment factor has limitations. First, the adjustment factor is a snapshot result since we determine it only once for a specific observation date. At the observation date, or $t = 0$, we know that the adjusted simulated values correspond exactly with the analytical value. This follows from filling in the definition of the adjusted cash flows in Equation 9.3 and taking the adjustment factor out of the expectation. However, the adjustment factor which ensures that the analytical value and the simulated value is exactly equal at $t = 0$, is most likely not the same as the adjustment factor at a later moment in time. Therefore, adjusting the simulated value with the adjustment factor determined at $t = 0$, might be wrong. Furthermore, it is interesting to see how this method performs for swaptions with shorter-term underlying swaps as opposed to swaptions on longer-swaps. We expect that the swaptions on swaps with a shorter length are less prone to errors caused by not updating the adjustment factor over time.

Secondly, the value of V_{sim} in Equation 4.33 can become zero for very deep Out-of-the-Money swaptions. If so, Equation 4.33 is not well defined. Even for relatively deep OTM swaptions with a few paths in which there is a strictly positive pay-off, V_{sim} is less reliable since only a moderate number of paths are used to obtain this value. Moreover, for values of V_{sim} close to zero, the adjustment factor might become very large. Therefore, depending on the moneyness of the swaption, we have to introduce a threshold, indicating whether we should modify the pay-offs with the adjustment factor. One way to obtain such a threshold is by comparing the analytical value of the to be simulated swaption to the analytical value of an ATM swaption. We could examine whether the ATM value of the swaption is not too large compared to the analytical value of our swaption of interest. Thus, only if $V_{analytical} > x\% \cdot V_{ATM}$, we will incorporate the adjustment factor C. As it is yet unspecified what a 'good' value of x should be, we emphasize that further research is necessary to develop a systematic procedure to determine an appropriate threshold.

Assessing the performance of our proposed method is not trivial. To get a better understanding how well our method performs, we designed a stylized experiment discussed below.

4.7 Experiment: The impact of miscalibration

As specified earlier, we are using flat ATM caplet volatilities to specify the dynamics of the LIBOR Market Model with the aim the value swaptions. We thus calibrate the LIBOR model using a certain type product to value another type of product. This is what we refer to as miscalibration. To assess the impact of this miscalibration, we designed a stylized experiment of our proposed method. In this experiment, we want to compare exposure profiles of a single swaption obtained by a hypothetically correctly calibrated model and by using our proposed method. As we expect that the duration of the contract might have an impact on the resulting EPE profiles, we will consider a 2Y5Y swaption and a 5Y10 swaption. Both swaptions evaluated are ATM and swap-settled. We use ATM swaptions, since ATM options are most responsive to changes in implied volatility.

Suppose that the swaption market is in reality driven by flat $x\%$ caplet volatilities. Of course, this is a hypothetical situation since we omit the existence of a volatility smile here. We can calibrate a model, Model X, exactly towards this volatility and obtain a simulated mark-to-market (MtM) and exposure profiles accordingly. Moreover, suppose that flat $y\%$ caplet volatilities drive the caplet market. We can construct a second model, Model Y, calibrated towards these flat $y\%$ volatilities and again obtain a simulated MtM and exposure profiles. Note that in this experiment, $x \neq y$. This naturally captures the cap/floor versus swaption basis which is commonly observed in the market. Then, we calculate the swaption-specific adjustment factor by dividing the simulated MtM of model X by the simulated MtM of model Y. We thus assume that the analytical swaption price is obtained from the simulated price. Subsequently; we multiply all cash flows resulting from model Y with this adjustment factor to get an adjusted EPE profile. As a result, we can compare the EPE profiles of the 'true' Model X with the profiles from Model Y, which is calibrated using our proposed method. We are going to assess the EPE profiles for different volatility levels x and y, to see to what extent our method can correct for the miscalibration and how severe differences in

EPE profiles are. The results are presented in Section 6.4. Note that this experiment is stylized by the fact that, we do not analyse the impact of miscalibration with respect to flat volatilities instead of volatilities that can render a smile. We only assess the impact of miscalibration with respect to the level of the volatilities. The impact of not incorporating the volatility smile on simulated exposure profiles is yet to be explored.

5 Credit Valuation Adjustment

The Credit Valuation Adjustment (CVA) can be seen as the difference between the risk-free value of a portfolio of assets and the value incorporating the risk of a potential default of the counterparty. CVA is thus the market value of counterparty credit risk. In this thesis, we will focus on the uncollateralized CVA of a portfolio of swaptions. By stating that the portfolio is uncollateralized, we mean that in case of a counterparty default, neither party will be able to seize any collateral. First, we will briefly introduce notation relevant to CVA. Subsequently, the mechanisms of LSMC introduced by Longstaff & Schwartz (2001) and its application to CVA calculation are discussed. Continuing, we will touch upon the expression of the EPE introduced in Joshi & Kwon (2016), which ensures that the EPE only features the sign of the regression function part of the LSMC algorithm rather than the magnitude of the estimated value. Then, we will present a new proof of this expression and discuss its implications for physically- and cash-settled swaptions.

Continuing within the same framework as earlier discussed, we assume a tenor structure $t \in [T_0, ..., T_\alpha, ..., T_\beta]$ with T_α the expiry date of the option and start date of underlying swap and T_β the maturity date of the contract. Moreover, we have $V(t)$ denote the value of a portfolio of products observed at time t. The positive exposure of an agent to a counterparty at time t is denoted by the positive part of the value of a contract. The positive exposure is given by $\max(V(t), 0)$.

The positive exposure can be seen as the loss in case of a counterparty default at time t. In addition, let T_τ be the default time of the counterparty. This is not necessarily the largest potential loss since there is also credit-contingent market risk. The value of a portfolio may change from the time of default up to the time of unwind of the portfolio. The indicator function $\mathbb{1}_{(T_\tau < T_\beta)}$ returns a value of 1 in case the counterparty defaults before the end date of the contract and zero otherwise. Therefore, the risk-neutral expectation of this function represents the risk-neutral probability of counterparty default within the life span of the contract. Moreover, let γ be the fractional loss given the default (LGD) of the counterparty, which we assume to be constant. CVA can be defined as the risk-neutral expectation of the time t-discounted exposures at T_τ, taking the loss given default into account. We thus have for a discrete-time grid that

$$CVA(t) = \mathbb{E}\left[\gamma \cdot \frac{N(t)}{N(T_\tau)} \cdot \mathbb{1}_{(T_\tau < T_\beta)} \cdot \max(V(T_\tau), 0) \middle| \mathcal{F}_t\right], \tag{5.1}$$

where $N(\cdot)$ is the numéraire with which we discount till the observation time. If we assume that the time of default of the counterparty is independent of other processes, we can approximate the above defined CVA by means of

$$CVA(t) = \gamma \cdot \sum_{i=1}^{\beta} q_i \cdot N(t)\mathbb{E}\left[N^{-1}(T_i) \cdot \max(V(T_i), 0) \middle| \mathcal{F}_t\right]. \tag{5.2}$$

Here, q_i denotes the risk-neutral probability of a default of the counterparty in the time interval $(T_{i-1}, T_i]$. In the process of calculating CVA, the probability of a counterparty default is usually assumed to be independent of the counterparty's exposure (J. Hull & White (2012)). Clearly, the probability of default plays an important role in Equation 5.2. One approach to acquire the default probability of a counterpart is by extracting implied probabilities from publicly quoted Credit Default Swaps (CDS) spreads. For a detailed explanation of the relation between CDS curves and implied default probabilities, we would like to refer to J. C. Hull (2003). Moreover, we will use market-based recovery rates to determine the fractional loss given a counterparty's default. In this thesis, we focus specifically on the expected positive exposure, which is the expectation of $\max(V(T_i), 0)$. Calculating an expression for the term $V(T_i)$ is non-trivial. We could decide to use a Nested Monte Carlo simulations to obtain an expression for the expected exposure. However, in practice, this is not desirable as these simulations require a significant amount of time, and speed is frequently of great importance in real-life situations. A trading desk wants to be able to obtain expressions for the CVA for each counterparty in real-time, rather than letting this exercise be an overnight job. Another method to obtain expressions for the expected exposures of a financial asset is using the Least Squares Monte Carlo method introduced in Longstaff & Schwartz (2001), which we will discuss shortly after touching upon the principle of netting agreements.

5.1 CVA Netting for Portfolios

Netting is an important aspect of CVA calculation. A netting agreement is a bilateral agreement stating that all trades between the agents included in the portfolio can be aggregated or combined in the case of a default. The exposure to a counterparty on portfolio level can be significantly reduced by incorporating a netting agreement in the contract. Netting entails offsetting the exposure of a single trade in the portfolio with the exposure of another trade, if possible. The exposure on a counterparty is thus evaluated at a netted portfolio level instead of for each trade separately.

Suppose we have a total portfolio value denoted by $V_{tot}(t) = \sum_{i=1}^{N} V_i(t)$, consisting of N different trades with a certain counterparty. The portfolio can exist out of various financial assets. If a netting agreement is included in the contract, the exposure of the total portfolio is smaller or equal to the sum of each exposure separately. We have that

$$\max(V_{tot}(t), 0) \leq \sum_{i=1}^{N} \max(V_i(t), 0). \tag{5.3}$$

This relation also holds for CVA, i.e.,

$$CVA_{tot}(t) \leq \sum_{i=1}^{N} CVA_i(t). \tag{5.4}$$

Intuitively, this makes sense. Suppose we have two outstanding trades with a counterparty. Both trades are interest rate swaps with identical contract specifications. In trade 1, the counterparty pays a fixed amount, and in trade 2 it receives a floating rate. Now let us have a positive exposure on trade 1, that is, $V_1(t)$ is greater than zero. By construction, the other swap has a negative exposure of $-V_2(t)$. If we incorporated a netting set in the contract with this counterparty, we would have that the total portfolio exposure is 0. However, the sum of individual positive exposures equals $V_1(t)$. Therefore, the CVA of the netted portfolio is smaller than the CVA of the sum of single trades. Since that $\max(\cdot)$ function is a non-linear operator, the sum of individual EPE's is not necessarily equal to the EPE of the portfolio as a whole. Therefore it is important to simulate the exposures for the whole portfolio. This characteristic causes us to calibrate the LMM to a portfolio instead of calibrating it towards individual trades, calculate exposures and subsequently combine results.

5.2 Least Squares Monte Carlo (LSMC)

In this section, we provide a summary of the Least Squares Monte Carlo method (LSMC) popularized by Longstaff & Schwartz (2001). LSMC was introduced to develop the optimal exercise strategy of American options. Every moment before the expiry date of an American option, the option holder must decide whether to exercise the option or keep it for another term. This choice is based on the holder's pay-off if exercised and the future expected pay-off. If the pay-off received at immediate exercise is larger than the future expected pay-off, the holder will exercise the option. The LSMC algorithm's idea is that an asset's future expected value could be approximated through a least-squares regression. The burden of a time-consuming Nested Monte Carlo technique can be avoided with such an approach.

Suppose we generate a total set of L LIBOR rate paths using the LIBOR Market Model derived in Section 3. Longstaff & Schwartz (2001) suggests that the continuation value of an option can be approximated as the risk-neutral conditional expectation of the discounted pay-offs from keeping the option alive, using cross-sectional information in the simulation. The LSMC algorithm aims to obtain approximations of the discounted portfolio values for every time period T_i for every path $l \in L$. The algorithm uses a backwards induction process, starting at time T_β and working back until time t. Longstaff & Schwartz (2001) states that the continuation value can be expressed as a linear combination of orthogonal basis functions. These basis functions are for example (a combination of) Power, Legendre and Laguerre polynomials. The theoretical argumentation of this claim is as follows. Longstaff & Schwartz (2001) consider the pay-off of an option to belong to the space of finite variance. Since the pay-offs belong to this space, the conditional expectations also belong in here (Herwig (2006)). Since the space of finite variance is a Hilbert space, any function that belongs to this space can be represented as a linear combination of orthogonal basis functions (Stentoft (2004)). The continuation value of an option at time T_i on path l, from here on denoted as $f_{T_i,l}$ can be denoted as

$$f_{T_i,l} = \sum_{j=1}^{p} \alpha_j \xi_j(\mathbf{x_l}). \tag{5.5}$$

In Equation 5.5, α_j represents the regression coefficient of the coefficient of the j-th basis function, ξ_j. The total number of p basis functions are polynomials of the set of simulated state variables along a path, in our setting LIBOR rates, given by $\mathbf{x_l}$. A selection of these polynomial basis functions is given in Appendix 9.5. Longstaff & Schwartz (2001) note that the choice of basis functions has an significant impact on the individual coefficients within the functions, but has little impact on the accuracy of the LSMC estimation. Moreover, Moreno & Navas (2003) states that the coefficients within the orthogonal Power function form a non-singular matrix, which means that the LSMC estimate should in principle be the same for all Power functions and set of polynomials functions. The regression coefficients are not previously known beforehand, but can be estimated by means of a linear regression. Moving backwards in time from T_β towards t, we minimize

$$\sum_{l \in L(T_i)} \left[\sum_{j=1}^{p} \alpha_{i,j} \xi_j(\mathbf{x_l}) - U_{T_i(l)} \right]^2, \tag{5.6}$$

where $U_{T_i}(l)$ represents the summation of the path l-discounted cash flows from moment T_i onward till T_β, thus

$$U_{T_i}(l) = N(l, T_i) \sum_{S \geq T_i}^{T_\beta} \frac{CF(l,S)}{N(l,S)}. \tag{5.7}$$

Continuing, a new set of paths is generated and the earlier obtained regression coefficients are used to obtain estimations of the continuation value of the portfolio.

