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Portfolio Optimization with Alternative Assets

Sevilha van der Tak (471931)

Supervised by: Dr. Rasmus Lonn,

Second assessor: Dr. Onno Kleen,

Company supervisor: Daniel Helder,

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Abstract

This paper estimates various portfolios with several asset allocation frameworks such as the Mean-Variance, Black-Litterman, and Risk Parity in combination with different optimization strategies. Additionally, several portfolios including factor returns are constructed. We construct portfolios using traditional assets and alternative assets including a self-constructed Private Equity index and Venture Capital index which are created from the cash flows of the funds, combined with tradable factors. The indices are drawn from a Bayesian Monte Carlo Markov Chain using the Gibbs Sampling and Metropolis-Hastings algorithm. Overall, portfolios including alternative assets outperform traditional portfolios based on several risk-return measures. The Mean-Variance framework is the optimal asset allocation framework for the mixed-asset portfolio. This portfolio shows that it is optimal to allocate a significant amount of weight to Private Equity and Venture Capital. Additionally, adding alternative assets to the factor portfolios ensures robust portfolio performance results for assessing the trade-off between risk-return within the frameworks.

Contents

1 Introduction

Diversification is an important aspect of portfolio optimization. Markowitz (1952) demonstrates that by using analytical Mean-Variance efficient allocation, investing in different assets can reduce volatility while increasing the expected returns of a portfolio. Markowitz (1952) and most papers limit their asset classes to more traditional assets such as stocks and bonds. Nowadays, investing in alternative assets, such as hedge funds, private capital, natural resources, real estate, and infrastructure is becoming increasingly popular. In addition to becoming more appealing to individual investors, alternative investments are also more accessible, allowing a far wider range of investors to invest in alternative assets.

The theory behind the Mean-Variance allocation, the Modern Portfolio Theory (MPT), is often applied to mixed-asset portfolios (Markowitz, 1952). The allocation method weighs the risk-return of the mixed-asset portfolio by searching for the highest return, conditional on the given risk, and the lowest risk conditional on the given return based on historical data (Fabozzi et al., 2002). To perform the strategic asset allocation other financial theories are also commonly used like the Capital Asset Pricing Model (CAPM) of Sharpe (1964). The CAPM uses the expected return and the systemic risk of the market to allocate the assets within the optimal portfolio. Webb et al. (1988) use the Mean-Variance framework as a framework for their research on mixed-asset portfolios, but this framework does not entirely cover the distinctive characteristics of a mixed-asset portfolio.

Alternative assets can enhance the overall risk-return aspect of a portfolio because of their unique characteristics, such as low correlation with traditional assets (Baird, 2013). However, investors should be aware of the risks when including alternative assets in their investment portfolio. An example is liquidity risk. Private Equity (PE) and Venture Capital (VC) are popular alternative assets that can present liquidity issues as they are not traded on exchanges or traded over the counter, compared to more traditional assets. Hence, they cannot be sold at any point in time, but only after the investment horizon. Analyzing PE funds presents several challenges that need to be considered before investing. Compared to traditional stocks and bonds, there are no transaction-based performance measures such as an index. The performance of PE is usually measured by various performance measures, making it difficult to compare PE investments to other traditional assets. Also, alternative assets are often privately traded which leads to more complications for the investors as there is often very little information on the investments available and they often come with additional transaction and advisory costs (Anson, 2002). Ang et al. (2014, 2018) circumvent most of these complications by creating their own PE time series using Bayesian Monte Carlo Markov Chain (MCMC). They use the cash flows of the investments and the factor returns from public capital markets to create the time series. Comparing PE to more conventional assets now becomes more feasible and practical.

The main goal of an investor is to create a more diversified optimal portfolio with strategic asset allocation. Therefore, this research explores the potential broadening of the investors' investment horizon by expanding the asset portfolio and adding alternative assets. This is followed by comparing the performance of the mixed-asset short-term portfolios for different strategic asset allocation techniques. Our research will answer the following question: How does the addition of alternative assets to a traditional portfolio affect the overall risk-return of the portfolio and what are the best techniques for allocating alternative assets? To answer the research question several subquestions are also stated:

1. For a diversified portfolio including alternative assets, how is the optimal portfolio constructed based on risk-return?

2. What is the impact of the portfolio performance based on risk-return for the different frameworks when adding alternative assets to the investment portfolio?

3. How do the different portfolios including alternative assets perform compared to portfolios including factor returns in terms of risk-return measures?

In this research, multiple short-term mixed-asset portfolios are created using different strategic asset allocation methods. We construct mixed asset portfolios using traditional assets: stocks and bonds and alternative assets: hedge funds and commodities. The indices are obtained from the Bloomberg Terminal. Furthermore, we also include PE and VC as additional alternative asset classes. PE and VC do not have an index such as most other assets. Consequently, we will delve more in-depth into the addition of PE and VC by creating PE and VC indices from the quarterly cash flow data of different PE and VC funds obtained from Preqin. The cashflows are combined with quarterly Fama French data and Pastor-Stambaugh factor data. The factor data is obtained from the Kenneth R. French Library. The indices are generated from a Bayesian MCMC using the Gibbs Sampling and Metropolis-Hastings algorithms. The sample period for the full data set ranges from 01-01-2000 until 31-12-2020.

For strategic asset allocation, this paper looks at various portfolio optimization frameworks and methods that focus on the risk-return trade-off. We construct the following three frameworks: Mean-Variance, Black-Litterman, and Risk Parity to create different optimal portfolios. For our portfolios, we focus on three optimization strategies representing specific risk preferences: Global Minimum Variance (GMV), Tangency (TAN), and Maximum Return (MR). The optimal portfolios, including alternative assets composited with the different frameworks, are compared to traditional portfolios to evaluate the performance.

Furthermore, we compose portfolios including factor returns to perform a robustness check of the different frameworks and optimization strategies. We use the Fama French factor data to construct factor portfolios. The factor returns are implemented in the best-performing optimization frameworks based on risk-return. We expand the factor portfolios by adding alternative assets, creating our alternative factor portfolios. From this, we can evaluate the addition of alternative assets to the factor portfolio. The performance of all the portfolios will be measured by different performance measures: Value at Risk (VaR), Expected Shortfall (ES), Maximum Drawdown, Sharpe Ratio, and the expected returns. The performance measures of the different portfolios are then evaluated and compared to one another.

This paper adds to the existing literature by including more than one alternative asset class in the portfolio optimization problem. Most research is focused on adding only one specific alternative investment to their asset class. This paper will study how adding PE, VC, hedge funds, and commodities to the diversified portfolio affects the risk-return performance. Because a more balanced portfolio is an important aspect for many investors, from institutional banks to individual investors. Furthermore, this study incorporates a self-constructed PE index and VC index, based on the methodology of Ang et al. (2014, 2018), into different optimization frameworks. This allows us to compare PE and VC to other assets in the mixed-asset portfolio, which provides a unique insight into the performance of these alternative assets.

Based on the research, we can state that for portfolios including alternative assets, the Mean-Variance optimal portfolios outperform the Black-Litterman and Risk Parity portfolios based on risk-return. For the optimal portfolio, a significant amount of weight is allocated to the self-constructed PE index and VC index. This indicates that adding these assets to the optimal portfolio adds value. Furthermore, the portfolios including alternative assets outperform the traditional portfolios. The diversified portfolios including alternative assets result in much higher returns and lower drawdowns in comparison to traditional portfolios. Lastly, the performance of the factor portfolios and the alternative factor portfolios are comparable to the performance of the traditional portfolios and portfolios including alternative assets, validating the robustness of adding alternative assets to the investment portfolio.

The organization of this paper is as follows. In section 2, a literature overview is presented. Section 3 introduces the methodology for creating the PE and VC indices, the portfolio allocation frameworks, the optimization strategies, the factor portfolios, and the performance measures. In section 4, the data is presented. We continue with section 5 which displays and evaluates the results of our methods. Lastly, we state the conclusion of our research in section 6.

2 Literature

Multiple studies have demonstrated the benefits of incorporating alternative investments into an investment portfolio. In Ziobrowski and Ziobrowski (1997) the goal was to reevaluate the benefits and renewal methods for diversification of real estate in mixed-asset portfolios. They conclude that adding real estate to the portfolio increases the returns regardless of the investor risk preference when the real estate returns are smoothed. Schweizer (2008) also shows promising results regarding the optimal weights of the portfolio including alternative assets in case a more flexible model that takes non-normality into account is used. Therefore, choosing a model that fits the characteristics of alternative assets is of great importance when considering adding alternative assets to the investment portfolio. On the other hand, the characteristics of the alternative assets also play a crucial role in the strategic asset allocation in the portfolio. For example, Fung and Hsieh (1997) state that the returns indices of hedge funds are not normally distributed and show higher than normal skewness and kurtosis. Nevertheless, hedge funds do have a low correlation with long-only portfolios, maintaining their diversification motive to invest in. Therefore, Schweizer (2008) states that all asset classes work together to achieve the best risk-return profile for a portfolio.