Within the framework of calculating CVA, we are also interested in the future value of a portfolio at a given moment in time. Referring to Equation 5.2, we are specifically interested in the risk-neutral time t conditional expectation of the portfolio at time T_i. To obtain an expression for this expectation, we can use the LSMC framework. Implementing our above-derived definition of the continuation value along a given path at T_i, we have that the CVA at time t can be calculated by

$$CVA(t) = \gamma \cdot \sum_{i=1}^{N} q_i \cdot N(t) \mathbb{E}\left[N^{-1}(T_i) \cdot \max(f_{T_i,l}, 0) \middle| \mathcal{F}_t \right]. \tag{5.8}$$

5.3 Joshi & Kwon LSMC (JKLSMC)

Note that the approximation of the CVA by means of Equation 5.8 requires the regression functions to be accurate, which could in some scenarios be difficult to achieve. For example, close to swaptions expiry date, the pay-off function of a swaption may be less accurately approximated with polynomial basis functions due to the non-smooth characteristic of the real pay-off at this exact moment. Joshi & Kwon (2016) propose a method to obtain an improved estimation of the CVA motivated by an alternative representation of the expression of the CVA, and more specifically the EPE. Referring to Equation 5.2, the EPE is stated as the term within brackets. We thus have that $CVA(t) = \gamma \cdot \sum_{i=1}^{N} q_i \cdot EPE(t, T_i)$. More specifically, we can rewrite the EPE as

$$\begin{aligned}
\text{EPE}(t, T_i) &= N(t)\mathbb{E}\left(\frac{\max(V(T_i), 0)}{N(T_i)}\right), \\
&= N(t)\mathbb{E}\left(\frac{V(T_i) \cdot \mathbb{1}_{\{V(T_i)>0\}}}{N(T_i)}\right), \\
&= N(t)\mathbb{E}\left[\frac{1}{N(T_i)} \cdot N(T_i)\mathbb{E}\left(\sum_{S \geq T_i}^{T_\beta} \frac{CF(S)}{N(S)} \middle| \mathcal{F}_{T_i}\right) \cdot \mathbb{1}_{\{V(T_i)>0\}}\right], \\
&= N(t)\mathbb{E}\left[\mathbb{E}\left(\sum_{S \geq T_i}^{T_\beta} \frac{CF(S)}{N(S)} \middle| \mathcal{F}_{T_i}\right) \cdot \mathbb{1}_{\{V(T_i)>0\}}\right], \\
&= N(t)\mathbb{E}\left[\mathbb{E}\left(\sum_{S \geq T_i}^{T_\beta} \frac{CF(S)}{N(S)} \cdot \mathbb{1}_{\{V(T_i)>0\}} \middle| \mathcal{F}_{T_i}\right)\right], \\
&= N(t)\mathbb{E}\left(\sum_{S \geq T_i}^{T_\beta} \frac{CF(S)}{N(S)} \mathbb{1}_{\{V(T_i)>0\}}\right).
\end{aligned} \tag{5.9}$$

Here, we substituted that value of $V(T_i)$ in the second line according to the value implied by the first fundamental theorem of asset pricing as in Appendix 9.1 Equation 9.3. We have used the law of total expectation to obtain the final expression from the second-last line. Though the formulation of the EPE in terms of cash flows and the expected sign of the continuation function is presented in Joshi & Kwon (2016), the paper does not include its derivation or any other form of formal proof. The idea behind Equation 5.9 is that the regression functions part of the LSMC algorithm are only used to determine signs of the deflated future cash flows, rather then their explicit value. In Equation 5.9 we are interested in $\mathbb{1}_{\{V(T_i)>0\}}$ rather than the regressed value $V(T_i)$. In this setting, we consider the realized simulated cash flows as dependent variable and again a set polynomial basis functions incorporating state dependent explanatory variables, in our case simulated LIBOR rates. Joshi & Kwon (2016) test their proposed method on swap-settled Bermudan swaptions and cancellable swaps. CVA estimations through JKLSMC for cash price-settled swaptions, collateralized cash price-setteld (CCP) swaptions and portfolios of cash- and/or physically-settled swaption has not yet been researched and is included in this thesis.

An interesting observation from Equation 5.9 is that the value of $EPE(t, T_i)$ is always positive by construction, while the expectation on the last line does not have to be. We could have that negative cash flows occur after T_i, while $V(T_i)$ was positive. This is not necessarily a contradiction. The fact that the EPE is strictly positive does not directly imply that an estimator hereof should also be.

5.4 EPE for European swaptions

One of the goals of this thesis is to provide a better insight into EPE estimations for regular and irregular physically- and cash-settled European swaptions. In this section, we will try to explain how one could obtain the EPE for these types of products.

5.4.1 EPE for physically settled European Swaption

In the scope of modelling the EPE of a swaption with exercise date T_α, we will simulate 5000 forward rate paths according to the LIBOR Market Model, spanning the previously used tenor structure $[T_0, ..., T_\alpha, ..., T_\beta]$. To obtain an expression for the EPE at time t in this structure, we will take the expectation of Equation 5.9 with respect to all paths. Hence, we obtain the following definition of the EPE

$$EPE(t,T) = N(t)\mathbb{E}\left(\sum_{T_i \geq T}^{T_\beta} \frac{CF(l,T_i)}{N(l,T_i)} \mathbb{1}_{V(l,T)\geq 0}\right). \tag{5.10}$$

Above, we take the expected value of all simulated paths. So we could replace the expectation sign with $\frac{1}{10000}\sum_{l=1}^{10000}$. To find the EPE for a vanilla European swaption, we have to plug in the cash flow of said swaption in Equation 5.10. The cash flow of a physically-settled European swaption on a given path l regardless of the notional amount, is given by

$$CF_{physical\_swaption}(l,T_i) = \mathbb{1}_{(\omega\cdot(S_{l,\alpha,\beta}(T_\alpha)-K)\geq 0)} \cdot \tau_i \cdot \omega \cdot (\tilde{L}_l(T_i,T_i) - K) \tag{5.11}$$

Here, $\tilde{L}_l(T_i, T_i)$ denotes the LIBOR rate at time T_i, for a given path l. We assume for ease of presentation, but without loss of generality, that the fixed and floating leg of the swap contract have the same frequency and schedule of payments.

After the expiry date, T_α, the cash flow of a European swaption will only exist if the option on the underlying swap is exercised on the specified exercise date. Therefore, an indicator function representing whether the swaption is exercised on T_α is included in the cash flow definition. The specifications on the exercise decision of swaptions have been discussed in Section 4. We can obtain the EPE for a swaption by plugging in Equation 5.11 in Equation 5.10.

5.4.2 EPE for cash price settled European Swaption

Next to a physical settlement of a swaption, settlement can also occur by single cash payment on the expiry date. Swaptions for which this type of settlement is agreed upon are commonly referred to as cash-settled swaptions. If a swaption is cash-settled, the two counterparties will exchange a cash amount on the exercise date, computed differently depending on the cash settlement method used: cash price or collateralized cash price (CCP). For cash price-settled swaptions, on the expiry date, in case the swaption is exercised, one receives a cash amount equal to the present value of the underlying swap as observed at the expiry date of the option. Following the definitions in Pietersz et al. (2020), the pay-off of a cash price-settled swaption regardless of the notional amount at time T_i on a given path l can be stated as

$$CF(l, T_i) = \mathbb{1}_{(\omega \cdot (S_{l,\alpha,\beta}(T_\alpha) - K) \geq 0)} \cdot \text{Ann}(S_{l,\alpha,\beta}(T_\alpha)) \cdot \tau_i \cdot \omega \cdot (S_{l,\alpha,\beta}(T_\alpha) - K). \tag{5.12}$$

Here $\text{Ann}(\cdot)$ denotes the cash-annuity given by

$$\text{Ann}(S_{l,\alpha,\beta}(T_\alpha)) = P(t, T_\alpha) \cdot \sum_{n=1}^{N} \frac{\frac{1}{m}}{(1 + \frac{1}{m} \cdot S_{l,\alpha,\beta}(T_\alpha))^n}. \tag{5.13}$$

In Equation 5.13, N represents the number of fixed payments in the underlying swap contract, m denotes the number of times a year a payment occurs, and $P(t, T_\alpha)$ is the t-discount factor for an expiry date T_α. Here, we choose the discount factor as the value of a zero-coupon bond with a unit notional. We thus have that $P(T_\alpha, T_\alpha) = 1$. We can obtain the EPE for a cash price-settled swaption by plugging the definition of the cash flow in Equation 5.12 into the definition of the EPE as in Equation 5.10.

5.4.3 EPE for Collateralized Cash Price swaptions

In addition to the cash price-settled swaption type discussed above, we will consider a third type of swaption, the so-called collateralized cash price-settled (CCP) swaption. The CCP is defined as the present value of an annuity, equal to the difference between the amount the fixed-rate payer pursuant to the agreed-upon fixed rate in the underlying swap contract and the amount payable by such a fixed-rate payer if the fixed rate were to be the swap rate. Following Appendix 9.1 Theorem

9.3, we have that the time T_β value of a CCP swaption observed at time t is defined as

$$\begin{aligned} V_{CCP}(t) &= \mathbb{E}\left(\frac{N(t)\cdot V_{CCP}(T_\beta)}{N(T_\beta)} \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}\left(N(t)\cdot \frac{\sum_{T_i \geq T_\alpha}^{T_\beta} CF(T_i)}{N(T_i)} \middle| \mathcal{F}_t \right). \end{aligned} \tag{5.14}$$

Because we are interested in the difference in cash flows using the fixed rate included in the contract and swap rate, we have that $CF(T_i) = \omega \cdot \tau_i \cdot (SR_{\alpha,\beta}(T_\alpha) - K) \cdot \mathbb{1}_{(\omega\cdot(S_{\alpha,\beta}(T_\alpha)-K)\geq 0)}$. Continuing from Equation 5.14 and rearranging terms, we obtain

$$\begin{aligned} V_{CCP}(t) &= \mathbb{E}\left(N(t)\cdot \frac{\sum_{T_i \geq T_\alpha}^{T_\beta} CF(T_i)}{N(T_i)} \middle| \mathcal{F}_t \right) \\ &= \omega \cdot (SR_{\alpha,\beta}(T_\alpha) - K) \cdot \mathbb{1}_{(\omega\cdot(S_{\alpha,\beta}(T_\alpha)-K)\geq 0)} \cdot \sum_{i=\alpha}^{\beta-1} \tau_i \cdot \mathbb{E}\left(\frac{N(t)}{N(T_i)} \middle| \mathcal{F}_t \right) \\ &= \xi \cdot \sum_{i=\alpha}^{\beta-1} \tau_i \cdot N(t)\mathbb{E}\left(\frac{1}{N(T_i)} \middle| \mathcal{F}_t \right) \\ &= \xi \cdot \sum_{i=\alpha}^{\beta-1} \tau_i \cdot N(t)\mathbb{E}\left(\frac{P(T_i,T_i)}{N(T_i)} \middle| \mathcal{F}_t \right) \\ &= \xi \cdot \sum_{i=\alpha}^{\beta-1} \tau_i \cdot N(t) \cdot \frac{P(t,T_i)}{N(t)} \\ &= \xi \cdot \sum_{i=\alpha}^{\beta-1} \tau_i \cdot P(t,T_i) \\ &= \xi \cdot A(t,T_\alpha,T_\beta). \end{aligned} \tag{5.15}$$

Here, for the sake of representation, we denoted $\xi = \omega \cdot (SR_{\alpha,\beta}(T_\alpha) - K) \cdot \mathbb{1}_{(\omega\cdot(S_{\alpha,\beta}(T_\alpha)-K)\geq 0)}$. The summation over the accrual and discount factor, $A(t,T_\alpha,T_\beta)$, is also known as the Present Value of a Basis Point, or PVBP. Note that the term $\mathbb{E}\left(\frac{N(t)}{N(T_i)} \middle| \mathcal{F}_t \right)$ thus represents a discount factor over the period t to T_i. We can find the EPE of a Collateralized Cash Price swaption by filling in the definition of the cash flows stated above in Equation 5.9 for all simulated paths.

6 Results

In this section, we will present and discuss the results of our research. We start by giving an overview of the swaptions and portfolios of swaptions we consider. Then, we present exposure profiles of single swaptions with and without including an adjustment factor to provide basis understanding. Subsequently, we will present the results of our experiment in which we analyse the impact of calibrating the LMM to flat volatilities of a different level than the hypothetically correct level. We then move on to portfolios of swaptions and show that our calibration method allows for portfolio extension without unexpectedly changing the exposure of the original portfolio substantially. Lastly, we will analyse and compare the impact of expected exposures on CVA in relation with the probability of default.

6.1 List of derivatives and portfolios

We will analyse how the previously introduced adjustment factor impacts the expected exposure profiles for three types of swaptions, i.e., swap-settled, cash price-settled CCP-settled swaptions. We will consider bought ITM, ATM, and OTM fixed payer swaptions with different tenor structures. Buying these swaptions gives us the right to exercise the swaption or not. Moreover, we will consider the payment frequency of the fixed and floating leg equal and set to semi-annual payments. The notional amount of the underlying is set to EUR 100 million. The list of swaptions with which we will construct portfolios of swaptions is presented in Table 2. We construct a total of 6 portfolios, which are presented in Table 3. We will use end-of-day data of 14 June 2022 to construct relevant interest and discounting curves. This data is obtained through a commercial market data provider.

Swap-settled	Tenor	Strike rate	Cash price-settled	Tenor	Strike rate	CCP-settled	Tenor	Strike rate
SS1	2Y5Y	0.20%	CS1	2Y5Y	0.20%	CCP1	2Y5Y	0.20%
SS2	2Y5Y	2.5%	CS2	2Y5Y	2.5%	CCP2	2Y5Y	2.5%
SS3	2Y5Y	6%	CS3	2Y5Y	6%	CCP3	2Y5Y	6%
SS4	5Y10Y	0.20%	CS4	5Y10Y	0.20%	CCP4	5Y10Y	0.20%
SS5	5Y10Y	2.7%	CS5	5Y10Y	2.7%	CCP5	5Y10Y	2.7%
SS6	5Y10Y	6%	CS6	5Y10Y	6%	CCP6	5Y10Y	6%

Table 2: For all types of settlements, we will consider a shorter (2Y5Y) and longer (5Y10Y) term structure. Moreover, we will consider ITM (0.20%), ATM (2.5%) and OTM (6%) swaptions. With these, we will construct portfolios consisting of various types of swaption with different moneyness and tenor.

Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5	Portfolio 6	Portfolio 7	Portfolio 8	Portfolio 9
SS1	SS2	SS3	CS1	CS2	CS3	CCP1	CCP2	CCP3
SS4	SS5	SS6	CS4	CS5	CS6	CCP4	CCP5	CCP6

Table 3: Specification of portfolios 1-6.

6.2 Simulated EPE profiles of a single swaption

Before we elaborate on our findings regarding the EPE's of portfolios of swaptions taking the adjustment factor into account, it may be beneficial to discuss first the simulated expected exposure profiles of single swaptions. By doing so, we hope the interpretation of the graphs shown later will become clearer. We will discuss the profiles of swap-settled, cash price-settled, and CCP-settled swaptions for all sorts of moneyness, i.e., ITM, ATM, and OTM. The corresponding exposure profiles are presented in Figure 3.

Figure 3a depicts the expected exposure profiles of an ITM 2Y5Y physical settled swaption. Until expiry, the EPE remains nearly constant. We observe this behaviour because of the risk-neutral nature of our simulated LIBOR rates. Because the LIBOR rates are martingales, we have that for time $T_0 \leq t \leq T_\alpha$ it holds that

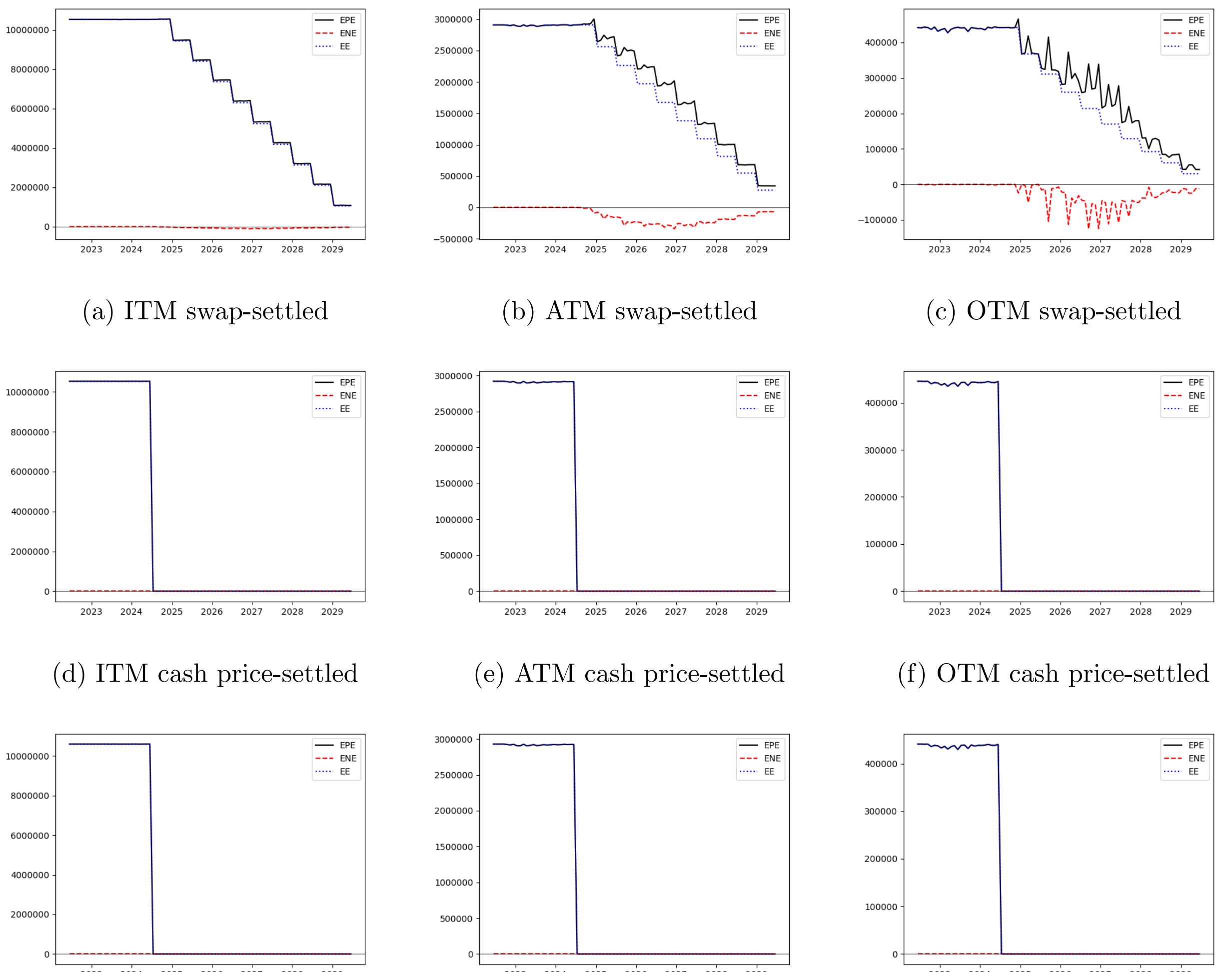
$$\begin{aligned} EPE(T_0, t) &= N(T_0)\mathbb{E}\left[\frac{\max(V(t), 0)}{N(t)} \middle| \mathcal{F}_{T_0} \right], \\ &= N(T_0)\mathbb{E}\left[\frac{V(t)}{N(t)} \middle| \mathcal{F}_{T_0} \right], \\ &= V(T_0). \end{aligned} \tag{6.1}$$

Here we have used that before expiry, the value of an option, $V(t)$, can not be negative. Therefore before expiry it holds that $\max(V(t), 0) = V(t)$. Then after expiry, if the option is exercised, the value of the underlying swap changes due to changes in the underlying LIBOR rates. Over time, the expected positive exposure of the contract eventually decreases due to the decreasing number of cash flows to be exchanged. If there are fewer remaining cash flows, the exposure logically approaches zero. At maturity, the EPE becomes zero, as no cash flows occur after this date. Since this swaption is deeply ITM, we observe that the Expected Negative Exposure, that is, the EPE from the point of view of the counterparty, is close to zero. The Expected Exposure (EE) is the sum of the EPE and ENE and follows the EPE closely due to the moneyness of the swaption. Moreover, we observe that the EPE profile of the swaption after expiry resembles the profile of the underlying interest rate swap, presented in Appendix 9.7. Since the option is very much ITM, there is little optionality, and the exposures of the underlying contained in the option are very much like the exposures of the underlying swap would it considered to be a trade on its own. The characteristic sawtooth pattern we observe is caused by payments of the counterparty. Whenever a payment is received, the exposure decreases by an amount equal to the amount paid relative to the notional amount.

Figure 3b shows the exposure profiles of an ATM 2Y5Y swap-settled swaption. Looking at the scale of this graph, it becomes apparent that the EPE and EE are much smaller in terms of size than the EPE and EE in Figure Figure 3a. From our point of view, the ATM trade is worth less than the ITM trade. Since the swaption is ATM, it should not be surprising that the EPE profile depicted is not as 'smooth' as the profile shown in Figure 3a. The total EPE of the trade is calculated as an average of the EPE over all simulated paths. Compared to the ITM trade, a fewer number of paths will generate a positive exposure on the counterparty. Therefore, the ATM EPE profile will turn out to be more jerky than the ITM profile. Moreover, the EPE profile's step pattern decreases and increases between payment dates. This is especially the case for the steps just after expiry. The increase reflects a payment from our side of the trade. As we have one less future payment, the exposure on the counterparty will increase. We do not see this increase for the ITM trade. This is because the amount to be paid from our side is very small compared to the amount we will receive from the counterparty. Note that the ITM trade is considered to be deeply ITM since the ATM rate is 2.5% and the fixed rate to be paid for the ITM trade is only 0.02%.

In Figure 3c the profiles of an OTM 2Y5Y physical-settled swaption are presented. With only a value of 0.004, the value of the EPE relative to the notional amount is quite low. This is expected for a deep OTM trade, in which we are obliged to pay 5% on the notional. The sawtooth pattern observed in the previous trades is not clearly visible here. Compared to Figure 3b, we observe in general more noise. Since we are considering a very deep OTM swaption, only a small number of simulated paths end up ITM. Therefore, changes along these paths have very much influence on the estimation of th exposure. This noise is also observed before the expiry date. Moreover, the value of the swaption is low because of its moneyness. Therefore, even small changes may have a large impact on the value of the exposure. The spiky pattern we observe is explained as follows. After payment from the counterpart side, exposure is decreased, but the value is only small compared to the value of the payments we make. The steep spikes upwards in Figure 3b are a result from payments from our side.

The second row in Figure 3 depicts the EPE profiles of the cash price-settled swaptions. The profiles differ much from the profiles in Figures 3a-3b. Up until expiry, we have an expected positive exposure on the counterparty. On expiry, the only exchange of cash flows included in the contract occurs. After this, no more cash flows between our side and the counterparty happen; therefore, the expected positive exposure is equal to zero. Profiles of trades varying in terms of moneyness only differ in the size of the exposure. Since CCP-settled swaptions also entail a single cash flow on expiry, exposure profiles of cash price-settled swaptions and CCP-settled swaptions look very similar. To the best of our knowledge, expected exposure profiles for cash price-settled and CCP-settled swaptions are not yet presented anywhere in the available academic literature.



(a) ITM swap-settled (b) ATM swap-settled (c) OTM swap-settled

(d) ITM cash price-settled (e) ATM cash price-settled (f) OTM cash price-settled

(g) ITM CCP-settled (h) ATM CCP-settled (i) OTM CCP-settled

Figure 3: The figure above depicts the EPE, ENE and EE profiles of a swap-, cash- and CCP-settled swaption for different values of moneyness.

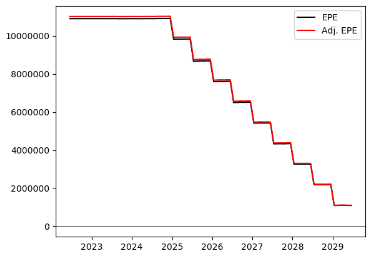
6.3 Single swaptions and the adjustment factor

Table 4 depicts the analytical and simulated mark-to-markets for the considered swaptions. Looking at the left side of the table, we note that analytical and simulated mark-to-markets for swap-settled swaptions do not differ much. The percentage difference between the simulated MtM and the analytical MtM is as much as 1.02% for SS1 and 0.99% for SS2. With 14.87%, the percentage difference is larger for trade SS3. Overall, we find that the percentage difference is larger for longer trades than for the shorter trades. Although is most likely not the only factor, we expect the difference to be partially caused by the fact that we calibrate the model towards ATM caplet volatilities instead of swaption volatilities that capture the volatility smile well. We find similar results for the cash and CCP-settled swaptions. Interestingly, we do not find that we systematically over-value or under-value trades. The column showing the swaption-specific adjustment factor indicates that some trades, for example, SS1, CS1, and CCP1 are under-valued, while other trades like SS2, CS2, and CCP2 are over-valued. Interestingly, we find that the tenor of the trade has impact on whether we under-value of over-value the swaption. For example, SS3 is under-valued using our model, while SS6 is overvalued. We find similar results for the cash price-settled and CCP-settled swaptions. For some of the trades, it may look like the percentage difference between de analytical mark-to-market and simulated mark-to-market is quite substantial. It is important to note that we use this method to modify the expected exposures in the context of CVA. Ideally, we would find a method that perfectly models the exposures of a swaption, such that the exposure of a set of swaptions is not changed when another swaption is added. While this ideal method is not found yet, we aim to find a method for which the condition of non-changing exposures is met. We present our method as an approach for this problem that can be used in practice and is reliable to some extent. In Section 6.4 we will analyse the impact of using flat caplet volatilities instead of flat swaption volatilities for the LMM calibration on simulated EPE profiles.

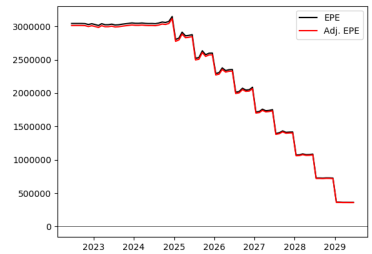
Swap	Ana. MtM	Sim. MtM	Adj. factor	Cash	Ana. MtM	Sim. MtM	Adj. factor	CCP	Ana. MtM	Sim. MtM	Adj. factor
SS1	11.019.258	10.907.409	1,0103	CS1	10.935.263	10.847.581	1,0081	CCP1	11.019.258	10.925.989	1,0085
SS2	3.014.352	3.044.257	0,9902	CS2	2.991.375	3.056.078	0,9788	CCP2	3.014.352	3.074.220	0,9805
SS3	528.449	454.846	1,1618	CS3	524.421	449.149	1,1676	CCP3	528.449	450.837	1,1722
SS4	20.969.432	20.436.689	1,0261	CS4	20.755.847	20.242.905	1,0253	CCP4	20.969.462	20.462.758	1,0248
SS5	6.482.049	5.584.718	1,1607	CS5	6.416.017	6.522.561	0,9837	CCP5	6.482.049	6.570.414	0,9866
SS6	1.729.182	1.883.826	0,9179	CS6	1.711.566	1.852.167	0,9241	CCP6	1.729.182	1.860.215	0,9296

Table 4: Table depicting the market observed mark-to-market, the simulated mark-to-market using the LMM and swaption specific adjustment factor. Here "Ana." stands for "Analytical" and "Sim." for "Simulated"

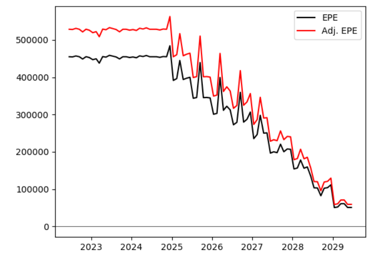
Now that we have simulated MtM values and swaption-specific adjustment factors, we can compose adjusted EPE profiles. The adjusted EPE profiles for the 2Y5Y swaptions are shown in Figure 4. Looking at the ITM and ATM profiles, we note that the adjusted EPE and unadjusted EPE profiles are similar. Looking carefully at Figure 4a and Figure 4b we observe that the adjusted EPE profile slowly converges to the unadjusted profile. This happens because we make use of a multiplicative adjustment factor. In Section 4.6 we discuss that every cash flow incorporated in the contract would be multiplied with the adjustment factor such that the analytical MtM and simulated MtM are equal on the observation date. Over time, since we have fewer cash flows outstanding, the impact of the adjustment factor on the EPE profile decreases. The corresponding exposure profiles for the single 5Y10Y trades are included in Figure 9 in Appendix 9.6. For all swaptions, we note that the difference in unadjusted and adjusted EPE profiles is slightly larger than the difference observed for the 2Y5Y swaptions. This is not strange; the tenor of 5Y10Y swaptions is longer than that of 2Y5Y swaptions. Therefore, adjusting the cash flows of the former swaption with some factor results in a larger impact on the EPE than if we would alter the cash flows of the latter swaption. In Figure 9a-9c the converging character of the adjusted EPE profile is well observable. While differences in adjusted en unadjusted EPE values can be seen at the start and middle of the contract, little to no difference is observed at the end.



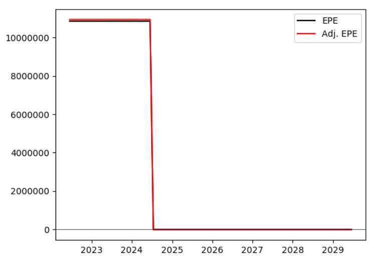
(a) ITM swap-settled



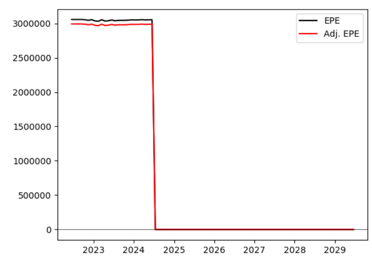
(b) ATM swap-settled



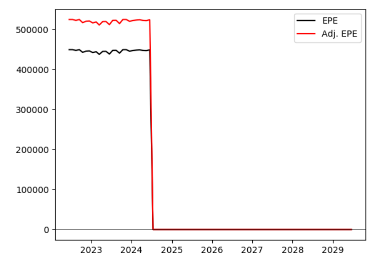
(c) OTM swap-settled



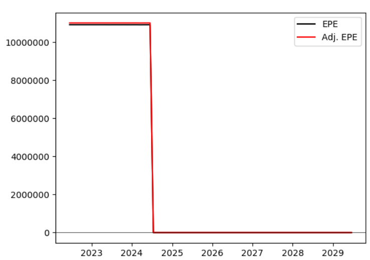
(d) ITM cash-settled



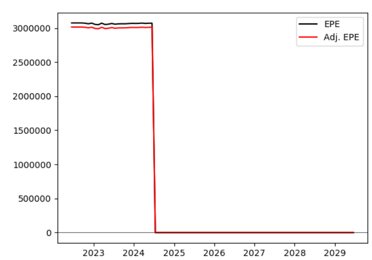
(e) ATM cash-settled



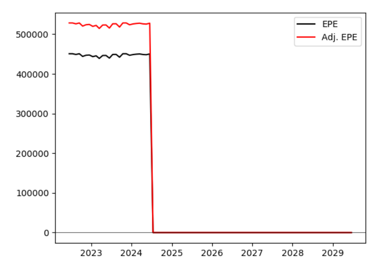
(f) OTM cash-settled



(g) ITM CCP-settled



(h) ATM CCP-settled

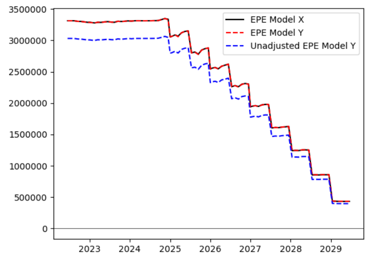


(i) OTM CCP-settled

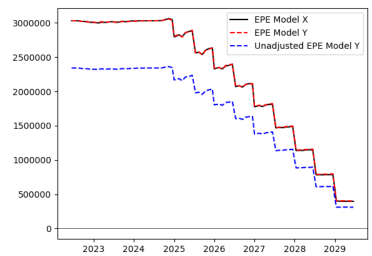
Figure 4: The figure above depicts the EPE and adjusted EPE profiles of 2Y5Y swap-, cash- and CCP-settled swaptions for different values of moneyness.