PE is an alternative asset class that has become more attractive to invest in. Cummming et al. (2013) claim that PE is based on three indices, namely listed PE, transaction-based PE, and appraisal value-based PE. Listed PE indices are based on PE funds that are listed on the stock exchange. Transaction-based PE indices are created with the cumulative cash flows of the firms in the fund portfolio of the non-listed PE funds. Appraisal value-based indices are created with a combination of the cumulative cash flows and the net asset value. This paper mainly focuses on transaction-based PE indices. In the paper of Cumming et al. (2013), they find that PE indices are not suitable for portfolio optimization, because they do not embrace all risk-return characteristics for the portfolio. This is because the results show that listed PE indices overestimate volatility which leads to sub-optimal allocation percentage to PE. Also, due to the delay of the data for PE transaction and appraisal value-based PE indices the calculated weights are often misallocated, especially during financial crises. On account of these implications, they create a benchmark to capture these characteristics more accurately. Their paper is solely based on PE allocation. A possible extension of this paper could entail the inclusion of additional alternative assets to this benchmark.

VC, alongside PE, is a crucial element to consider when adding alternative investments to a portfolio. Unlike the more mature PE landscape, VC demands a more nuanced evaluation of factors such as technological trends and unpredictability. The early-stage, high-growth enterprises require a unique perspective in order to successfully implement them as alternative investments. Recent research has suggested that adding VC to a traditional portfolio has a significant favorable effect on the risk-return (Moretta, 2021).

The portfolio theory by Markowitz (1952) has been used for a long time by investors to manage their portfolios and choose their assets. MPT incorporates the findings of Markowitz (1952, 1959), Tobin (1958), and Sharpe (1964). Early papers such as Merton (1971) use the MPT as a framework for their asset allocation model. Papers nowadays such as Dimmock et al. (2019) also use the MPT framework but incorporate alternative assets. Despite being commonly used throughout time, the MPT has limits. Several papers call the MPT in question. For example, the research of Blom and Warglou (2016) states a shortcoming of the MPT, namely using the variance as a risk measure where the dependence is given by a linear correlation. In their paper, they use copulas as an instrument to measure the dependence. A copula function connects marginals to form a joint distribution of variables, from which the dependence can be interpreted. Traditional MPT portfolios are then compared to copula-based portfolios.

Curtis (2004) also emphasizes the shortage of practical application of the theoretical method. The paper states that the MPT requires a lot of predetermined conditions such as continuous prices, free markets, and a specific rational outlook of the investor on the market and their portfolio. He claims that the outlook of the investor must always be to maximize its capital. However, investors always hold a specific risk tolerance. Creating a diversified portfolio with for example alternative assets is a way of exploring investment opportunities while adhering to the risk tolerance and the goals of the investor.

Black-Litterman takes one of the limits of MPT into consideration. It extends the MPT framework by incorporating the subjective view of the investor. Black and Litterman (1992) state in their paper: "Our approach allows us to generate optimal portfolios that start at a set of neutral weights and then tilt in the direction of the investor's views." The Black-Litterman framework uses the CAPM equilibrium distribution as a base and then incorporates the subjective behavior of the investor retrieved from additional market information (He and Litterman, 2002). According to Da Silva et al. (2009), the most important characteristic of the Black-Litterman framework is the application of a Bayesian approach by combining active and equilibrium investment views. The expected returns are random variables in the Bayesian approach for which we can only derive the probability distribution. This approach has demonstrated the ability to create portfolios that are more resilient and less affected by inaccuracies in the expected excess return data.

An alternative method that proves insightful for this paper due to the focus on risk-return, is the Risk Parity. This method allocates its assets based on the risk the asset brings to the portfolio. Chavez et al. (2011) compare Risk Parity portfolios and other different strategic asset allocation methods. They conclude that Risk Parity portfolios outperform minimum-variance and Mean-Variance portfolios, however, investors need to be critical of the different asset classes they select. Volatility can be used as a risk measure for asset allocation based on Risk Parity. Other measures such as VaR and Conditional Value at Risk (CVaR) are also often utilized. Boudt et al. (2013) measure the risk of their portfolios using the CVaR risk measure to directly amend downside risk and to reduce the tail risk compared to using the VaR measure.

Instead of relying on the conventional methods of asset allocation, Ang et al. (2014) propose an approach that determines the allocation and risk decisions based on a bundle of risk factors rather than solely examining the attributes of particular asset classes. This is because, despite a portfolio being well-diversified across different asset classes, the portfolio may still be largely exposed to the same risk factors that apply to multiple asset classes. Consequently, factor investing aims to expand the range of asset classes to create a more diversified portfolio that identifies and captures systematic risk factors. Previous empirical studies have shown that several types of portfolios outperform the market portfolio. Fama and French (1992) and Carhart (1997) demonstrate that you can capture the impact of a risk factor or strategy by building a factor portfolio. This, in turn, creates additional risk-return management opportunities for the investor.

According to Bender et. al (2013), factors can be divided into three primary categories: macroeconomic, statistical, and fundamental. Macroeconomic factors entail for example inflation or the yield curve. Statistical factors are measures based on historical data for assets, which can be derived using principal components analysis (PCA) for example. Fundamental factors are more commonly used and are based on the characteristics and technical indicators of the assets. One of the most popular, fundamental multi-factor models is created by Fama and French (1992, 1993). Initially, the factor model consisted of three factors: the market factor, which is based on the

CAPM model; the size factor and the value factor. The three-factor model is extended by Fama and French (2015) to a five-factor model by adding the investment and profitability factors. Fama and French (2015) find that the five-factor model outperforms the three-factor model in explaining the cross-section of stock returns. Other factors are often considered as well, such as the momentum factor of Carhart (1997) which extends the CAPM model, and the Stambaugh illiquidity factor which focuses on the liquidity of an asset (Pástor and Stambaugh, 2003)

The concept of ARP extends the factor investing approach. Roncalli (2017) explains that factor investing is only used for equity risk factors, whereas ARP broadens this approach by incorporating alternative assets. With the ARP asset allocation strategy, portfolio returns are augmented by diversifying the portfolio with alternative assets and different factors. The risk premium is inserted as compensation for the additional risk of the alternative assets as this cannot be hedged. Roncalli (2017) distinguishes two types of strategies for ARP: pure risk premia and market anomalies. He states that market anomalies are the factors that are based on the linkage between past performance and systematic risk, for example, the momentum factor.

Research from Mainik et al. (2015) points out the practicality of comparing different optimization techniques, such as the Extreme Risk Index (ERI), to determine the optimal portfolio weights. The ERI, as performed by Mainik et al. (2015), uses Extreme Value Theory (EVT) to minimize significant losses in the portfolio. Their findings demonstrate that when using the ERI technique, portfolios tend to outperform the equally weighted and minimum variance portfolio, especially in the presence of large tails.

Another optimization tool is a copula. A copula is a function that connects marginals to form a joint distribution of variables. This allows us to interpret the dependence of the variables and therefore the risks. Subsequently, when the correlation between assets is known, there is additional information for the portfolio weight allocation. Huang and Hsu (2015) use a copula simulation to predict portfolio loss distributions and to derive optimal asset allocation weights by minimizing portfolios based on the CVaRs.

3 Methodology

This research is conducted in the following manner. Initially, we create PE and VC indices using different factor models. Subsequently, we explore different mathematical frameworks with optimization strategies for portfolios including traditional and alternative assets where we add the created PE and VC index to our portfolio. Thirdly, we construct portfolios utilizing factor returns with distinct equity-based factors to perform a robustness check. In the fourth phase, we expand these factor portfolios by adding alternative assets to the factor portfolio. Lastly, we conduct a comparative analysis of the portfolios constructed with different frameworks to the factor portfolios using different performance measures based on risk-return.

3.1 Private Equity and Venture Capital Indices

There are many alternative assets to invest in, from commodities like gold and oil to hedge funds. Because there are indices for these alternative assets, they can be more easily compared to traditional assets. This is not the case for PE and VC as there are no indices available for them. To make PE and VC investments more comparable to different assets, we adopt the methodology of Ang et al. (2014, 2018) to estimate PE and VC indices based on the cash flows of the investors. The cash flows consist of the investments and the distributions. The investments (I_t) are the cash that flows out for the investor and the distributions (D_t) are the cash inflow. Ang et al. (2014, 2018) start with the following construction: A PE fund has κ investments of amount I_l at time t_l, where $l \in 1,...,\kappa$. The investments pay dividend D_{Tj} at time T_l, where $j \in 1,...,\kappa$. Subsequently, we obtain for each investment l:

$$D_{T,l} = I_l R_{i,t+1,l} ... R_{i,T,l}, \tag{1}$$

where $R_{i,t}$ represents the discount rate of PE at time t or the realized return which is one plus the rate of return of the investment during time period T. To derive the model we look at the derivation of $R_{i,t}$:

$$R_{i,t} = R^e_{i,t} + r_{f_t}. \tag{2}$$

$R^e_{i,t}$ is the excess return compared to the risk-free rate r_{f_t}. To derive the PE risk premium we use the following model:

$$R^e_{i,t} = \alpha + \beta' F_t + f_t, \tag{3}$$

where

$$f_t = \psi f_{t-1} + \sigma_f \epsilon_t, \tag{4}$$

and $\epsilon_t \sim$ i.i.d $N(0, 1)$. The model in equation 3 encompasses the systematic risks of the different types of investments of the PE funds by using the tradable factors from the public market, denoted by $F_t = [F_{1,t}, ...F_{r,t}]$. Different factor models are used for the tradable factors. β is the corresponding common factor loading. α is the average of PE returns in excess of its systematic component of the PE return. f_t is an asset class-specific latent factor that can be derived as an AR(1) model.