6.4 Experiment: Impact of miscalibration

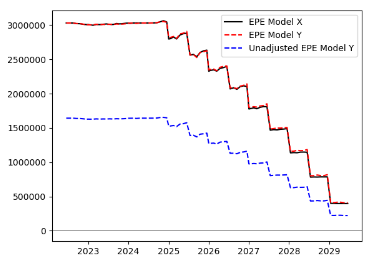
In Figure 11 the results of the experiment designed in Section 6.4 are presented for a 2Y5Y ATM swap-settled swaption for values of x larger than y. This implies that the flat caplet volatilities that hypothetically drive the swaption market, x, are larger than the flat volatilities y that drive the caplet market. In Appendix 9.8 we have included results for $x < y$, which yield similar outcomes. Figure 11a depicts the scenario where swaptions are in a hypothetical reality driven by flat caplet volatilities at 22%, and 20% flat volatilities drive the caplet market. We observe that the difference in EPE profiles is very small after adjusting the simulated value resulting from Model Y towards the simulated value obtained from model X. The largest difference relative to the notional amount of 100 million EUR is lower than 0.4%. Therefore, in this scenario, we deem that calibrating the model towards caplet volatilities of 22% and adjusting accordingly, while 'true' volatilities are 20%, has very little impact on the EPE. We observe that the unadjusted EPE profile of Model Y is much smaller than the hypothetically correct and adjusted profile. The adjustment factor clearly contributes towards a better modelled EPE. Figure 11b depicts the EPE profile in the scenario that $x = 20\%$ and $y = 15\%$. Since we lowered volatilities, we observe that the expected positive exposure is also decreased. Similar to the scenario in Figure 11a, we note that there is only a small difference in EPE profiles, even though volatility levels differ by as much as 5%. Again, the Model Y's unadjusted EPE profile is seriously off. In practice, we generally observe a difference between caplet and swaption volatilities between 0 and 5 percent. Therefore, the situation depicted in Figure 11a and Figure 11b can be seen as a proxy to real scenario, though still stylized with respect to the volatility smile. For both scenarios, EPE's obtained using a miscalibrated LMM in combination with an adjustment factor are very similar to the hypothetically correct EPEs. Moving forward to Figure 11e, we see discrepancies between the Model X's EPE profile and Model Y's. However, differences are still small relative to the notional amount. That we observe differences is not surprising. After all, we are simulating forward LIBOR rates with volatilities of 1%, while the hypothetically correct volatilities are 20%. This is a considerable difference, which will seldom be seen in practice. Still, it is interesting to see that the adjustment factor still manages to correct the EPE profiles of Model Y, such that both profiles are still quite close.



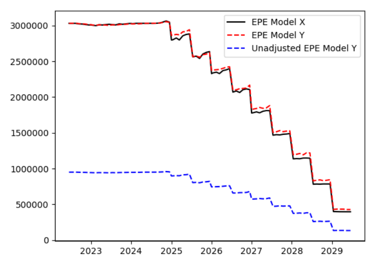
(a) Model X: 22%; Model Y: 20%



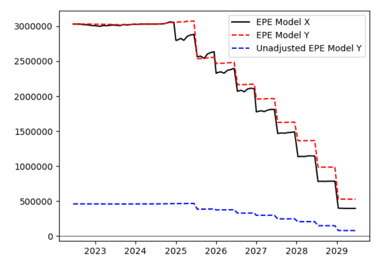
(b) Model X: 20%; Model Y: 15%



(c) Model X: 20%; Model Y: 10%



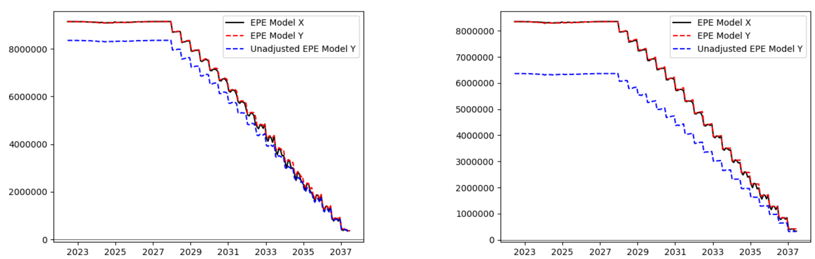
(d) Model X: 20%; Model Y: 5%



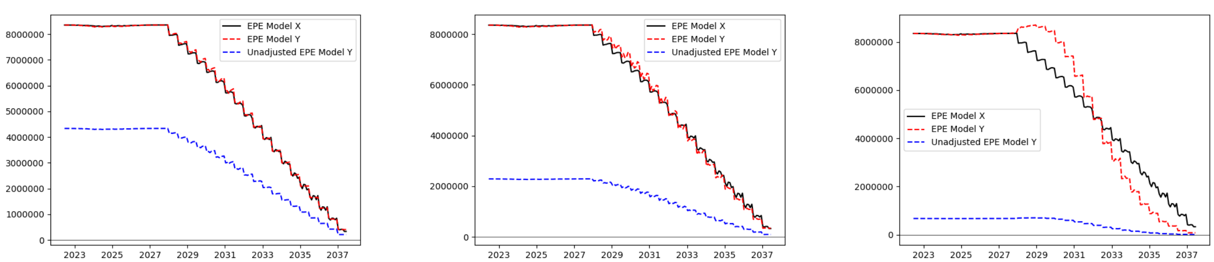
(e) Model X: 20%; Model Y: 1%

Figure 5: Exposure profiles for a 2Y5Y ATM swap-settled swaption obtained using different flat volatilities of x% and y%, where $x > y$

Appendix 9.8 Figure 12 depicts the result of the experiment for a 5Y10Y swaption. Since the maturity of this swaption is larger than that of the 2Y5Y swaption, we expect larger differences in EPE profiles. Looking at Figure 12a, we note that EPE profiles are similar till midway through the underlying swap, i.e., around 2031. From here, we observe discrepancies, but still, relative to the notional of 100 million EUR, these are small. The largest difference between Model X's EPE and Model Y's is smaller, approximately 0.17%. We see similar results for the scenario depicted in Figure 12b. Again, for all scenarios the benefit of adjusting the EPE resulting from Model Y is clearly visible. The result presented in Figure 12e is different from the result we have seen in Figure 11e. Apparently, a longer swap tenor, in combination with a large difference between x and y, yields substantial differences in the EPE after expiry. Up to expiry, the EPE is the same. This is a result of the martingale property of the value of the swaption. From a practical point of view, this is a very convenient characteristic. It means that up to expiry, the hypothetically correct EPE is exactly the same as the EPE obtained from the miscalibrated model using an adjustment factor. In terms of magnitude, we have that the exposure on a counterparty is the largest up to expiry. By construction, our proposed method yields the 'hypothetically' correct EPE up to expiry, which is a positive result. Although the difference in EPE's depicted in Figure 12e is quite substantial, the difference relative to the notional amount of the swaption is marginal. The results shown in Figure 11 and Figure 12 suggest that our proposed calibration method in combination with an adjustment factor can be considered quite reliable. Only for very large differences in volatility levels we observe substantial differences in EPE profiles. A difference of this size is rarely seen in the market. Note that in our experiment, we assumed hypothetically flat volatilities that drive the swaption market. In reality, a volatility smile is present. The difference between a model that captures this smile and a model calibrated and adjusted according to our method is yet to be explored.



(a) Model X: 22%; Model Y: 20%   (b) Model X: 20%; Model Y: 15%



(c) Model X: 20%; Model Y: 10%   (d) Model X: 20%; Model Y: 5%   (e) Model X: 20%; Model Y: 1%

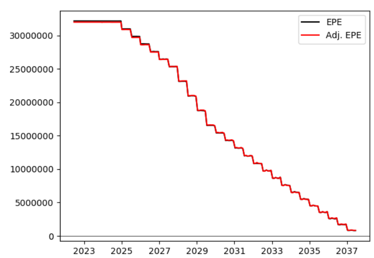
Figure 6: Exposure profiles for a 5Y10Y ATM swap-settled swaption obtained using different flat volatilities of x% and y%, where $x > y$

6.5 Portfolios of swaptions and the adjustment factor

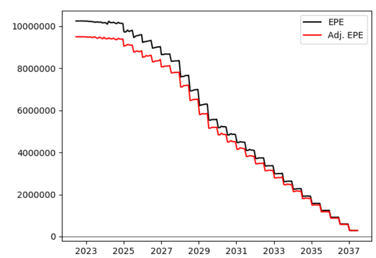
In this section, we will discuss the EPE profiles of portfolios of swaptions. In Section 6.5.1 we will first show the results for a portfolio consisting of two swaptions, with the same settlement method and moneyness but differing in tenor. Then we will extend these portfolios and discuss implications in Section 6.5.2.

6.5.1 Portfolios of two swaptions

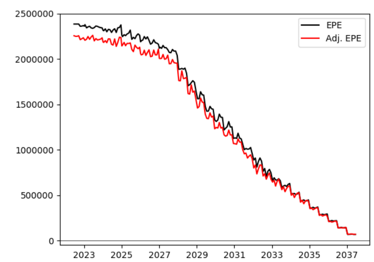
In Figure 7 the EPE profiles for a portfolio of two swaptions are depicted. Looking at the Figures 7a-7b, we can clearly see the exposure profile changing after 2027. This is exactly when the 5Y10Y swaption expires. Then the exposure profile changes again around 2029. This is when the 2Y5Y swaption matures and only the 5Y10Y swaption is alive. Note again that the differences in EPE profiles for the adjusted and unadjusted cases are small, and the two lines converge over time. For the cash-settled and CCP-settled swaptions, we can clearly see when the 2Y5Y and 5Y10Y swaptions expire by to the sudden steep drops in exposure.



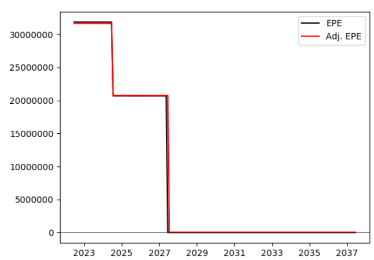
(a) Portfolio 1: ITM Swap-settled



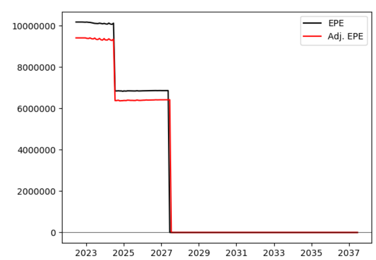
(b) Portfolio 2: ATM Swap-settled



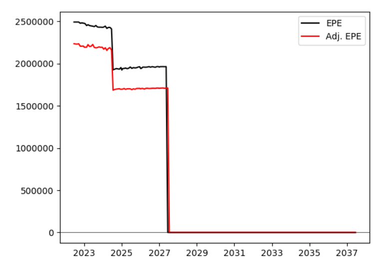
(c) Portfolio 3: OTM Swap-settled



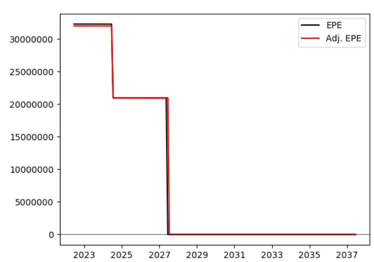
(d) Portfolio 4: ITM Cash-settled



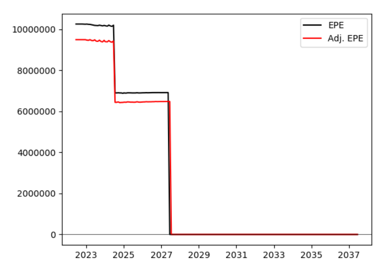
(e) Portfolio 5: ATM Cash-settled



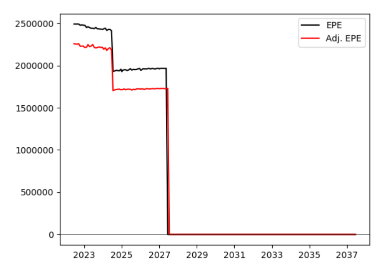
(f) Portfolio 6: OTM Cash-settled



(g) Portfolio 7: ITM CCP-settled



(h) Portfolio 8: ATM CCP-settled



(i) Portfolio 9: OTM CCP-settled

Figure 7: The figure above depicts the EPE and adjusted EPE profiles of a portfolio of a 2Y5Y and 5Y10Y swaptions. We consider portfolios with same moneyness and settlement method, but different tenor structure.

6.5.2 Extending a portfolio of two swaptions

In section 4 we propose to use Black implied volatilities as instantaneous LIBOR rate volatilities. Theoretically, the exposure of a portfolio of swaptions is not modified if we were to extend the portfolio with an irrelevant longer trade. In this section, we will extend portfolio 2, a portfolio of two swap-settled ATM swaptions with an irrelevant swaption with a longer maturity. We choose to extend this portfolio because, due to its moneyness, the portfolio is more subject to changes in volatility. The current longest tenor in the portfolio is the 5Y10Y swaption, which matures in 2037. We will add a 10Y15Y swap-settled swaption on a notional of 1 EUR to the portfolio, thus increasing the tenor significantly. A plot showing the EPE profiles of the original portfolio and the extended portfolio are shown in Figure 8 and the numerical values are presented in Table 8 in Appendix 9.10.

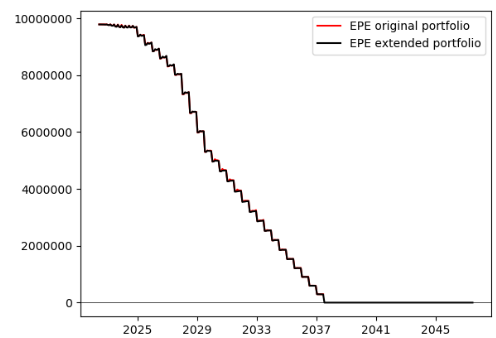


Figure 8: EPE profiles of the original and extended portfolio

From Figure 8, it becomes apparent that although the two lines look similar, very small differences exist. Looking at Table 8, we see that over time, the exposure profiles of the original and extended portfolio are not exactly equal. Even though the original and extended portfolios' differences are way below 1% for most dates, equivalence should hold theoretically. We suspect that the difference may be caused by Monte Carlo noise. Our simulation is constructed in a way where we generate random numbers (RN) along each path. If we increase the tenor of the portfolio, the way our random numbers are distributed changes. This is graphically displayed in Appendix 9.9. The suspicion that the difference in EPE profile is caused by Monte Carlo noise is substantiated when we compare the EPE profiles generated by a fewer number of simulation paths. We find that for a lower number of paths, the difference between the original and extended portfolio is much larger than for a higher number of paths. The values presented in Table 8 are obtained using 50.000 simulated paths. We expect that the differences will approach zero for a very large number of paths. However, we have not tested this due to computational complexities.

7 Conclusion

The purpose of the thesis is to address the continuity problem practitioners face when adding a relative insignificant swaption with a longer maturity to a portfolio of relevant swaptions with a shorter maturity. We have presented an approach to solving the continuity problem. The problem possibly arises in the process of CVA calculation. To calculate CVA, we need to obtain expressions for the expected positive exposure (EPE). The EPE is dependent on future pay-offs, which in the case of interest rate swaptions, are functions of interest rates. In this thesis, we simulated forward interest rates using the LIBOR Market Model. If the LIBOR Market Model is calibrated using swaption volatilities, instantaneous LIBOR volatilities corresponding to a portfolio before extension are likely not equal to the LIBOR volatilities corresponding to an extended portfolio. The LIBOR volatilities determine the dynamics of the LIBOR rates, which are used directly to calculate the pay-offs of the trades included in the portfolio. If pay-offs change, the expected positive exposure of both the individual swaptions and the portfolio as a whole will also change. Ultimately, with the expected positive exposures, we can calculate CVA charges. For practitioners, like banks and other financial institutions, it is undesirable that CVA charges change significantly after a new trade with a marginal impact on the EPE is added to an already existing portfolio. Sudden unexplainable CVA charges may cause a counterparty to lose trust in a practitioner. Besides that, since CVA directly influences the profit and loss statement, unaccountable changes are undesirable.

Our approach to the continuity problem entails calibrating the LIBOR Market Model using ATM market quoted caplet prices. These prices are quoted in terms of their Black implied volatility. Due to the analytical equivalence of the Black'76 formula and the LIBOR Market Model, we can set instantaneous LIBOR volatilities to corresponding Black implied caplet volatilities. By doing so, we can theoretically ensure that the instantaneous LIBOR volatilities affiliated to the portfolio before extension remain the same even after extending the portfolio. Figure 2 depicts the non-extended and extended portfolios and the corresponding instantaneous LIBOR rate volatilities. Theoretically, since LIBOR volatilties over the tenor of the non-extended portfolio remain the same even after extension, an irrelevant addition to the portfolio should not impact the exposures of the non-extended portfolio. Figure 8 and Table 8 depict the exposure profiles of a portfolio consisting of two ATM swap-settled swaptions and an extension of this portfolio. Although simulated exposures of the extended portfolio over the tenor of the first portfolio differ to a very small extent, equivalence should hold theoretically. We suspect that this difference is caused by Monte Carlo noise and can be decreased by increasing the number of simulated paths.

Since we calibrate the LIBOR Market Model using flat caplet volatilities, we will not be able to construct market observed swaption prices. We correct this mistake to some extent by introducing a swaption-specific adjustment factor. This adjustment factor is derived by dividing the analytical swaption price by the simulated price. The adjustment factor is then used to adjust swaption pay-offs, which leads to an adjusted expected positive exposure. Since the adjustment factor is applied multiplicatively, its impact on the expected exposure slowly decreases over time, which is a favourable characteristic. Table 4 shows the analytically calculated and simulated prices for the swaptions considered in this research. We observed that the difference between the analytical and simulated swaption price is relatively larger for OTM trades than for ITM and ATM trades. We expect the fewer number of simulation paths with which we calculate the EPE may be causing the larger mismatch we observe for OTM trades. We then discussed expected positive exposure profiles for portfolios consisting of swap-settled, cash price-settled, and collateralized cash price-settled swaptions. For the expression of the expected positive exposure, we used the definition stated in Joshi & Kwon (2016). In Equation 5.9 we provided a formal proof that supports this definition.

Additionally, we designed an experiment to analyse the impact of using a LIBOR Market Model calibrated to caplet volatilities instead of swaption volatilities. We constructed a model, Model X, in which we assumed that the swaption market was hypothetically driven by flat caplet volatilities of a specific level. We then built a model, Model Y, where we calibrated the model to flat caplet volatilities of another level. Subsequently, we would adjust the EPE profiles resulting from Model Y such that the simulated MtM's of both models would be equal on the date of observation. The comparison of the EPE profiles shown in Figure 5 and Figure 6 suggests that the impact of calibrating using volatilities of different levels is marginal if we adjust EPE profiles with an adjustment factor. Because our proposed method solves the continuity problem while maintaining reliable results, we deem our research both relevant from a theoretical and practical point of view.

To conclude this section, we would like to point out that our proposed method is interest rate derivative agnostic. Previously, we have stated that the continuity problem occurs when extending a portfolio of swaptions with another trade. We have shown that exposure profiles remained nearly similar when this other trade was a swaption. However, we could have also extended the portfolio with another type of interest rate derivative, e.g., caps, floor, or swaps. This derivative agnostic characteristic makes our proposed method well-suited for practitioners who often have multiple different kinds of trades with counterparties.

8 Limitations and Future Research

In this section, we would like to address limitations in the setup and methods used in this research. While doing this, we also propose topics that can be explored in further research. We start by touching upon the construction of the LIBOR Market Model we use in this thesis. In Section 3 we formally derived how we can construct correlated Brownian motions to simulate correlated LIBOR rates. In practice, we used a single Brownian motion to simulate forward LIBOR rates. Therefore, the correlation between Brownian motions was assumed to be constant and equal to 1. In the literature, we find that LIBOR Market Models are generally constructed, taking LIBOR rate correlations into account. Among many others, Schoenmakers (2002), Brigo & Mercurio (2006) and Fries (2007) discuss calibration of the LIBOR Model with respect the the instantaneous volatilities and correlations. Correctly calibrating the LIBOR correlations especially makes sense in the context of evaluating swaptions. We have seen that swap rates are weighted averages of LIBOR rates. It may be expected that the price of a swaption will change if the correlation between forward LIBOR rates alters, as this will also lead to a change in volatilities. Incorporating and finding LIBOR rate correlations has been well researched in the literature. However, the topic of calibrating the LMM to find correlations when a portfolio of swaptions is extended is not yet explored. If we were to assume a constant correlation between LIBOR rates, extending a portfolio will not influence the correlation structure. However, if we were to use some parametric approach to determine parameters to construct the correlation structure, we may run into difficulties. Future research might specify how a new correlation structure, changed after portfolio expansion, influences the exposure profiles of swaption included in the original portfolio.

In Section 4 we have briefly mentioned an approach to solving the continuity problem where we would create a very large swaption matrix and calibrate accordingly. If we can construct such a model, extending a portfolio with a swaption not already included in the covered tenor is not likely to happen. However, this method also brings disadvantages, also explained in Section 4. As we did not explore the differences and impact on EPEs between such an approach and the approach we present in this thesis, we do not know which method yields more accurate exposure profiles. This research is the first to address the continuity problem and suggests a practical solution to the continuity problem. Future research could potentially find other approaches and, more importantly, specify which of the proposed methods is preferred under which circumstances.

We have only considered LIBOR Market Model calibration using ATM Black implied caplet volatilities. In our conducted experiment, we analysed how well our proposed method performs in a situation where we know that the swaption market is driven by flat caplet volatilities of a certain level. We neglected the existence of a smile in associated volatilities. Our experiment, therefore, does not provide evidence on how well our proposed method works in a real scenario, where we generally observe a non-flat implied volatility structure (see e.g., Piterbarg (2015)). The conclusion that our proposed method is reliable therefore only holds in the stylized environment of our experiment. However, as the results of the experiment suggest that our proposed method is quite robust, our approach may very well present reliable results in reality. We suggest further research into an experiment to test how well our method performs in case the hypothetical real-life volatility structure captures the volatility smile well.

In this research, we have modelled forward LIBOR rates assuming these were log-normally distributed. By doing so, we implied a flat volatility smile. Extensions to the LIBOR model have been proposed to render the volatility smile correctly. Examples of these are discussed in Andersen & Andreasen (2000), and P. Hagan & Lesniewski (2008). Our proposed method to solve the continuity problem does not co-integrate well with modified versions of the LIBOR model that can capture the volatility smile since we use ATM volatilities directly. It is yet to be researched how we can extend a portfolio of swaptions, such that exposures are not altered, while also capturing the volatility smile in the course of LIBOR rate simulation.

9 Appendix

9.1 Mathematical framework

In this section, we will briefly discuss the relevant mathematical background of this thesis. The reason for including this section is two-fold. Firstly, for the sake of readability, we will frequently refer to the equations and theorems discussed in this section instead of including these in the main part of the thesis. Secondly, the section provides an overview of the background knowledge and ensures that no confusion will occur concerning the definitions to be discussed. We start by discussing theorems and definitions related to financial markets and probability theory.

Definition 9.1 (**Probability measure**). *Let a sample space of a random experiment be defined as $(S, \mathcal{F})$ where S is the set of outcomes of an experiment and $\mathcal{F}$ denotes the collection of events, which is a subset of S. While running the experiment, a certain event A either occurs or does not occur. A probability measure $\mathbb{P}$ on the sample space $(S, \mathcal{F})$ is a function defined on the collection for which the following axioms hold*

- $\mathbb{P}(A) \geq 0$ *for every event A.*
- $\mathbb{P}(S) = 1$.
- *If $\{A_i : i \in I\}$ is a countable, pairwise disjoin collection of events then*

$$\mathbb{P}\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mathbb{P}(A_i). \tag{9.1}$$

The third axiom states that the union of a countable infinite collection of disjoint events is the sum of the corresponding probabilities. A common measure used in mathematical finance is the risk-neutral measure $\mathbb{Q}$. This probability measure is such that the value of a financial asset is equal to the discounted expectation of this asset under this measure. Intuitively this makes sense; however, in reality, investors have risk preferences resulting in asset values being somewhat different from the discounted expected future returns.

Definition 9.2 (**Money-market account**). *The money-market account, denoted by $M(t)$ is the time-t value of a bank account. The process of this $M(t)$ is driven by the short rate $r(t)$. The short-rate represents the interest rate at which one can earn a risk-free profit on an infinitely short period of time. Therefore, the process of $M(t)$ is given by*

$$dM(t) = r(t)M(t)dt, \tag{9.2}$$

where the initial value of the bank account, $M(t_0)$, equals 1.

Definition 9.3 (Numéraire). *A numéraire is a positive asset used to measure the value of another tradable asset. Put differently, a numéraire represents in which unit prices of tradable assets are measured. The risk-neutral measure $\mathbb{Q}$ has the money-market account as numéraire.*

Definition 9.4 (Arbitrage). *Under the term arbitrage, we understand the process of earning a risk-free profit by simultaneously purchasing and selling financial assets in different markets and exploiting discrepancies.*

Definition 9.5 (The fundamental theorem of asset pricing). *The value at time t, $V(t)$, of a contingent claim with cash flow at time T, CF(T), is given by*

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)} \cdot CF(T) \middle| \mathcal{F}_t\right], \tag{9.3}$$

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^T V(T)} \middle| \mathcal{F}_t\right], \tag{9.4}$$

where $\mathcal{F}_t$ denotes the filtration set up until time t. The filtration set models the evolution of information through time. For the fundamental theorem of asset pricing to hold, it is assumed that there are no arbitrage opportunities in the market available. Moreover, markets are complete, meaning they are frictionless, and all agents active on the market have access to all relevant information.

Definition 9.6 (Martingales). *We define the stochastic process X to be a $\{\mathcal{F}_t\}$-martingale under the risk-neutral measure if:*

- *X is adapted to $\{\mathcal{F}_t\}$, informally meaning that X cannot see into the future time past t and that the value of X is known at time t when the information contained in the filtration set $\mathcal{F}_t$ is presented.*
- *$\mathbb{E}^{\mathbb{Q}}[X(t)] < \infty$ for all $t \in [0, T]$.*
- *$\mathbb{E}^{\mathbb{Q}}[X(t)|\mathcal{F}_s] = X(s)$ for all $0 \leq s \leq t \leq T$.*

The property of X being a martingale thus entails both the filtration and the probability measure. It could very well be the case the X is a martingale under a certain probability measure but is not under another one.

Definition 9.7 (Equivalent Measures). *Let $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on the sample space S. The two measures are equivalent, and we denote $\mathbb{Q} \sim \mathbb{P}$ whenever for all events A it holds that*

$$\mathbb{Q}(A) > 0 \Leftrightarrow \mathbb{P}(A) > 0. \tag{9.5}$$

Definition 9.8 (**(Geometric) Brownian Motion**). *The stochastic process* $X = (X(t), t \geq 0)$ *is a* $\mathbb{P}$*-Brownian motion if it adheres to all of the four stated condition below*

- $X(0) = 0$
- X *is continuous*
- *Under the probability measure* $\mathbb{P}$, $X(t) \sim N(0, t)$
- *Under* $\mathbb{P}$, *the increments* $X(s+t) - W(s) \sim N(0, t)$ *are independent of the history of* X *up to time* s.

A vector of N *independent Brownian motions is called a multidimensional Brownian motion or multidimensional Wiener process. The following stochastic process*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \tag{9.6}$$

with solution

$$X(t) = X(0)exp\left(\mu t + \sigma W(t) - \frac{\sigma^2 t}{2}\right), \tag{9.7}$$

is called a geometric Brownian motion. Here, the drift term is denoted by μ *and* σ *denotes the volatility of the stochastic variable* $X(t)$.

Theorem 9.1 (**Black 1976 Formula**). *Suppose the dynamics of the stochastic process* X *satisfy*

$$dX(t) = \sigma X(t)dW(t),$$

with σ *representing the constant volatility of the stochastic variable* X *and* $W(t)$ *a standard Wiener process, such that* $X(t)$ *follows a drift-less geometric Brownian Motion, and its solution is given by*

$$X(t) = X(0) \cdot \mathcal{E}\left(\int_0^t \sigma(s)dW(s)\right), \tag{9.8}$$

with $\mathcal{E}(\cdot)$ *the Doléans-Dade exponential. Let* V *denote the value of a European call option, such that* V *is given by* $\mathbb{E}^{\mathbb{Q}}[X(T) - K]^+$. *Then the value of* V *can be computed by the Black'76 formula with* $\Phi(\cdot)$ *denoting the standard normal distribution*

$$V = X(0)\Phi(d_1) - K\Phi(d_2) \tag{9.9}$$

with

$$d_1 = \frac{ln\left(\frac{X(0)}{K}\right) + \frac{\sigma T}{2}}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

9.2 Probability measures in the LIBOR Market Model framework

Here, we will briefly introduce probability measures in the LIBOR Market Model framework relevant for this thesis. In Definition 9.1 and Definition 9.3, we have already touched upon the risk-neutral measure and how to price derivatives using this measure.