To simulate the model we will use a Bayesian MCMC method. If the model is accurate, the cash flows meet the following Net Present Value (NPV) requirement:

$$E[\sum_t I_{lt}\delta_{it}] = E[\sum_t D_{lt}], \tag{5}$$

where δ_{lt} is the cumulative discount rate derived from

$$\delta_{lt} = \delta_{l,t-1}(1 - R_{i,t})^{-1}. \tag{6}$$

The proportion of the present value of investments to the present value of distributions follows a log-normal distribution by assumption:

$$ln \frac{E[\sum_t I_{lt}\delta_{lt}]}{E[\sum_t D_{lt}\delta_{lt}]} \quad \sim N(-\frac{1}{2}\sigma^2, \sigma^2). \tag{7}$$

μ is set to $-\frac{1}{2}\sigma^2$ because we assume the mean is equal to zero for the log-likelihood ratio. Using the likelihood function in equation 7 and state equation 3, it is possible to estimate $\theta = (\alpha,\ \beta,\ \psi,\ \sigma_r,\ \sigma)$. To estimate θ and ultimately $R^e_{i,t}$ and $R_{i,t}$, we perform the Bayesian MCMC as below.

3.1.1 Bayesian Markov Chain Monte Carlo

For the Bayesian MCMC, we also adopt the simulation method of Ang et al. (2014, 2018). The Bayesian MCMC is implemented as outlined. The PE returns are retrieved by estimating the parameters in the state equation via different simulations. The state equation and the observation equation are set up by combining equations 3 and 4 resulting in the following setup:

$$R^e_{i,t} = (1-\psi)\alpha + \psi R^e_{i,t-1} + \beta'(F_t - \psi F_{t-1}) + \sigma_r \epsilon_t, \tag{8}$$

$$f_t = R^e_{i,t} - (\alpha + \beta' F_t). \tag{9}$$

To retrieve the returns, we need to estimate the following set of parameters $\theta = (\alpha, \beta, \psi, \sigma_r, \sigma)$. Then $\theta-$ is the parameter set without the single parameter that is being estimated. Furthermore, we have a set of time-dependent parameters: investments, distributions, and tradable factors, from which we obtained the data: $Y_t = ({I^h}_{i,t}$, $D^h_{i,t}$, $F_t)$, h is the amount of funds included in our model. The total set of parameters is estimated during four different states. In each state, one or more parameters are estimated by calculating the posterior distribution of the parameter. The values of the parameters are then drawn from the posterior distribution. To obtain the posterior distribution, we use the likelihood function and the prior of the parameter while we condition on the other given parameters within the set θ. Accordingly, the four states can be summarized as:

- State 1: Estimate the PE returns by the following posterior distribution: $p(R^e_{i,t}|\theta, Y)$.
- State 2: Estimate the parameters β, α, and ψ by the following posterior distribution: $p(\beta, \alpha, \psi|\theta-, R^e_{i,t}, Y)$.
- State 3: Estimate the standard deviation of the PE returns by the following posterior distribution: $p(\sigma_r|\theta-, R^e_{i,t}, Y)$.
- State 4: Estimate the standard deviation of the likelihood by the following posterior distribution: $p(\sigma|\theta-, R_{i,t}, Y)$.

To estimate the model and ultimately the index returns, two types of simulations are used. We employ the MCMC Gibbs sampling algorithm, this method iterates over the different states and draws samples from the posterior distributions. The second type is the Metropolis-Hastings algorithm, which only updates the first state using a proposal value for the new state and the acceptance ratio. A burn-in period of 2,500 draws and a sample of 10,000 draws are implemented for the simulation to draw the parameter from the posterior distributions. The priors are acquired from the literature from Ang et al. (2014, 2018). The VC index is generated similarly to the PE index. The simulation is as follows.

Starting with initial values θ_0, we iterate over the burn-in period and the sample period. The calculation steps per state per iteration for the simulation are explained below:

- *State 1: Estimate the PE returns by the following posterior distribution: $p(R^e_{i,t}|\theta, Y)$.*
  To estimate the parameter $R^e_{i,t}$, the following derived joint posterior distribution is employed:
  $$p(R^e_{i,t}|\theta, Y) \propto p(Y|R^e_{i,t}, \theta)p(R^e_{i,t}|R^e_{i,t-1}, \theta, Y)p(R^e_{i,t+1}|R^e_{i,t}, \theta, Y)p(R^e_{i,t}). \tag{10}$$
  The posterior distribution can be divided into three likelihood functions and the prior of $R^e_{i,t}$. The first likelihood function $p(Y|R^e_{i,t},\ \theta)$ is equal to the likelihood function in equation 7. The second and third likelihood function $p(R^e_{i,t}|R^e_{i,t-1}, \theta, Y)$ and $p(R_{i,t+1}|R^e_{i,t}, \theta, Y)$ can be combined into the following distribution:
  $$p(R^e_{i,t}|R^e_{i,t}, \theta, Y) \propto p(Y|R^e_{i,t}, \theta)\text{exp}(-\frac{(R^e_{i,t} - \mu)^2}{2\sigma_r^2}(1 + \psi^2))p(R^e_{i,t}). \tag{11}$$
  Hence, a suggestion for $R^e_{i,t}$ is drawn from the posterior distribution. Moreover, we employ the Metropolis-Hasting algorithm in this state. This means that after drawing from the joint posterior distribution the proposal density is estimated. The proposal density using μ_t is as follows,
  $$q(R^e_{i,t}) \propto \exp(-\frac{(R^e_{i,t} - \mu_t)^2}{2\sigma_r^2}(1 + \psi^2)), \tag{12}$$
  $$\mu_t = \frac{\psi(R^e_{i,t-1} + R^e_{i,t+1} + (1-\psi)\alpha + \beta'((1+\psi^2)F_t - \psi(F_{t+1} + F_{t-1}))}{1 + \psi^2}. \tag{13}$$
  The proposal density proposes a new value for $R^e_{i,t}$, the proposed value will then be incorporated provided the acceptance probability is attained. The acceptance probability for the $(k+1)^{th}$ iteration is
  $$\min(\frac{p(Y|R^{e,k+1}_{i,t}, R^e_{i,t}, \theta)}{p(Y|R^{e,k}_{i,t}, R^e_{i,t}, \theta)}, 1). \tag{14}$$
  If the acceptance probability is met, $R^e_{i,t}$ will be updated at each iteration and new parameters in θ can be drawn from the updated $R^e_{i,t}$.

- *State 2: Estimate the parameters β, α, and ψ by the following posterior distribution: $p(\beta, \alpha, \psi|\theta-, R^e_{i,t}, Y)$.*

  The posterior distribution can be derived by multiplying the likelihood function with the priors of the parameters. In this state, a normal conjugate prior is employed and the posterior of β can be stated as follows:

$$p(\beta|\theta-, R^e_{i,t}, Y) \propto p(Y|\beta, \theta-, R^e_{i,t}|\beta, \theta-)p(\beta),$$

$$p(\beta, R^e_{i,t}, Y) \propto p(R^e_{i,t})|\beta, \theta-)p(\beta). \tag{15}$$

  We estimate the posterior distribution per parameter. Consequently, β can be drawn from the posterior distribution. The parameters α and ψ are estimated in a similar manner.

- *State 3: Estimate the standard deviation of the PE returns by the following posterior distribution: $p(\sigma^2_r|\theta-, R^e_{i,t}, Y)$.*

  To estimate σ_r, a conjugate inverse Gamma draw is employed. We now have the following prior:

$$p(\sigma^2_r) \sim \mathrm{IG}(\frac{a_0}{2}, \frac{b_0}{2})1[10^{-6}, 1]. \tag{16}$$

  Following the literature, a_0=2 and b_0=10^{-6}. The posterior distribution of σ_r is as follows:

$$p(\sigma^2_r|\theta-, Y) \sim IG(\frac{A_0}{2}, \frac{B_0}{2})1[10^{-6}, 1], \tag{17}$$

  where $1[10^{-6}, 1]$ is the indicator function of the range, A_0 = a_0 + T - 1, B_0 = b_0 + u and u can be calculated by:

$$u = \Sigma(R^e_{i,t} - (1-\psi)\alpha - \psi R^e_{i,t} - \beta'(F_t - F_{t-1}))^2. \tag{18}$$

- *State 4: Estimate the standard deviation of the likelihood by the following posterior distribution: $p(\sigma^2|\theta-, R^e_{i,t}, Y)$.*

  The parameter σ is also estimated from the conjugate inverse Gamme. To estimate σ, we can use the same prior and posterior distribution as σ_r.

$$p(\sigma^2|\theta-, Y) \sim IG(\frac{A_0}{2}, \frac{B_0}{2})[10^{-6}, 1], \tag{19}$$

  However, different values are applied for the probability parameters: a_0 = 10^{-6} and b_0 = 10^{-6} for the prior, and A_0 = a_0 + h and B_0 = b_0 + s for the posterior distribution, where s is derived from the following equation: $s = \Sigma_h ln\frac{PV(D^h)}{PV(I^h)}^2$.

3.2 Portfolio allocation

This section looks at several portfolio optimization frameworks that focus on the trade-off between risk-return and their constraints for creating different types of portfolios. The portfolios are constructed over time with weights re-balancing every quarter using an expanding rolling window during an in-sample period ranging from 01-01-2000 until 29-06-2012 and an out-of-sample period ranging from 28-09-2012 until 31-12-2020.

3.2.1 Modern Portfolio Theory

MPT provides the Mean-Variance framework for portfolio allocation and optimization based on the predicted performance of investments and the investor's risk behavior (Markowitz, 1952). For this theory, conventional Mean-Variance optimization is used as a mathematical framework for creating portfolios. The purpose of this framework is to select the set of weights that maximize the expected returns while minimizing the risks. The framework of the Mean-Variance is as follows. Consider N assets in the portfolio with R_i the return of each asset. The expected return of the portfolio $E(R_p)$ and the variance σ_p^2 are as follows:

$$E(R_p) = \sum_{i}^{n} w_i E(R_i) = \mu, \tag{20}$$

$$\sigma_p = \sqrt{\omega' \Sigma \omega}, \tag{21}$$

where ω represents the portfolio weights and Σ the variance-covariance matrix. The objective function of the Mean-Variance maximizes the expected returns of the portfolio while simultaneously minimizing the risk. Hence, the Mean-Variance model is structured as follows:

$$\max \quad \mu' \omega - \frac{1}{2} \omega' \Sigma \omega. \tag{22}$$

The objective equation subjects to the following constraints:

$$\sum_{i=1}^{N} \omega_i = 1, \tag{23}$$

$$w_i \geq 0 \qquad \forall \quad i, \tag{24}$$

$$(\omega_{Gold} + \omega_{Oil}) \leq Max(\omega_{Commodites}), \tag{25}$$

$$\omega_{Hedgefunds} \leq Max(\omega_{HedgeFunds}), \tag{26}$$

$$(\omega_{PE} + \omega_{VC}) \leq Max(\omega_{PE} + \omega_{VC}), \tag{27}$$

$$\omega_{VC} \leq Max(\omega_{VC}). \tag{28}$$

The Mean-Variance framework gives us the optimal weight sets for the given portfolio and constraints. The first constraint ensures that the weights sum up to one, which means that the portfolio is fully invested. The second constraint restraints the portfolio from short positions. In order to avoid extreme allocations we constraint the weights on the alternative assets. The last four constraints relate to the maximum amount that can be invested in alternative assets. The constraint in equation 25 states a cap on the amount of investment in commodities. Daskalaki and Skiadopoulos (2011) explore the possibility of adding commodities to their portfolios. They state to allocate a maximum of 5% to 20% to commodities. As the focus of our research mainly lies on the addition of the PE index and VC index, our portfolio is restricted by setting a maximum allocation constraint of 5% to the weights of commodities. In Kooli and Selam (2010) they add hedge funds as an additional asset to their Black-Litterman model, they set a maximum allocation constraint of 5% to the weights of hedge funds. We follow their paper and restrict hedge funds with 5% as stated in equation 26. The weight constraints on both the PE index and VC index are stated in equations 27 and 28. Following Brown et al. (2021), we set a maximum allocation constraint of 20% on PE and VC combined. Consequently, we set a tighter constraint of 10% on VC. The tighter constraint for VC is attributable to the heightened prevalence of survivorship bias. Aspects such as the high failure rate of start-ups themselves, higher risk profile compared to other assets, and the differing investment horizons explain this bias.

3.2.2 Black-Litterman

By resolving the issues of the Mean-Variance optimization by accounting for uncertainty, the Black-Litterman model seeks to improve asset allocation decisions (Bessler et al., 2017). Their papers state that Black-Litterman: "reduces the sensitivity of portfolio weights", by using implied returns (Π) from the market and subjective returns implied by the investor. The subjective returns consist of the views of the investor on the market and can be seen as additional information. A view is a statement on the performance of an asset. We follow the paper of He and Litterman (2002) for the

Black-Litterman model. We assume N assets that have normally distributed returns $R_i \sim N(\mu, \Sigma)$. The framework of the Black-Litterman approach is as follows:

$$\max \quad \omega'\Pi - \frac{\delta}{2}\omega'\Sigma\omega, \tag{29}$$

where ω is the weights assigned to the portfolio, Σ is the variance-covariance matrix and δ represents the risk behavior[1] of the investor. The weighted market capitalization[2] (ω_{mc}) represents the market size of an asset, weighted across the entire portfolio. The implied returns are calculated from the weighted market capitalization of the different assets, the risk behavior of an investor, and the average risk-free rate over time:

$$\Pi = \delta\omega'_{mc}\Sigma\omega_{mc} + \bar{r}_{f_t}. \tag{30}$$

To combine the implied returns of the market and the view of the investor in the implied excess returns based on the CAPM, we follow the Bayesian approach for which the expected return is equal to:

$$\mu = \Pi + \epsilon^{(e)} \qquad \epsilon^e \sim N(0, \tau\Sigma). \tag{31}$$

$e^{(e)}$ the random error term and τ represents the uncertainty of the CAPM model which is set to 0.05.

Furthermore, V is the total number of views of the investor. Q is a Vx1 vector including the views of the investor and P is a VxN matrix including the weights of the views in the portfolio. Thus, we have:

$$Q' = (q_1, q_2 ... q_v), \tag{32}$$

[1]Following He and Black-Litterman (2002), δ is set to 2.5 which represents the world average risk aversion.

[2]As some alternative assets cannot be encompassed into a market capitalization value, we use alternative market capitalization values based on market and statistics reports of the specific assets in 2020. The market capitalization of the assets are as follows: stocks: 109.21 (World Federation of Exchanges, 2021), bonds: 128.30 (Bond Market Size ICMA, 2020), hedge funds: 3.83 (Assets Managed by Hedge Funds Globally 2023 | Statista, 2023), gold: 12.10 (Market Cap of Gold (Precious Metal), n.d.), oil: 0.34 (Oil Market Report, 2020), PE: 0.17 (TalkingPoints the S&P Listed Private Equity Index, 2019).

$$P = \begin{bmatrix} p1,1 & p1,2 & .. & p1,N \\ p2,1 & p2,2 & .. & p2,N \\ .. & .. & .. & .. \\ pV,1 & pV,2 & .. & pV,N \end{bmatrix}, \tag{33}$$

$$P\mu = Q + \epsilon^{(v)} \qquad \epsilon^{(v)} \sim N(0, \Omega). \tag{34}$$

The views of the investor (the subjective returns) are then combined with the implied returns to calculate the Black-Litterman model. For the views, we employed relative views based on the average performance of the out-of-sample performance. The relative views are as follows: stocks will increase by 2.76%, bonds will increase by 0.84%, PE will increase by 11.70% and VC will decrease by 0.11%. We do not add views on hedge funds and commodities as our focus is mainly on adding PE and VC to the investment portfolio. The Black-Litterman model is now as follows,

$$\bar{\mu} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q], \tag{35}$$

and

$$\bar{M}^{-1} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}, \tag{36}$$

where Ω is the diagonal variance-covariance matrix of the two error terms $\epsilon^{(e)}$, and $\epsilon^{(\nu)}$. Consequently, $R \sim N(\bar{\mu}, \bar{\Sigma})$, where $\bar{\Sigma} = \Sigma + \bar{M}^{-1}$. The Black-Litterman framework then uses the same estimation method and constraints as the Mean-Variance framework to estimate the optimal weights with the new mean and variance-covariance matrix.