Definition 9.9 (**T-forward measure**). *Under the so called $T-forward$ measure, a zero-coupon bond with expiry T is used as numéraire. We denote this specific bond by $P(\cdot, T)$. Using Definition 9.1, we find the value of a financial derivative*

$$V(t) = \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[\frac{P(t,T)}{P(T,T)} \cdot V(T) \bigg| \mathcal{F}_t\right], \tag{9.10}$$

$$= P(t,T)\mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[V(T) | \mathcal{F}_t\right], \tag{9.11}$$

because $P(T,T) = 1$. In the framework of the LIBOR Market Model, as we will see in Section 3, the LIBOR rate $L(t, T_i, T_{i+1})$ is a martingale under the T_{i+1} forward measure. Put differently, we have that

$$L(t, T_i, T_{i+1}) = \mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[L(T_i, T_i, T_{i+1}) | \mathcal{F}_t\right]. \tag{9.12}$$

Definition 9.10 (**Swap measure**). *In Section 4.2 we introduce the annuity factor $A(\cdot)$ given by $A(t, T_\alpha, T_\beta) = \sum_{i=\alpha}^{\beta-1} \tau_i P(t, T_{i+1})$. The annuity factor consists of multiple discount bonds, which are on themselves positive processes. Therefore, $A(t, T_\alpha, T_\beta)$ itself is a positive process as well and thus justified to be used as numéraire. The probability measure characterised by choosing $A(t, T_\alpha, T_\beta)$ as numéraire is the so called swap measure, denoted by $\mathbb{Q}^{\alpha,\beta}$. Using this measure in combination with its numéraire yields the following pricing formula for a derivative*

$$V(t) = A(t, T_\alpha, T_\beta)\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\left[\frac{V(T)}{A(T, T_\alpha, T_\beta)} \bigg| \mathcal{F}_t\right]. \tag{9.13}$$

Definition 9.11 (**Spot measure**). *The tenor structure used in the LIBOR Market Model framework is of a discrete nature. Therefore, a numéraire accruing interest in a continuous setting would not suit well in this setting. Hence Jamshidian (1997) introduced the discrete-time equivalent of the money market account as numéraire. The probability measure that corresponds with this numéraire is known as the sport measure, noted by $\mathbb{Q}^S$. The discrete money market account, $N(t)$, is composed as follows. We invest a value of 1 at time T_0=0 into the T_1 zero-coupon bond. Next, after the maturity of this bond, we invest the proceeds, denoted by $\frac{1}{P(T_0,T_1)} = 1 + \tau_0 L(T_0, T_0, T_1)$, into a zero-coupon bond with a maturity T_2. Hence the portfolio at this time consists of $\frac{1}{P(T_0,T_1)} \frac{1}{P(T_1,T_2)} \times P(T_1, T_2)$. Generalising this process, we thus re-invest the proceeds of the T_j bond in the T_{j+1} bond. We will* *do so, until time T_{N-1}. Formally the discrete-time implied money market account is given by*

$$\begin{aligned}
N(0) &= 1 \\
N(1) &= \frac{1}{P(T_0, T_1)} P(T_1, T_2) \\
N(2) &= \frac{1}{P(T_0, T_1)} \frac{1}{P(T_1, T_2)} P(T_2, T_3) \\
&\vdots \\
N(t) &= P(t, T_{i+1}) \prod_{j=0}^{i} \frac{1}{P(T_j, T_{j+1})} = P(t, T_{i+1}) \prod_{j=0}^{i} 1 + \tau_j L(T_j, T_j).
\end{aligned} \tag{9.14}$$

Under the so called spot measure we find the risk-neutral value of a financial derivative by

$$V(t) = N(t)\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V(T)}{N(T)}\right| \mathcal{F}_t\right]. \tag{9.15}$$

9.3 Derivation of the drift term under the Spot measure

In this section, we will discuss the derivation of the drift term in the LIBOR Market model under the spot measure. Previously, we have seen that the dynamics of the T_i-th LIBOR rate follows driftless Brownian motion under the T_{i+1} forward measure and the LIBOR rate itself is a martingale. However, under the spot probability measure, we have to implement a drift term to ensure rates are martingales. Therefore, we define the process of the T_i-th LIBOR rate under the terminal measure as follows

$$d\tilde{L}(t,T_i) = \mu^S(t,T_i)\tilde{L}(t,T_i)dt + \sigma_L(t,T_i)\tilde{L}(t,T_i)dW^S \quad \text{for } i = 0, ..., N-1. \tag{9.16}$$

The exercise left is to derive the drift term $\mu^S(t,T_i)$. For this we will follow very closely the derivation of the drift-term presented in Fries (2007), though we consider a shifted LIBOR rate to allow for negative interest rates. For the spot measure, we will consider the rolled over one period bond as numéraire. That is, we invest a value of 1 at time T_0 into the T_1-bond and invest the proceedings into the the bond of the next period. Thus, we invest the proceedings of the T_j-bond in the T_{j+1}-bond. We define $q(t) = \max[i \ : \ T_i \leq t]$ such that the numéraire $N(t)$ at time t corresponding to the equivalent martingale spot measure is denoted as

$$\begin{aligned} N(t) = P(t,T_{q(t)+1}) \prod_{j=1}^{q(t)+1} \frac{1}{P(T_{j-1},T_j)} &= P(t,T_{q(t)+1}) \prod_{j=0}^{q(t)} (1+\tau_j \tilde{L}(T_j,T_j)), \\ = \frac{P(T_{j-1},T_{j-1})}{P(T_{j-1},T_j)} &= 1 + \tau_{j-1}\tilde{L}(T_{j-1},T_{j-1})) \end{aligned} \tag{9.17}$$

We will consider the processes of numéraire relative prices of traded products, in our case $P(t,T_i)$ bonds and as earlier, we know that these processes have zero drift under the Spot measure $\mathbb{Q}^S$. We have that

$$\begin{aligned} \frac{P(t,T_i)}{N(t)} &= \frac{P(t,T_i)}{P(t,T_{q(t)+1})} \prod_{j=0}^{q(t)} (1+\tau_j \tilde{L}(T_j,T_j))^{-1}, \\ &= \prod_{j=q(t)+1}^{i-1} (1+\tau_j \tilde{L}(t,T_j))^{-1} \cdot \prod_{j=0}^{q(t)} (1+\tau_j \tilde{L}(T_j,T_j))^{-1}. \end{aligned} \tag{9.18}$$

Therefore, the dynamics of $P(t,T_i)/N(t)$ are

$$\begin{aligned} & d\left(\prod_{j=q(t)+1}^{i-1} (1+\tau_j \tilde{L}(t,T_j))^{-1} \cdot \prod_{j=0}^{q(t)} (1+\tau_j \tilde{L}(T_j,T_j))^{-1} \right), \\ &= d\left(\prod_{j=q(t)+1}^{i-1} (1+\tau_j \tilde{L}(t,T_j))^{-1} \right) \cdot \prod_{j=0}^{q(t)} (1+\tau_j \tilde{L}(T_j,T_j))^{-1}. \end{aligned} \tag{9.19}$$

To ensure zero drift under the Spot measure it holds that

$$\text{Drift}_{\mathbb{Q}^S}\left[\prod_{k=q(t)+1}^{i-1}(1+\tau_k\tilde{L}(t,T_k))^{-1}\right]=0. \tag{9.20}$$

To determine the drift of the expression above, we are considering the dynamics of $\prod_{k=q(t)+1}^{i-1}(1+\tau_k\tilde{L}(t,T_k))^{-1}$. For this we need the product rule for Ito processes. Suppose X, Y, and $X_1,...,X_N$ are Ito processes then

$$d(XY)=YdX+XdY+dXdY \tag{9.21}$$

$$d\left(\prod_{i=1}^{N}X_i\right)=\sum_{i=1}^{N}\prod_{\substack{k=1\\k\neq i}}^{N}X_kdX_i+\sum_{\substack{i,j=1\\j>i}}^{N}\prod_{\substack{k=1\\k\neq i,j}}^{N}X_kdX_idX_j. \tag{9.22}$$

Here the second equation follows by iteratively applying the first equation. Therefore, $\forall i=0,...,N$

$$d\left(\prod_{k=q(t)+1}^{i-1}(1+\tau_k\tilde{L}(t,T_k))^{-1}\right) \tag{9.23}$$

$$=\sum_{j,l=q(t)+1}^{i-1}\prod_{\substack{k=q(t)+1\\k\neq j}}^{i-1}\frac{1}{1+\tau_k\tilde{L}(t,T_k)}\cdot\left(\frac{-\tau_jd\tilde{L}(t,T_j)}{(1+\tau_j\tilde{L}(t,T_j))^2}+\frac{\tau_j^2d\tilde{L}(t,T_j)d\tilde{L}(t,T_j)}{(1+d\tilde{L}(t,T_j))^3}\right)$$

$$+\sum_{\substack{j,l=q(t)+1\\l<j}}^{i-1}\prod_{\substack{k=q(t)+1\\k\neq j,l}}^{i-1}\frac{1}{1+\tau_k\tilde{L}(t,T_k)}\left(\frac{\tau_jd\tilde{L}(t,T_j)}{(1+\tau_j\tilde{L}(t,T_j))^2}+\frac{\tau_j^2d\tilde{L}(t,T_j)d\tilde{L}(t,T_j)}{(1+d\tilde{L}(t,T_j))^3}\right)$$

$$\cdot\left(\frac{\tau_ld\tilde{L}(t,T_l)}{(1+\tau_l\tilde{L}(t,T_l))^2}+\frac{\tau_l^2d\tilde{L}(t,T_l)d\tilde{L}(t,T_l)}{(1+d\tilde{L}(t,T_l))^3}\right)$$

$$=\prod_{k=q(t)+1}^{i-1}\frac{1}{1+\tau_k\tilde{L}(t,T_k)}$$

$$\cdot\left(\sum_{j=q(t)+1}^{i-1}\frac{-\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}+\frac{\tau_j^2d\tilde{L}(t,T_j)d\tilde{L}(t,T_j)}{(1+d\tilde{L}(t,T_j))^2}+\sum_{\substack{j,l=q(t)+1\\l<j}}^{i-1}\frac{-\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}\cdot\frac{-\tau_ld\tilde{L}(t,T_l)}{1+\tau_l\tilde{L}(t,T_l)}\right)$$

$$=\prod_{k=q(t)+1}^{i-1}\frac{1}{1+\tau_k\tilde{L}(t,T_k)}\sum_{j=q(t)+1}^{i-1}\left(\frac{-\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}+\sum_{l=q(t)+1}^{j}\frac{\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}\cdot\frac{\tau_ld\tilde{L}(t,T_l)}{1+\tau_l\tilde{L}(t,T_l)}\right).$$

Therefore, in order to have a zero drift term, it must holds that

$$\sum_{j=q(t)+1}^{i-1}\text{Drift}_{\mathbb{Q}^S}\left[\frac{-\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}+\sum_{l=q(t)+1}^{j}\frac{\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}\cdot\frac{\tau_ld\tilde{L}(t,T_l)}{1+\tau_l\tilde{L}(t,T_l)}\right]=0, \tag{9.24}$$

and thus $\forall j=0,...,N-1$

$$\text{Drift}_{\mathbb{Q}^S}\left[\frac{-\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}+\sum_{l=q(t)+1}^{j}\frac{\tau_jd\tilde{L}(t,T_j)}{1+\tau_j\tilde{L}(t,T_j)}\cdot\frac{\tau_ld\tilde{L}(t,T_l)}{1+\tau_l\tilde{L}(t,T_l)}\right]=0. \tag{9.25}$$

We can fill in the expression for the j-th LIBOR rate in Equation 9.16 in Equation 9.25 to derive an expression for $\mu^S(t,T_j)$. For this we also need an expression for $d\tilde{L}(t,T_j)d\tilde{L}(t,T_l)$. Fries (2007) shows that

$$d\tilde{L}(t,T_j)d\tilde{L}(t,T_l) = \tilde{L}(t,T_j)\tilde{L}(t,T_l)\sigma(t,T_j)\sigma(t,T_l)\rho_{j,l}(t)dt, \tag{9.26}$$

where $\rho_{j,l}(t)$ is defined in 3.12 and $\sigma(t,\cdot)$ denotes the instantaneous volatility of the j-th and l-th LIBOR rate. Therefore, by filling in the expressions in Equation 9.25 and rearranging terms, we obtain

$$-\mu^S(t,T_j)\frac{\tau_j\tilde{L}(t,T_j)}{(1+\tau_j\tilde{L}(t,T_j))} + \sum_{l=q(t)+1}^{j}\frac{\tau_j\tilde{L}(t,T_j)}{(1+\tau_j\tilde{L}(t,T_j))}\cdot\frac{\tau_l\tilde{L}(t,T_l)}{(1+\tau_l\tilde{L}(t,T_l))}\cdot\sigma(t,T_j)\sigma(t,T_l)\rho_{j,l}(t) = 0. \tag{9.27}$$

Dividing both sides by $\frac{\tau_j\tilde{L}(t,T_j)}{(1+\tau_j\tilde{L}(t,T_j))}$ and moving terms to the right hand side yields the solution for $\mu^S(t,T_j), i.e.,$

$$\mu^S(t,T_j) = \sum_{l=q(t)+1}^{j}\frac{\tau_l\tilde{L}(t,T_l)}{(1+\tau_l\tilde{L}(t,T_l))}\sigma(t,T_j)\sigma(t,T_l)\rho_{j,l}(t). \tag{9.28}$$

9.4 Derivation Black model for caplets

In this section we will derive the Black'76 formula to price caplets. Equation 4.4 yields

$$V^{LMM}_{Capl}(0,T_i,K,\sigma(\cdot,T_i)) = \tau_i P(0,T_i)\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(\tilde{L}(T_i,T_i)-\tilde{K})^+\right]. \tag{9.29}$$

To derive an analytical expression of V^{LMM}_{Capl}, we have to work out the expectation $\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(\tilde{L}(T_i,T_i)-\tilde{K})^+\right]$.

$$\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(\tilde{L}(T_i,T_i)-\tilde{K})^+\right] = \int_0^\infty (l-\tilde{K})^+ p_{\mathbb{Q}_{T_{i+1}}}(l)dl, \tag{9.30}$$

where $p_{\mathbb{Q}_{T_{i+1}}}(\cdot)$ is the marginal distribution of $\tilde{L}(T_i,T_i)$ under $\mathbb{Q}_{T_{i+1}}$. We now have to find an expression for $p_{\mathbb{Q}_{T_{i+1}}}(l)$. The Black'76 Model assumes a constant volatility of the forward rate, thus $\sigma(t,T_i)=\sigma$. Under this assumption and under $\mathbb{Q}_{T_{i+1}}$, the solution of process describing the dynamics of the LIBOR rate stated in Equation 3.16 yields

$$\begin{aligned}\tilde{L}(T_i,T_i) &= \tilde{L}(0,T_i)\cdot\exp(-\frac{1}{2}\sigma^2(T-t)+\sigma(W^{T_{i+1}}(T)-W^{T_{i+1}}(t))), \qquad (9.31)\\ &= \tilde{L}(0,T_i)\cdot\exp(-\frac{1}{2}\sigma^2 T_i+\sigma W^{T_{i+1}}(T_i)).\end{aligned}$$

Here, $W^{T_{i+1}}(T_i)\sim N(0,T_i)$. Now if we define $Y=\frac{W^{T_{i+1}}(T_i)}{\sqrt{T_i}}\sim N(0,1)$, we have that

$$\tilde{L}(T_i,T_i) = \tilde{L}(0,T_i)\cdot\exp\left(-\frac{1}{2}\sigma^2T_i+\sigma\sqrt{T_i}Y\right), \tag{9.32}$$

and

$$\log(\tilde{L}(T_i,T_i)) \sim N\left(\log(\tilde{L}(0,T_i) - \frac{1}{2}\sigma^2 T_i, \sigma^2 T_i\right), \tag{9.33}$$

such that

$$p_{\mathbb{Q}_{T_{i+1}}}(l) = \frac{1}{l\sqrt{2\pi\sigma^2 T_i}}\exp\left(-\frac{1}{2}\frac{(\log(l/\tilde{L}(0,T_i)) + \frac{1}{2}\sigma^2 T_i)^2}{\sigma^2 T_i}\right). \tag{9.34}$$

Given that $l = \tilde{L}(0,T_i)\exp\left(-\frac{1}{2}\sigma^2 T_i + \sigma\sqrt{T_i}y\right)$,

$$dl = l\sigma\sqrt{T_i}dy \tag{9.35}$$

$$\phi(y) = l\sigma\sqrt{T_i}p_{\mathbb{Q}_{T_{i+1}}}(l). \tag{9.36}$$

Moreover, it holds that $\lim_{l\to 0} y = -\infty$ and $\lim_{l\to\infty} y = \infty$. Substituting expressions for l, dl and $p_{\mathbb{Q}_{T_{i+1}}}(l)$, we find that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(\tilde{L}(T_i,T_i) - \tilde{K})^+\right] &= \int_0^\infty (l-\tilde{K})^+ p_{\mathbb{Q}_{T_{i+1}}}(l), \qquad (9.37)\\ &= \int_{-\infty}^\infty \left(\tilde{L}(0,T_i)\exp\left(-\frac{1}{2}\sigma^2 T_i + \sigma\sqrt{T_i}y\right) - \tilde{K}\right)^+ \phi(y)dy.\end{aligned}$$