3.2.3 Risk Parity

The goal of Risk Parity optimization is to divide the volatility proportionally over the assets (Chaves et al., 2011). We follow the Risk Parity framework from Costa and Kwon (2019) which starts with the Markowitz (1952) framework as stated in equations 20 and 21. Accordingly, they follow Maillard et al. (2010) to calculate the risk of the portfolio (σ_p) and the volatility the asset contributes to the risk of the portfolio (σ_i). The portfolio risk and asset volatility can be calculated using the following model:

$$\sigma_p = \sqrt{\omega' \sum \omega} = \sum_{i=1}^{n} \sigma_i, \tag{37}$$

$$\sigma_i = \omega \frac{\partial \sigma_p}{\partial \omega_i} = \omega_i \frac{(\Sigma\omega)_i}{\sqrt{\omega'\Sigma\omega}}. \tag{38}$$

The goal is to minimize risk. Therefore, our optimization problem is stated as:

$$\min \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(\omega_i(\Sigma\omega)_i - \omega_j(\Sigma\omega)_j)^2, \tag{39}$$

which is subjected to four constraints,

$$\sum_{i}^{N} \omega_i = 1, \tag{40}$$

$$\omega \geq 0 \quad \forall \quad i, \tag{41}$$

$$(\omega_{Gold} + \omega_{Oil}) \leq Max(\omega_{Commodites}). \tag{42}$$

$$\omega_{Hedgefunds} \leq Max(\omega_{HedgeFunds}), \tag{43}$$

$$(\omega_{PE} + \omega_{VC}) \leq Max(\omega_{PE} + \omega_{VC}). \tag{44}$$

$$\omega_{VC} \leq Max(\omega_{VC}). \tag{45}$$

Thus, the same constraints on the portfolio as the Mean-Variance framework and Black-Litterman framework are again followed for the Risk Parity framework. The Risk Parity is performed with a target volatility equal to the volatility of the three different optimization strategies.

3.2.4 Optimization Strategies

We examine different optimization techniques for allocating assets across different portfolios. Given our emphasis on the trade-off between risk-return, we optimize different frameworks and extract three types of portfolios based on different risk preferences. We create GMV portfolios, TAN portfolios, and MR portfolios. The GMV portfolio portrays a risk-averse investor preference, whereas the TAN portfolio reflects a risk-neutral investor preference, and the MR portfolio depicts a risk-seeking investor preference. The GMV, TAN, and MR approaches restrict the objective function to enhance the performance of the portfolios for a given strategy and expand the flexibility of the investor risk-return preference. For the Mean-Variance the constraints are already nested in the optimization framework. The GMV optimization focuses on the following constraint for the three frameworks:

$$\min \qquad \omega'\Sigma\omega. \tag{46}$$

This constraint minimizes the risk and thus reduces the overall variance of the portfolio. The TAN portfolios revolve around the Sharpe Ratio:

$$\max \qquad \frac{\omega'\mu - \bar{r}_{f_t}}{\sqrt{\omega'\Sigma\omega}}. \tag{47}$$

The objective certifies that the calculated set of weights maximizes the Sharpe Ratio (Kourtis, 2016). Lastly, the MR constraint seems self-explanatory, it maximizes the portfolio returns regardless of the risk:

$$\max \qquad \omega'\mu. \tag{48}$$

3.3 Robustness check

3.3.1 Factor Portfolios

A factor model establishes a relation between expected returns and a set of factors. A multi-factor portfolio provides diversification benefits. To assess the robustness of the portfolios including alternative assets for the different frameworks and optimization strategies, we create a portfolio existing of solely factors which we call the factor portfolio. We use the factors of the Fama French five-factor model (Fama French, 2015). The factor returns are then implemented in the best-performing optimization frameworks based on risk-return. Subsequently, we compare the factor portfolio to the portfolios including alternative assets to validate the portfolio performance.

3.3.2 Alternative Factor portfolios

Roncalli (2017) defines the ARP as an extension of factor investing. He states that the objective of the ARP strategy is to increase portfolio returns and decrease risk. This is achieved by diversifying the portfolio through the inclusion of alternative assets such as hedge funds as an additional factor. We compose portfolios by including factor returns and alternative assets in a manner resembling ARP, emphasizing the addition of alternative assets. As a result, we have a portfolio that exists of the factor returns and the alternative assets which we call alternative factor portfolios. The returns are again implemented in the best-performing optimization frameworks, but now with the existing constraints on the alternative assets and compared to the factor portfolios and the constructed portfolios including alternative assets.

3.4 Performance measures

After creating the optimal portfolio combination for each framework and the factor portfolios, we calculate different performance measures of the portfolios. We examine different risk measures of the portfolios, such as the VaR, the ES, and the Maximum Drawdown. In addition, we evaluate performance measures with a focus on return such as the Sharpe Ratio and the expected return of the portfolio (μ_p). We compare the performance measures of the portfolios with alternative assets to traditional portfolios to deduce the effects of incorporating alternative assets into the portfolio. We also compare the performance measures of the portfolios with alternative assets to the performance measures of the factor investing portfolios, both including and excluding alternative assets. The measures can be calculated as follows:

1. Value at Risk:

   For loss $L_p \sim N(\mu_p, \sigma_p)$ and normal distribution $\Phi^{-1}(\alpha)$,

$$\alpha = Pr(L_p \leq VaR_\alpha) = Pr(\frac{L_p - \mu_p}{\sigma_p} \leq \frac{VaR_{a,p} - \mu_p}{\sigma_p}), \tag{49}$$

$$VaR_{a,p} = \mu_p + \sigma_p \Phi^{-1}(\alpha).$$

   The VaR value expresses the maximum expected loss of the portfolio for the given time period.

2. Expected Shortfall:

$$ES_p = E(L|L \geq VaR_\alpha) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_{u,p}du. \tag{50}$$

   The ES value expresses the maximum expected loss of the portfolio after the VaR is reached for the given time period.

3. Maximum Drawdown:

$$MD_p = \max_{t<t*} \frac{R_{i,t} - R_{i,t*}}{R_{i,t}}. \tag{51}$$

   The Maximum Drawdown represents the biggest percentage loss of the portfolio from the highest to the lowest point before a new high is reached.

4. Expected return:

$$\mu_p = \frac{1}{T}\sum_{t=1}^{T} E(R_i). \tag{52}$$

It represents the combined average expected return of all the assets over the time period

6. Sharpe Ratio:

$$SR_p = \frac{\mu_p - \bar{r}_{f_t}}{\sigma_p}. \tag{53}$$

The Sharpe Ratio is the expected return of the portfolio in relation to the risk level of the portfolio. The listed performance measures can also be calculated for the asset indices.

4 Data

In this paper, multiple optimal portfolios are created using different types of asset classes and different types of allocation methods. Firstly, we use quarterly cash flow data for the PE and VC index, obtained from Preqin. In addition, quarterly Fama French and Pastor-Stambaugh factor data is used as tradable factors for the estimation of the PE and VC returns. The Fama French and Pastor-Stambaugh factor data is obtained from the Kenneth R. French library. Furthermore, we use quarterly index returns of the different traditional and alternative assets obtained from Bloomberg. The quarterly Fama French factor data is also used for the factor portfolios and the ARP factor portfolios. The data covers the following sample period for all the assets: 01-01-2000 until 31-12-2020. We use quarterly returns because PE and VC cash flow data are only available per quarter as funds give updates about investment returns every quarter.

4.1 Cash Flow and Tradable Factor data

The performance of a PE fund is measured differently from other assets. The investment setup is more complex and due to the fact that most investments are private, less public data is available. The General Partner (GP) manages the PE fund and applies the chosen fund strategy, whereas the Limited Partner (LP) is the investor in the PE funds. The performance of a PE fund is based on the company investments in the fund portfolio. PE funds invest in non-listed companies. VC is a type of strategy used in

PE. VC funds solely invest in startups and new fast-growing companies. A GP requires capital to set up an investment fund. The capital is provided by LPs who are the investors in the funds. The GP selects the companies to invest in, before announcing a capital call which recalls LPs to bring in their investments. The GP then utilizes this capital to invest in the selected companies. The investments in the companies generate profits which are then distributed to the LP again.

We use the cash flows (C_t) derived from the investments (I_t) and distributions (D_t) of 1826 PE funds and 250 VC funds for the full sample period to create the indices. We use clean data from Bluemetric which is obtained from the Preqin cash flow data set. The data set includes the following information for the vintages 1980-2020. Firstly, the name of the funds and their fund manager. The fund manager is the entity behind a fund. The fund manager is able to possess several funds. Secondly, the transaction type which refers to a capital call or a distribution. It holds the numbers for cumulative contributions and cumulative distributions. It also contains the status of the funds which means whether the fund is still raising money or not. We restrict the data set in several ways. We only include PE funds that employ the following strategies: Buyout, Growth, Turnaround, Secondaries, Mezzanine, Distressed, and Co-Investment. We include funds with an initial investment occurring later than 01-01-2000. A fund must also hold a closed status, so there is no additional cash inflow or outflow. Throughout the set period, we require that a distribution takes place every quarter.

The restricted data is used to create three indices: an index created by only using the cash flows of PE funds, an index created by only using the cash flows of VC funds, and a total index created by both PE and VC cash flows. Table 1 shows statistics of the cash flows created by the PE and VC funds along with the vintages of the funds. The vintage denotes the year the PE or VC fund was founded. However, it is not always the case that the vintage year matches the year the firm made its initial investments. As we take into account investments starting from 01-01-2000, the vintage year could be set before the first investments. Furthermore, Table 1 reports the number of investments and distributions of the different funds in that specific vintage year. A fund can have multiple investments every quarter, but distributions are at most once every quarter. This explains the lower amount of distributions compared to investments. We can indicate from Table 1 that PE and VC became more popular over the years, as the amount of investments grew over the vintage years. A PE fund's performance is measured by a number of standard metrics, such as the Multiple of Invested Capital (MOIC), which compares the investments made to their original value. Another measurement is the Internal Rate of Return (IRR) which is used to evaluate the profitability of the investments of the PE fund. The IRR can be calculated from the NPV:

$$NPV = \sum_{t=1}^{T} \frac{C_t}{(1+IRR)^t} - C_0. \tag{54}$$

The goal of our simulation is similar as we produce the NPV by using the estimated returns and cash flows, which correspond to the investments and distributions.

The PE and VC indices are estimated using publicly traded factors. We estimate the indices several times, using four different types of factor models: the CAPM model, the three-factor Fama French model, the four-factor model using Fama French factors and the Pastor-Stambaugh illiquidity factor, and the Fama French five-factor model. The results of one model are selected to include in the frameworks. We take the following factors into account in the market analysis: market, size, value, illiquidity, profitability, and investments. In addition, the average risk-free rate is obtained from the risk-free rate of the Kenneth Library for the full sample period from 01-01-2000 until 31-12-2020.

4.2 Traditional and Alternative Assets

The data for creating the different portfolios includes the quarterly return indices of the following asset classes: stocks, bonds, hedge funds, commodities, and the self-constructed PE index and VC index. To provide an overview of the world equity market, we use the weighted average of the quarterly returns of the following three indices: the Morgan Stanley Capital International All Country World Index (MSCI ACWI), the MSCI World Index, and the MSCI Emerging Markets Index. The weighting scheme for the equity market is based on the size of the market capitalization of the index for the full sample period. We take the weighted market value of four different bond indices to create a single index for the bond market which compasses different types of bonds such as government bonds, corporate bonds, treasury bonds,

Table 1: Overview of the cash flows created by the included Private Equity and Venture Capital funds for the vintages ranging from 1996 until 2020.

Vintage	Funds		Investments		Distributions		Vintage	Funds		Investments		Distributions	
	PE	VC	PE	VC	PE	VC		PE	VC	PE	VC	PE	VC
1996	1	1					2009	76	10	921	205	586	106
1997	1	0					2010	94	10	1181	230	822	158
1998	5	5					2011	151	14	1469	254	1086	184
1999	7	4					2012	138	11	1766	256	1356	197
2000	12	16	38	34	13	8	2013	153	12	1982	244	1542	185
2001	16	14	63	54	29	7	2014	172	15	2332	259	1838	196
2002	15	7	90	70	37	16	2015	185	22	2569	274	2068	190
2003	21	17	126	77	71	29	2016	203	26	2678	245	2145	163
2004	31	20	176	86	107	51	2017	171	23	2846	268	2320	181
2005	67	31	260	115	156	73	2018	160	21	2886	239	2405	149
2006	119	20	460	155	261	94	2019	125	16	2710	245	2222	153
2007	126	31	643	192	379	115	2020	29	5	1718	145	1439	94
2008	146	20	924	198	570	118							
Average	39	14	309	109	180	57		138	15	2088	239	1652	163
Total	506	186	2780	981	1623	511		1657	185	25058	2864	19829	1956

Notes: PE represents Private Equity and VC represents Venture Capital.

high-yield bonds, and bonds from emerging markets. The following bond indices are incorporated: the Bloomberg Global Aggregate Index, the Global Aggregate Treasures Index, The Global High Yield Index, and the Emerging Markets Aggregate Index. For the alternative assets, we incorporate the quarterly returns of the HFRX Global Hedge Fund Index, the Gold index, and the Brent Oil index. Furthermore, we create our own PE and VC index which is subsequently added to the list of alternative assets.

For all return indices, we calculate the percentage change per quarter ($R_{i,t}$). Table 2 shows the descriptive statistics of the quarterly returns of all the assets besides PE and VC for the full sample period 01-01-2000 until 31-12-2020. The Table also shows the properties of a listed PE index, making it possible to compare it to our created PE index and VC index. Listed PE is added to the Table as a comparison to the PE and VC index, but the asset is not included in the portfolio optimization. The descriptive statistics show skewness and kurtosis results of approximately zero and three for most indices which indicates a normal distribution. This is not the case for the oil index and the listed PE index as those hold very high kurtosis results. The standard deviation, Maximum Drawdown, and mean results for the listed PE index are also higher compared to the other asset classes, but this can be explained by the sub-optimal estimation of volatility and the complex composition of the index. Stocks and oil also hold high Maximum Drawdown values which descend from their more volatile character. The Sharpe Ratios of commodities are remarkably high compared to the other assets.

Table 2: Summary of the statistical properties of the quarterly returns of the assets for the full sample period 01-01-2000 until 31-12-2020.

Assets	Mean	Std.dev.	Skewness	Kurtosis	MD	SR
Stocks	0.0159	0.0727	-0.8922	3.8433	-0.3032	0.2012
Bonds	0.0088	0.0335	0.7681	3.4827	-0.1417	0.2239
Hedge funds	0.0050	0.0490	0.0420	3.5561	-0.2396	0.0746
Gold	0.0249	0.0659	-0.6836	4.1990	-0.3340	0.3574
Oil	0.0209	0.0242	2.6467	11.5321	-0.2349	0.8104
Listed PE	0.0242	0.1217	-0.6321	8.3402	-0.6669	0.1876

Notes: MD represents the Maximum Drawdown and SR represents the Sharpe Ratio.
PE represents Private Equity.

4.3 Factor Data

To examine the robustness of the portfolios including alternative assets we also create portfolios including factor returns instead of traditional assets. We implement the following factors: market, size, value, profitability, and investments. Firstly, the market factor represents the return on the market. The size factor represents the return differences between companies with a small and large market capitalization. The value factor represents the return difference between companies with a high and low book-to-market ratio. The profitability factor represents the return difference between companies with robust and weak profitability. Lastly, the investment factor represents the return difference between companies with conservative and aggressive investments. An overview of the statistical properties of the factor returns is stated in Table 3. The factor returns show closely related means and standard deviations. The kurtosis values of the factors are remarkably high and in combination with the skewness results it can be stated that the factors are not normally distributed. Furthermore, there are large differences between the factors for the Maximum Drawdown values and the Sharpe Ratio values.

Table 3: Summary of the statistical properties of the quarterly returns of the factors for the factor portfolios for the full sample period 01-01-2000 until 31-12-2020.

Factors	Mean	Std.dev.	Skewness	Kurtosis	MD	SR
Market	0.0060	0.0549	1.2769	8.8288	-0.3015	0.0846
Size	0.0047	0.0279	2.1764	12.8239	-0.1733	0.1195
Value	0.0052	0.0374	1.9335	12.9834	-0.2584	0.1035
Profitability	0.0035	0.0222	-0.4351	12.0404	-0.1777	0.0988
Investments	0.0022	0.0123	1.5300	6.9216	-0.0695	0.0682

Notes: MD represents the Maximum Drawdown and SR represents the Sharpe Ratio.

5 Results

5.1 Private Equity and Venture Capital Index Returns

The cash flows of 1827 PE funds and 250 VC funds are used to estimate the PE and VC return indices. The returns are simulated using the Bayesian MCMC. To perform our simulation, we consider priors and initial values for the parameters in θ for the different factor models. We consider four different factor models: the CAPM model, the Fama French three-factors model, a four-factor model including the Fama French three-factors and the Pastor-Stambaugh factor, and the Fama French five-factor model. The results of the parameter set θ for the four different factor models, after performing the simulation for 2,500 burn-in draws and 10,000 samples, are presented in Table 4 and in Table 11 in Appendix B. The returns are calculated based on the probable θ, to which the simulation converges, and the average $\bar{\theta}$ of the sample period: the average of all the sampled values of θ after the burn-in period. The results displayed in Table 4 are based on the probable θ and the results in Table 11 in Appendix B are based on the average $\bar{\theta}$. As our data includes funds with different investment strategies (Buyout, Growth, VC, and others.), it is necessary to use proper priors. We consider the priors and initial values set by Ang et al. (2014, 2018). For the priors Ang et al. (2014, 2018) use the average parameter results representing different investment strategies from several papers[3].

With the priors, we create the indices that are incorporated into the portfolio frameworks. The prior of α is set to 0.01 as the prior of α has little effect on the estimation. The priors of β are based on the weighted average of the different strategies and are set to 1.3 for the market factor, 0.55 for the size factor, 0.05 for the value factor, 0.5 for the illiquidity factor, 0.49 for the profitability factor and 0.16 for the investments factor. As ψ should be bounded by 0 and 0.9, the prior is set to 0.45. Lastly, the prior of the returns R_{it} is uninformative $p(R^e_{i,t}) \propto 1$.