The option is only exercised if the expression within brackets is positive. We have that this holds for values of y^* greater or equal to

$$y^* = \frac{\log(\tilde{K}/\tilde{L}(0,T_i)) + \frac{1}{2}\sigma^2 T_i}{\sigma\sqrt{T_i}}. \tag{9.38}$$

Thus, the expectation of interest can be denoted as

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(\tilde{L}(T_i,T_i) - \tilde{K})^+\right] &= \int_{y^*}^\infty \left(\tilde{L}(0,T_i)\exp\left(-\frac{1}{2}\sigma^2 T_i + \sigma\sqrt{T_i}y\right) - \tilde{K}\right)\phi(y)dy \qquad (9.39)\\ &= \underbrace{\int_{y^*}^\infty \tilde{L}(0,T_i)\exp\left(-\frac{1}{2}\sigma^2 T_i + \sigma\sqrt{T_i}y\right)\phi(y)dy}_{1} - \underbrace{\int_{y^*}^\infty \tilde{K}\phi(y)dy}_{2}.\end{aligned}$$

Here, if we apply a change of variable with respect to y in the form of $y := y - \sigma\sqrt{T_i}$ the first part of the equation results in the analytical expression

$$\begin{aligned}\int_{y^*}^\infty \tilde{L}(0,T_i)\exp\left(-\frac{1}{2}\sigma^2 T_i + \sigma\sqrt{T_i}y\right)\phi(y)dy &= \tilde{L}(0,T_i)\int_{y^*}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\left(y-\sigma\sqrt{T_i}\right)/2\right)dy, \qquad (9.40)\\ &= \tilde{L}(0,T_i)\int_{y^*-\sigma\sqrt{T_i}}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-y^2/2\right)dy,\\ &= \tilde{L}(0,T_i)\left(1-\Phi\left(y^*-\sigma\sqrt{T_i}\right)\right),\\ &= \tilde{L}(0,T_i)\Phi\left(-y^*+\sigma\sqrt{T_i}\right)\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal CDF, and for the second part of Equation 9.39

$$\int_{y^*}^{\infty} \tilde{K}\phi(y)dy = \tilde{K}\int_{y^*}^{\infty} \phi(y)dy, \tag{9.41}$$
$$= \tilde{K}\left(1-\Phi(y^*)\right) = \tilde{K}\Phi\left(-y^*\right)$$

We can work out the expression for y^* and $-y^* + \sigma\sqrt{T_i}$

$$-y = \frac{\log(\tilde{L}(0,T_i)/\tilde{K}) - \frac{1}{2}\sigma^2 T_i}{\sigma\sqrt{T_i}} = d_2, \tag{9.42}$$
$$-y + \sigma\sqrt{T_i} = \frac{\log(\tilde{L}(0,T_i)/\tilde{K}) + \frac{1}{2}\sigma^2 T_i}{\sigma\sqrt{T_i}} = d_1$$

Combining terms results into an analytical expression for the expectation term of interest.

$$\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(\tilde{L}(T_i,T_i) - \tilde{K}^+\right] = \tilde{L}(0,T_i)\Phi\left(d_1\right) - K\Phi(d_2). \tag{9.43}$$

Therefore, substituting the analytical expression of the expectation in Equation 4.4, we find the Black value of the $i-th$ caplet with constant volatility σ_i

$$V_{Capl}^{LMM}(0,T_i,K,\sigma_i) = \tau_i P(0,T_i)(\tilde{L}(0,T_i)\Phi(d_1) - K\Phi(d_2)]). \tag{9.44}$$

Since caplet prices are quoted using the Black'76 formula, it holds that the Black implied volatility equals the constant caplet volatility, $\sigma_i^{Black} = \sigma_i$. However, in the LIBOR Market Model, we do not assume constant volatilities for the LIBOR rates. The solution of Equation 3.16 now becomes

$$\tilde{L}(t,T_i) = \tilde{L}(0,T_i)e^{-\frac{1}{2}\int_0^t(\sigma(s,T_i))^2 ds + \int_0^t \sigma(s,T_i)dW^{T_{i+1}}(s)}, \quad 0 \leq t \leq T_i, \tag{9.45}$$

as earlier stated in Equation 4.3. Following the lines of the derivation above, under this time-dependent volatility $\sigma(\cdot,T_i)$, the i-th caplet can be priced according to

$$V_{Capl}^{LMM}\left(0,T_i,K,\sigma(\cdot,T_i)\right) = \tau_i P(0,T_i)(\tilde{L}(0,T_i)\Phi(d_1) - \tilde{K}\Phi(d_2)]),$$
$$d_1 = \frac{ln\left(\frac{\tilde{L}(0,T_i)}{\tilde{K}}\right) + \frac{1}{2}\int_0^{T_i}\sigma(s,T_i)^2 ds}{\sqrt{\int_0^{T_i}\sigma(s,T_i)^2 ds}},$$
$$d_2 = d_1 - \sqrt{\int_0^{T_i}\sigma(s,T_i)^2 ds}. \tag{9.46}$$

Comparing Equation 9.42 with Equation 9.46, we see that the the Black volatility and instantaneous volatility are related through

$$\sigma_i^{Black} = \sqrt{\frac{1}{T_i}\int_0^{T_i}\left(\sigma(s,T_i)\right)^2 ds}, \tag{9.47}$$

which concludes the derivation.

9.5 LSMC polynomial basis functions

In the Least Squares Monte Carlo algorithm, we will use the Power function as a polynomial basis function to estimate the continuation value. Other functions include Legendre, Laguerre and Hermite A functions. These polynomials can be expressed in three different ways, i.e., Rodrigues' formula, the explicit expression and the recurrence law. The format used to define a polynomial is decided such that the pricing procedure becomes operationally more practical. Moreno & Navas (2003) state that for the Power and Laguerre polynomials the explicit form is applied, Hermite A is expressed through the recurrence law and the Legendre polynomial is written through Rodrigues' formula. Rodrigues' formula describes orthogonal polynomials. The formula is defined as

$$\xi_n(x) = \frac{1}{a_n g(x)} \frac{\delta^n}{\delta x^n} [\rho(x)(g(x))^n], \tag{9.48}$$

where n denotes the polynomial degree ($n \geq 0$). The coefficients for the above stated polynomial functions in the context of Rodrigues' formula are shown in Table 5.

	a_n	$\rho(x)$	g(x)
Power	$(2n)!/n!$	x^{2n}	1
Legendre	$(-1)^n 2^n n!$	1	$1-x^2$
Laguerre	$n!$	e^{-x}	x
Hermite A	$(-1)^n$	e^{-x^2}	1

Table 5: Coefficient for Rodrigues' formula for Power, Legendre, Laguerre and Hermite A polynomials

The explicit expression for the orthogonal polynomials is defined as

$$\xi_n(x) = d_n \sum_{m=0}^{N} c_m \cdot g_m(x) \tag{9.49}$$

The coefficients for the above stated polynomial functions in the context of Rodrigues' formula are shown in Table 6.

Lastly, the recurrence law can also be used to define the polynomials. The recurrence law is defined as

$$A_{N+1} \cdot \xi_{n+1}(x) = (a_n + b_n x) \cdot \xi_n(x) - a_{n-1} \cdot \xi_{n-1}(x). \tag{9.50}$$

The coefficients for the above stated polynomial functions in the context of Rodrigues' formula are shown in Table 7.

	N	d_n	c_m	$g_m(x)$
Power	0	1	1	x^n
Legendre	n/2	2^{-n}	$(-1)^m \cdot \binom{n}{m} \cdot \binom{2n-2m}{n}$	x^{n-2m}
Laguerre	n	1	$\frac{(-1)^m}{m!} \cdot \binom{n}{n-m}$	x^m
Hermite A	n/2	$n!$	$(-1)^m \cdot \frac{1}{m!(n{\cdot}2m)!}$	$(2x)^{n-2m}$

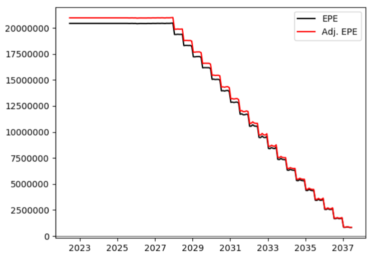
Table 6: Coefficient for the explicit expression for Power, Legendre, Laguerre and Hermite A polynomials

	a_{n+1}	a_n	b_n	a_{n-1}	$\xi_0(x)$	$\xi_1(x)$
Power	1	0	1	0	1	x
Legendre	n+1	0	2n+1	n	1	x
Laguerre	n+1	2n+1	-1	n	1	1-x
Hermite A	1	0	2	2n	1	2x

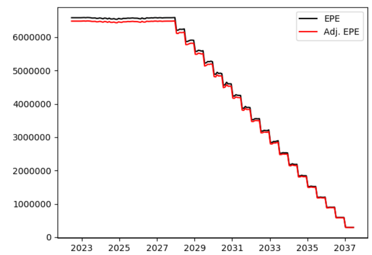
Table 7: Coefficient for the recurrence law for Power, Legendre, Laguerre and Hermite A polynomials

9.6 Exposure profiles single 5Y10Y swaptions

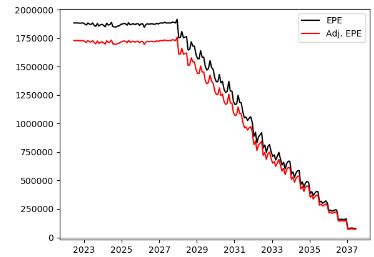
Figure 9 shows the adjusted and unadjusted EPE profiles for the 5Y10Y swaptions.



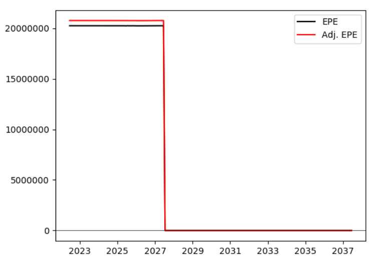
(a) Exposure profiles for a 5Y10Y ITM swap-settled swaption



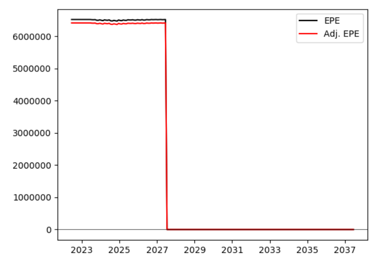
(b) Exposure profiles for a 5Y10Y ATM swap-settled swaption



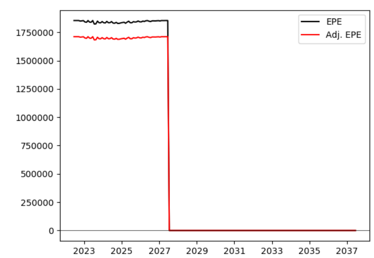
(c) Exposure profiles for a 5Y10Y OTM swap-settled swaption



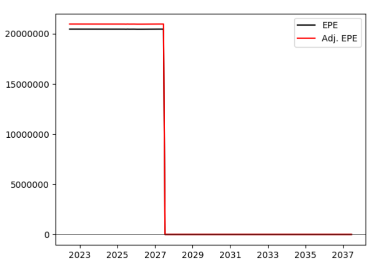
(d) Exposure profiles for a 5Y10Y ITM cash-settled swaption



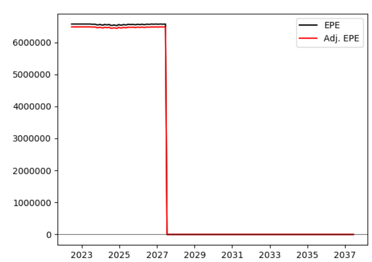
(e) Expected exposure profiles for a 5Y10Y ATM cash-settled swaption



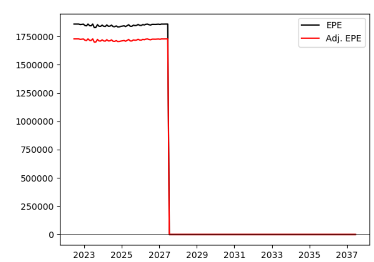
(f) Exposure profiles for a 5Y10Y OTM cash-settled swaption



(g) Exposure profiles for a 5Y10Y ITM CCP-settled swaption



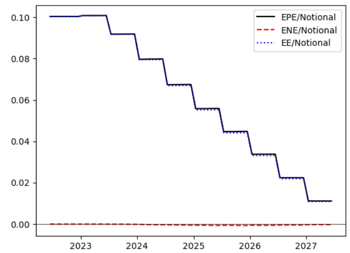
(h) Exposure profiles for a 5Y10Y ATM CCP-settled swaption



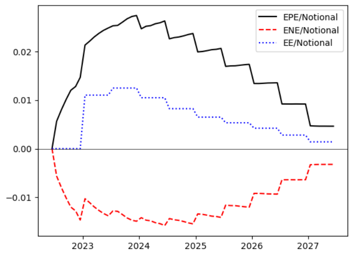
(i) Exposure profiles for a 5Y10Y OTM CCP-settled swaption

Figure 9: The figure above depicts the EPE and adjusted EPE profiles of 5Y10Y swap-, cash- and CCP-settled swaptions for different values of moneyness.

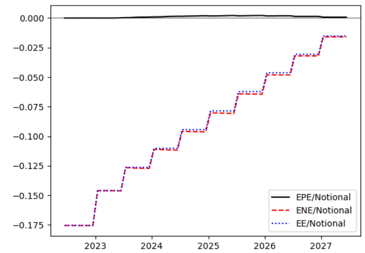
9.7 Exposure profiles of regular interest rate swaps



(a) Expected exposure profiles for an ITM interest rate swap



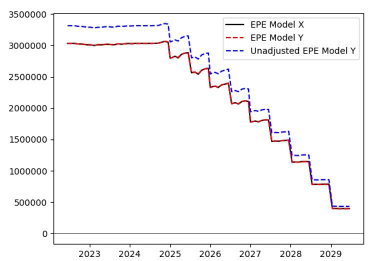
(b) Expected exposure profiles for an ATM interest rate swap



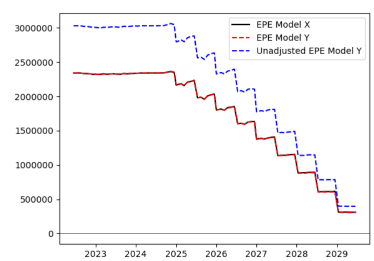
(c) Expected exposure profiles for an OTM interest rate swap

Figure 10: The figure above depicts the EPE, ENE and EE profiles of an interest rate swap for different values of moneyness.