The prior values attributed to the results of θ for the four-factor models. The θ results for the three indices are presented in Table 4. The results include the PE index, the VC index, and the total index of which the latter is created by using all the PE and VC funds combined. The Table shows that the β values for the PE indices are closely related for all factor models. The β values of the VC indices and the total indices differ. One clarification for this is the fact that both indices incorporate VC funds and VC funds tend to have more extreme performance results, resulting in more

[3]The average weighted priors of Ang et al. (2014, 2018) are based on the parameter values of the following papers: Brav, and Gompers (1997), Driessen, Lin, and Phalippou (2012), Ewens, Jones and Rhodes-Kropf (2013), Korteweg, and Sorensen (2010), Cao, and Lerner (2007), Driessen, Lin, and Phalippou (2012), Ewens, Jones and Rhodes-Kropf (2013), Franzoni, Nowak, and Phalippou (2012), Jegadeesh, Kräussl, and Pollet (2009), Chiang, Lee and Wissen (2005), Derwall et al. (2009), Lin, and Yuang (2004) and Elton et al. (2001).

outlier values. It can be seen that the β_{market} factor is very explanatory as the β values of CAPM model are closely related to the four- and five-factor model. Furthermore, the β results of the four-factor model and the five-factor are on average more closely related. The results of parameter ψ are again similar for the CAPM, four-factor, and five-factor model. The standard deviation of the state equation and the likelihood ratio are higher for the VC indices. This can be explained by the lower amount of VC funds used to create the index, which is 250, compared to the 1827 PE funds. This makes the results more sensible to outliers, resulting in higher standard deviations. Overall, results offered by the four-factor model and the Fama French factor models show superior insights compared to those of the CAPM model and the three-factor model due to the extra information provided by the additional factors.

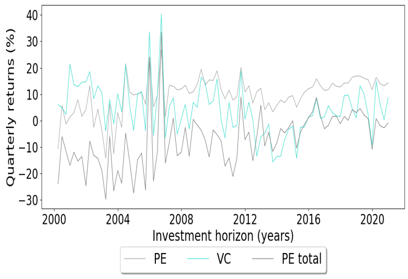
The returns of the indices for the different factor models using the probable θ are shown in figure 1. The indices based on the average $\bar{\theta}$ are shown in figure 7 in Appendix B. It can be seen that the returns based on $\bar{\theta}$ show higher quarterly returns compared to the returns incorporating the probable θ. The return difference is also larger between indices for the average $\bar{\theta}$. The graphs of the probable θ show higher returns before the financial crisis than after. This can be explained by the high deal value before the crisis and the decrease in deal volume in this asset during the period after. After the financial crisis, the interest in PE and VC rose again; leading to higher quarterly returns. The beginning of 2020 shows lower results which can be explained by the COVID crisis and the economic downfall it brought. Overall, the indices show an upward trend over the past 5 years.

Due to the fact that the simulation draws values for θ from the posterior distribution, the simulated θ values converge to the most probable set of θ values. As a result, we will focus on the return indices created by the probable θ. Furthermore, we will continue with the indices of only one factor model. Because the Fama French five-factor model includes more factors than the other model, it has a higher explanatory power, which is why we use its results in the continued analysis. As we intend to add the indices to the mathematical frameworks we will focus solely on the PE and VC returns. The cumulative returns of the PE and VC index, starting from one hundred, are shown in figure 2a, and the cumulative returns of the total portfolio are shown

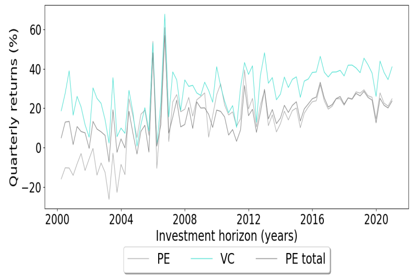
Table 4: Results of the Bayesian Markov Chain Monte Carlo Simulations of the parameter set θ per factor model for a burn-in period of 2,500 draws and 10,000 sample draws for the full sample period 01-01-2000 until 31-12-2020.

θ	α	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	$\beta_{profitability}$	$\beta_{investments}$	ψ	σ_R	σ
5-factor model										
Total	0.0002	0.2783	0.1178	0.0107		0.1049	0.0343	0.0964	0.7066	0.1008
PE	0.0004	0.4966	0.2101	0.0191		0.1872	0.0611	0.1719	0.5941	0.1250
VC	0.0006	0.7474	0.3162	0.0287		0.2817	0.0920	0.2587	0.7325	0.3633
4-factor model										
Total	0.0002	0.2783	0.1178	0.0107	0.1071			0.0964	0.7059	0.1008
PE	0.0004	0.5276	0.2232	0.0203	0.2029			0.1826	0.6091	0.1244
VC	0.0003	0.3908	0.1653	0.0150	0.1503			0.1353	0.6432	0.3310
3-factor model										
Total	0.0007	0.8709	0.3684	0.0335				0.3015	0.2155	0.1267
PE	0.0004	0.5476	0.2317	0.0211				0.1896	0.6012	0.1245
VC	0.0002	0.3168	0.1340	0.0122				0.1096	0.3234	0.3142
CAPM										
Total	0.0002	0.2783						0.0964	0.7057	0.1008
PE	0.0004	0.4996						0.1729	0.5808	0.1258
VC	0.0003	0.3832						0.1327	0.6416	0.3315

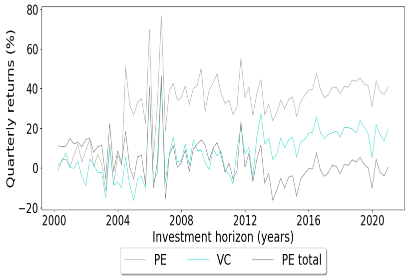
Notes: PE represents Private Equity and VC represents Venture Capital.



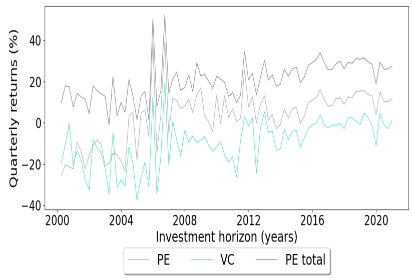
(a) Five-factor model



(b) Four-factor model



(c) Three-factor model

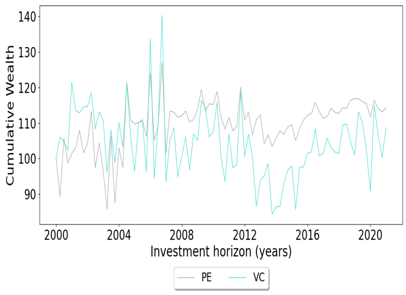


(d) CAPM model

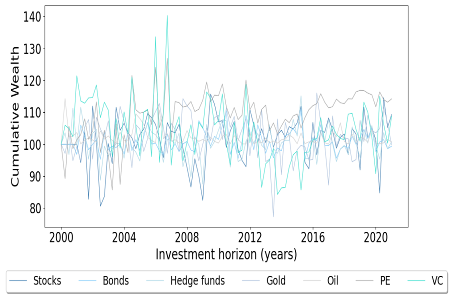
Figure 1: The fluctuations of the total index, the Private Equity Index, and the Venture Capital index over the full sample period from 01-01-2000 until 31-12-2020 using the probable θ.

in figure 2b. The cumulative returns of the PE index and VC index follow the same pattern, but the VC index results in larger fluctuations which is in line with the riskier characteristics of VC. To compare the returns of the PE index and VC index to other assets included in our portfolio, the correlations of the quarterly returns between the assets are presented in the heat map in figure 3 and the statistical properties of the PE index and VC index are presented in Table 5. The correlations show diversification opportunities as the assets exhibit low correlations. For example, stocks and commodities are negatively correlated and bonds and commodities also have a low correlation. The aforementioned indices are moderately correlated, as the cumulative returns in figure 2b suggests. We again incorporated the listed PE index as a comparison, but there is no correlation found between listed PE and the created PE index and VC index. This can be due to the different natures of public and private companies. Listed PE and stocks are strongly correlated as they are both based on public tradable companies.

The correlations between PE and VC and traditional assets are low. A higher correlation between stocks and PE and VC was expected as they are both equity-based. On the other hand, a low correlation with bonds is expected. Overall, we can state that the correlation between alternative assets is slightly higher than between alternative assets and traditional assets. When evaluating the statistical properties of the PE and VC, the mean returns and the standard deviations of the PE index and VC index are much higher in comparison to the other assets. The Maximum Drawdown is also high for the PE and VC index. This is in line with the cumulative returns which indicate large upward and downward fluctuations of the indices. The Sharpe Ratio of the PE index is significantly higher in comparison to the other assets which can be explained by the diversified nature of the PE index: considering different types of companies and strategies.



(a) Private Equity and Venture Capital



(b) Total portfolio

Figure 2: The cumulative returns of the Private Equity index and Venture Capital index (left) and the total portfolio (right) for the full sample period 01-01-2000 until 31-12-2020.

Table 5: Summary of the statistical properties of the quarterly returns of the Private Equity index and Venture Capital index for the full sample period 01-01-2000 until 31-12-2020.