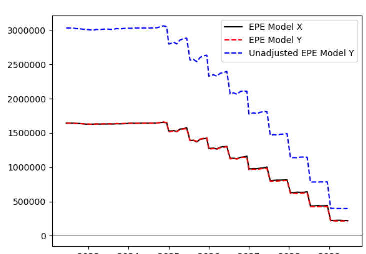
9.8 Experiment: Impact of miscalibration - x < y



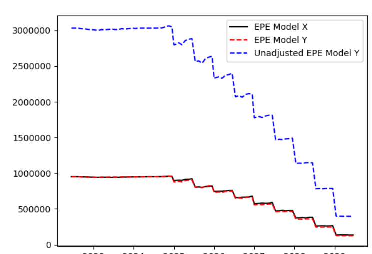
(a) Model X: 20%; Model Y 22%



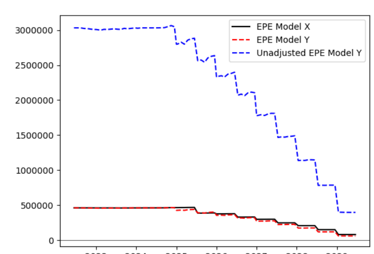
(b) Model X: 15%; Model Y 20%



(c) Model X: 10%; Model Y 20%

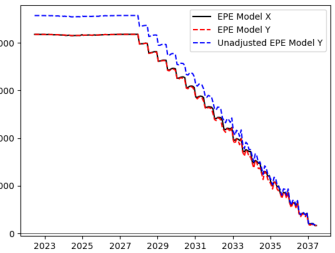


(d) Model X: 5%; Model Y 20%

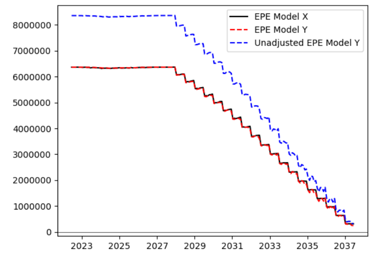


(e) Model X: 1%; Model Y 20%

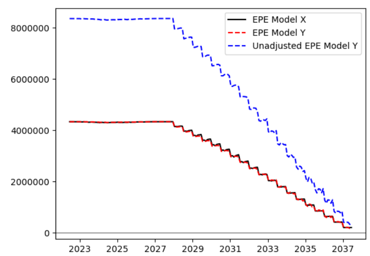
Figure 11: Exposure profiles for a 2Y5Y ATM swap-settled swaption obtained using different flat volatilities of $x\%$ and $y\%$



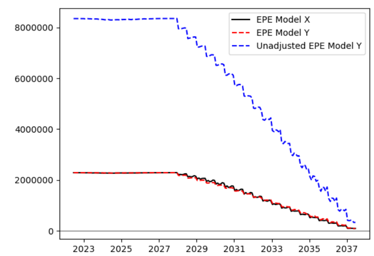
(a) Model X: 20%; Model Y 22%



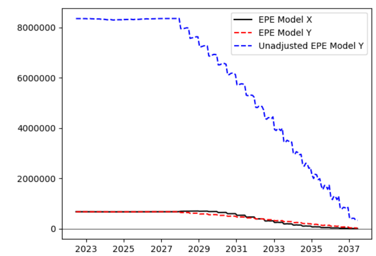
(b) Model X: 15%; Model Y 20%



(c) Model X: 10%; Model Y 20%



(d) Model X: 5%; Model Y 20%



(e) Model X: 1%; Model Y 20%

Figure 12: Exposure profiles for a 5Y10Y ATM swap-settled swaption obtained using different flat volatilities of $x\%$ and $y\%$

9.9 Graphical display of Monte Carlo noise

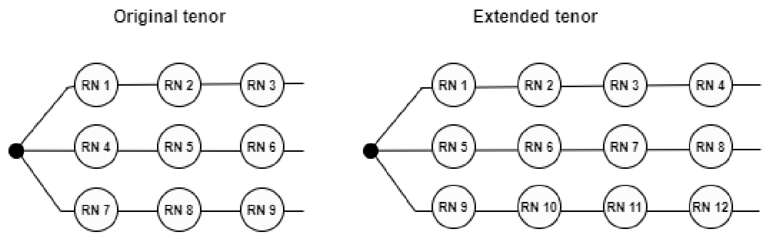


Figure 13: Random numbers (RN) in the simulation for the original tenor are distributed differently than in the simulation for the extended tenor.

9.10 Differences in EPEs original and extended portfolio

Date	Portfolio 1	Portfolio 2	Difference	Difference (%)	Date	Portfolio 1	Portfolio 2	Difference	Difference (%)
06-22	9787739.52	9784530.50	3209.02	0.03%	05-30	5008991.81	4991012.96	17978.85	0.36%
07-22	9787739.52	9784530.50	3209.02	0.03%	06-30	4995013.57	4977518.30	17495.28	0.35%
08-22	9787739.52	9784530.50	3209.02	0.03%	07-30	4635549.01	4617637.60	17911.41	0.39%
09-22	9787739.52	9784530.50	3209.02	0.03%	08-30	4636184.79	4614151.82	22032.98	0.48%
10-22	9786285.29	9782458.58	3826.72	0.04%	09-30	4702202.57	4642661.52	59541.05	1.27%
11-22	9784453.92	9779160.51	5293.41	0.05%	10-30	4662877.82	4644492.10	18385.72	0.39%
12-22	9787536.28	9784505.38	3030.90	0.03%	11-30	4665730.93	4646365.98	19364.95	0.42%
01-23	9769761.99	9767788.99	1973.00	0.02%	12-30	4659302.65	4643315.30	15987.34	0.34%
02-23	9764212.32	9758650.58	5561.74	0.06%	01-31	4284855.23	4264559.82	20295.42	0.47%
03-23	9787628.64	9784530.50	3098.14	0.03%	02-31	4282511.76	4264852.57	17659.18	0.41%
04-23	9747587.55	9741615.34	5972.21	0.06%	03-31	4339585.50	4282020.02	57565.49	1.33%
05-23	9745166.48	9737950.59	7215.89	0.07%	04-31	4309197.48	4286931.30	22266.18	0.52%
06-23	9783792.56	9783239.02	553.53	0.01%	05-31	4310406.15	4288089.55	22316.60	0.52%
07-23	9721620.81	9709755.46	11865.35	0.12%	06-31	4307886.31	4291629.99	16256.32	0.38%
08-23	9714035.77	9702867.62	11168.15	0.11%	07-31	3927322.13	3901168.95	26153.17	0.67%
09-23	9781478.93	9771551.71	9927.22	0.10%	08-31	3924030.79	3905026.98	19003.81	0.48%
10-23	9706583.06	9698156.92	8426.15	0.09%	09-31	3979463.35	3913062.54	66400.81	1.67%
11-23	9696056.05	9683287.96	12768.09	0.13%	10-31	3949766.33	3918223.75	31542.58	0.80%
12-23	9769247.76	9747215.61	22032.15	0.23%	11-31	3944199.41	3925697.60	18501.81	0.47%
01-24	9696647.93	9682103.30	14544.63	0.15%	12-31	3942249.78	3928612.30	13637.48	0.35%
02-24	9676030.88	9670551.57	5479.31	0.06%	01-32	3567710.84	3539678.25	28032.59	0.79%
03-24	9746115.42	9740756.70	5358.72	0.05%	02-32	3567106.65	3549895.02	17211.63	0.48%
04-24	9689868.70	9680137.27	9731.43	0.10%	03-32	3585481.83	3551855.60	33626.23	0.94%
05-24	9680755.75	9675025.99	5729.76	0.06%	04-32	3585442.50	3562494.44	22948.07	0.64%
06-24	9745869.85	9739370.74	6499.12	0.07%	05-32	3588065.83	3566287.46	21778.36	0.61%
07-24	9687216.69	9677748.85	9467.84	0.10%	06-32	3582159.76	3566937.19	15222.57	0.42%
08-24	9681381.92	9682432.97	1051.05	0.01%	07-32	3207411.21	3187085.63	20325.58	0.63%
09-24	9735843.02	9735798.49	44.53	0.00%	08-32	3207667.91	3190616.53	17051.38	0.53%
10-24	9668853.30	9674061.42	5208.11	0.05%	09-32	3230483.06	3205359.99	25123.07	0.78%
11-24	9677097.08	9682354.59	5257.50	0.05%	10-32	3229420.41	3211642.28	17778.13	0.55%
12-24	9698316.35	9698555.15	238.81	0.00%	11-32	3226956.14	3214508.92	12447.21	0.39%
01-25	9375812.64	9362335.78	13476.86	0.14%	12-32	3253351.35	3221588.00	31763.35	0.98%
02-25	9385751.40	9374163.25	11588.15	0.12%	01-33	2871922.56	2857243.54	14679.02	0.51%
03-25	9423607.72	9427366.33	3758.62	0.04%	02-33	2873190.26	2862674.47	10515.78	0.37%
04-25	9393195.00	9382929.46	10265.54	0.11%	03-33	2894574.65	2871520.94	23053.70	0.80%
05-25	9404008.96	9394016.69	9992.27	0.11%	04-33	2888201.09	2876307.80	11893.30	0.41%
06-25	9420841.38	9409927.61	10913.77	0.12%	05-33	2887630.75	2879218.92	8411.83	0.29%
07-25	9078526.98	9056571.94	21955.03	0.24%	06-33	2911599.62	2884994.19	26605.43	0.91%
08-25	9087905.38	9068740.33	19165.05	0.21%	07-33	2532242.17	2521286.78	10955.39	0.43%
09-25	9137500.37	9132558.98	4941.39	0.05%	08-33	2529958.72	2523512.83	6445.90	0.25%

10-25	9117339.58	9103927.49	13412.09	0.15%	09-33	2553405.06	2531882.89	21522.17	0.84%
11-25	9128240.48	9115670.66	12569.82	0.14%	10-33	2542847.15	2536056.96	6790.19	0.27%
12-25	9147930.01	9149638.18	1708.17	0.02%	11-33	2541171.58	2537919.98	3251.60	0.13%
01-26	8838269.92	8836168.66	2101.26	0.02%	12-33	2544369.93	2541417.57	2952.36	0.12%
02-26	8850403.82	8847742.62	2661.20	0.03%	01-34	2190987.94	2183687.17	7300.78	0.33%
03-26	8897317.24	8915567.50	18250.26	0.21%	02-34	2188462.69	2187620.33	842.36	0.04%
04-26	8881918.11	8893054.42	11136.31	0.13%	03-34	2213469.39	2195792.36	17677.03	0.80%
05-26	8899165.11	8901030.10	1864.99	0.02%	04-34	2202836.55	2198716.44	4120.11	0.19%
06-26	8926094.37	8936203.70	10109.32	0.11%	05-34	2201595.88	2199776.35	1819.52	0.08%
07-26	8577983.55	8590616.76	12633.21	0.15%	06-34	2204700.66	2203618.45	1082.21	0.05%
08-26	8595553.17	8599620.15	4066.98	0.05%	07-34	1855057.82	1849101.61	5956.21	0.32%
09-26	8639149.56	8655289.41	16139.84	0.19%	08-34	1851193.53	1851136.65	56.88	0.00%
10-26	8619521.71	8623443.52	3921.81	0.05%	09-34	1873235.06	1856934.28	16300.79	0.87%
11-26	8624825.11	8630112.52	5287.41	0.06%	10-34	1862365.12	1858555.50	3809.63	0.20%
12-26	8664166.44	8679230.46	15064.01	0.17%	11-34	1860591.13	1857753.59	2837.54	0.15%
01-27	8302527.81	8315970.07	13442.26	0.16%	12-34	1861781.91	1859256.02	2525.89	0.14%
02-27	8314087.18	8321157.61	7070.43	0.09%	01-35	1534212.47	1528235.45	5977.02	0.39%
03-27	8348385.16	8358224.14	9838.98	0.12%	02-35	1529187.57	1529986.23	798.66	0.05%
04-27	8331813.16	8338032.40	6219.24	0.07%	03-35	1546734.38	1533146.51	13587.87	0.88%
05-27	8339375.93	8344862.41	5486.48	0.07%	04-35	1536796.67	1535409.92	1386.75	0.09%
06-27	8370786.15	8382506.65	11720.50	0.14%	05-35	1536019.84	1535837.61	182.23	0.01%
07-27	8009909.53	8016562.39	6652.86	0.08%	06-35	1538653.74	1536107.24	2546.51	0.17%
08-27	8007868.78	8015273.59	7404.81	0.09%	07-35	1212606.69	1209208.43	3398.27	0.28%
09-27	8038704.75	8051746.97	13042.22	0.16%	08-35	1211198.54	1208472.54	2726.00	0.23%
10-27	8027389.95	8035653.91	8263.96	0.10%	09-35	1222583.97	1213057.56	9526.41	0.78%
11-27	8029701.18	8041240.88	11539.70	0.14%	10-35	1216222.21	1213752.62	2469.59	0.20%
12-27	8040567.05	8051560.82	10993.76	0.14%	11-35	1213977.84	1214075.92	98.09	0.01%
01-28	7323070.04	7336482.25	13412.21	0.18%	12-35	1224603.38	1209103.77	15499.61	1.27%
02-28	7336510.56	7339244.11	2733.55	0.04%	01-36	904405.86	902954.01	1451.85	0.16%
03-28	7375748.64	7399450.40	23701.76	0.32%	02-36	902929.02	903245.06	316.04	0.04%
04-28	7370245.11	7383008.00	12762.88	0.17%	03-36	909431.45	904809.50	4621.95	0.51%
05-28	7381104.03	7386320.89	5216.86	0.07%	04-36	905389.27	904810.70	578.57	0.06%
06-28	7400806.00	7403741.66	2935.66	0.04%	05-36	905392.38	905317.15	75.23	0.01%
07-28	6662377.64	6673799.27	11421.63	0.17%	06-36	911287.47	900491.53	10795.93	1.18%
08-28	6662546.91	6674859.57	12312.66	0.18%	07-36	594247.80	594765.71	517.92	0.09%
09-28	6715929.87	6721436.02	5506.14	0.08%	08-36	593782.14	594374.77	592.63	0.10%
10-28	6706506.69	6711601.25	5094.56	0.08%	09-36	596962.27	594875.00	2087.27	0.35%
11-28	6710142.08	6718694.35	8552.27	0.13%	10-36	595201.62	595145.29	56.33	0.01%
12-28	6710826.47	6703093.86	7732.60	0.12%	11-36	594398.35	594520.04	121.69	0.02%
01-29	5977956.98	5988816.64	10859.66	0.18%	12-36	596714.44	587646.42	9068.02	1.52%
02-29	5986924.65	5992529.59	5604.93	0.09%	01-37	295673.17	294958.98	714.19	0.24%
03-29	6028319.32	6038359.65	10040.33	0.17%	02-37	294448.31	294493.67	45.36	0.02%
04-29	6015250.22	6022174.09	6923.88	0.12%	03-37	295056.83	294055.27	1001.56	0.34%
05-29	6027971.68	6030351.92	2380.25	0.04%	04-37	294936.31	293438.19	1498.12	0.51%
06-29	6027020.83	6024407.49	2613.34	0.04%	05-37	294315.10	292848.55	1466.56	0.50%
07-29	5296511.85	5306571.66	10059.82	0.19%	06-37	294315.10	288188.65	6126.45	2.08%
08-29	5311128.14	5300404.26	10723.88	0.20%	07-37		0.118232056	0.12	
09-29	5352437.50	5346045.53	6391.96	0.12%	08-37		0.117558464	0.12	
10-29	5341470.20	5337559.24	3910.97	0.07%	09-37		0.119419365	0.12	
11-29	5350247.99	5335930.94	14317.06	0.27%	01-47		0.004368345	0.00	
12-29	5343173.10	5327573.45	15599.65	0.29%	02-47		0.004368345	0.00	
01-30	4960948.03	4957718.83	3229.21	0.07%	03-47		0.004368345	0.00	
02-30	4967168.92	4965386.04	1782.87	0.04%	04-47		0.004368345	0.00	
03-30	5045810.38	4994036.00	51774.38	1.03%	05-47		0.004368345	0.00	
04-30	5010439.34	4988307.86	22131.48	0.44%	06-47		0.004368345	0.00	

Table 8: Table of the EPE profiles presented in Figure 8

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