Assets	Mean	Std.dev.	Skewness	Kurtosis	MD	SR
PE	0.0970	0.0711	-0.9429	4.9736	-0.3250	1.3461
VC	0.0443	0.0973	0.6519	4.6603	-0.3993	0.4422

Notes: PE represents Private Equity and VC represents Venture Capital.
MD represents the Maximum Drawdown and SR represents the Sharpe Ratio.

5.2 Optimal Portfolios

We optimize our strategic asset allocation based on the in-sample period for the different frameworks. The weights of the assets are re-balanced every quarter using an

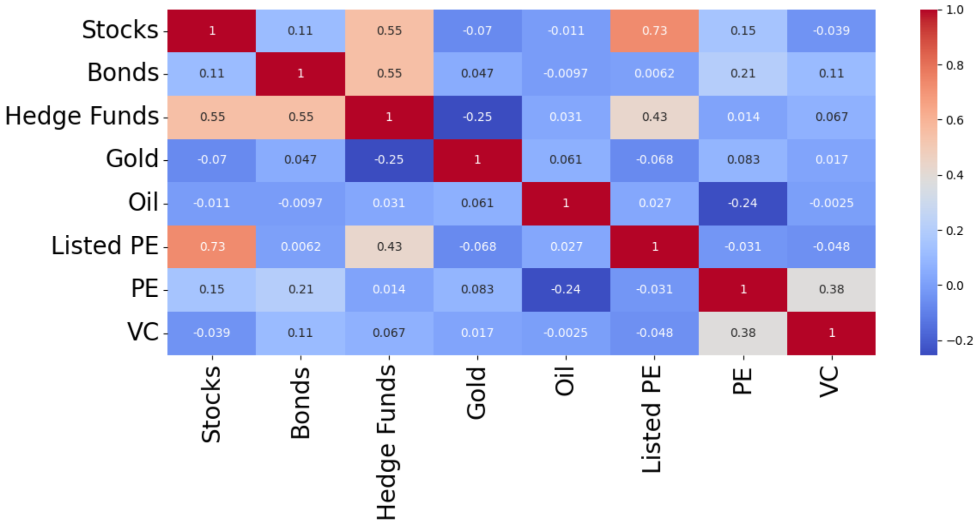


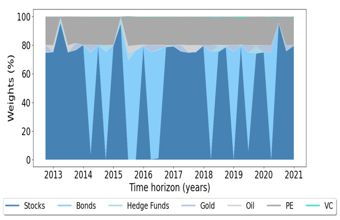
Figure 3: Heat map of the correlations of the quarterly returns of the assets including the Private Equity index (PE) and the Venture Capital index (VC) for the full sample period 01-01-2000 until 31-12-2020.

expansion rolling window. The estimated weights are used to simulate the out-of-sample portfolios. The estimated optimal weights for the out-of-sample portfolios for the three frameworks are shown in figure 4, 5, and 6. The weights of all three Mean-Variance portfolios show a strong presence of alternative assets. The Mean-Variance GMV, TAN, and MR portfolios allocate a substantial amount of weight to PE with a dispersion of weights in other alternative assets. This can be explained by the higher cumulative returns in figure 2b and the higher Sharpe Ratio of the PE index compared to the other assets shown in the statistical properties in Table 2. The Mean-Variance TAN and MR portfolio results in slightly higher weight allocations to VC. The optimal allocation to commodities could be higher and more diversified for all three portfolios because the cumulative returns and the statistical properties show high returns on average for commodities. However, the model was limited to assigning a cumulative 5% to gold and oil. The high allocation to stocks in the portfolios can be explained by the out-performance of the stocks index compared to the other asset indices in figure 2b.

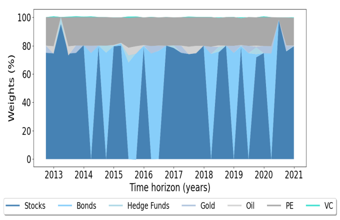
For the second framework, the Black-Litterman, we imposed the same constraints as the Mean-Variance framework but included subjective views on the movements of the assets. All three Black-Litterman optimal portfolios are heavily invested in bonds. The MR portfolio holds a higher allocation in stocks. This is expected as stocks tend to have higher returns than bonds and are thus more appealing to risk-seeking investors. The GMV portfolio allocates a low amount to alternative assets. This can be explained by the higher risk characteristics of the alternative assets. The amount of weight allocated to alternative assets is higher for the MR portfolio, but still lower than for the Mean-Variance portfolio. The additional view on the increase in stocks and PE does not seem to affect the strategic asset allocation, as most of the portfolio is allocated to bonds.

The Risk Parity framework focuses more on the dispersion of risk. The weights show almost equal allocation over time for the GMV portfolio and an almost steady allocation to alternative assets for all three portfolios. Again, a large amount of allocation in bonds. Furthermore, the Risk Parity portfolios show an almost equal distribution to the PE index and VC index and also a maximum allocation amount for the other alternative assets. This can be clarified by the fact that the Risk Parity explores an equal diversion of risks and benefits of allocating weights to a lot of different assets.

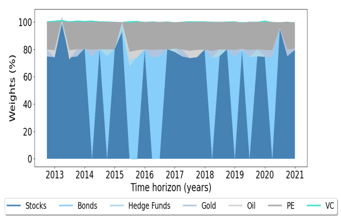
Overall, the performance of the frameworks shows that to obtain the optimal portfolio, a significant amount of weight should be allocated to the PE index and VC index, especially for the Mean-Variance framework. This indicates a positive effect of adding PE and VC as alternative assets to the investment portfolio.



(a) GMV

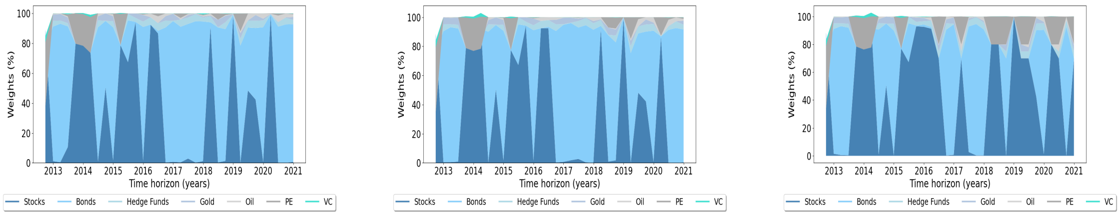


(b) TAN



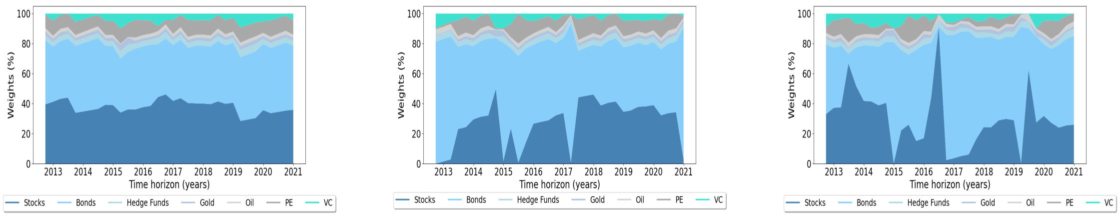
(c) MR

Figure 4: Weight allocation of the Mean-Variance portfolios for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.



(a) GMV **(b)** TAN **(c)** MR

Figure 5: Weight allocation of the Black-Litterman portfolios for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.



(a) GMV **(b)** TAN **(c)** MR

Figure 6: Weight allocation of the Risk Parity portfolios for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

5.3 Portfolio Performance

The results of the performance measures of the portfolios are based on the three different frameworks for the out-of-sample period. These are shown in Table 6, 7, and 8. We evaluated performance measures based on risk and return. For the risk measures, we incorporated the VaR for two different confidence rates of 95% and 99%. For the Mean-Variance portfolios, the VaR and ES for both confidence rates and the expected return are in line with the set risk preference. The GMV risk-averse portfolio expects the lowest maximum loss whereas the MR portfolio expects the highest expected return. Conversely, the risk-seeking MR portfolio shows slightly lower results for the Maximum Drawdown and higher results for the Sharpe Ratio in comparison to the risk-averse strategy and the risk-neutral strategy.

The Black-Litterman portfolios lead to overall lower results. The Maximum Drawdown of the GMV portfolios for the Mean-Variance and Black-Litterman differ by almost 8%. This is quite a large drawdown difference. The VaR and ES values are low, but the Risk Parity TAN and MR portfolio still outperforms the Black-Litterman MR portfolio for the VaR and ES with a confidence rate of 95%. Furthermore, the Sharpe Ratio and the expected returns are also significantly lower, resulting in an unappealing portfolio performance.

The Risk Parity portfolios show low VaR and ES values for both confidence rates. It is noteworthy that the amount of expected loss of the portfolio decreases for both confidence rates, and the expected returns decrease for the more risk-seeking portfolios. The difference between the portfolios is negligible but hints at an inverted frontier. The Maximum Drawdown values are significantly lower than the other frameworks, resulting in a low level of risk for the portfolio with an average expected return.

We find that the Black-Litterman portfolio performs worse than the Mean-Variance portfolios and the Risk Parity portfolios, based on the given performance measures. The risk preference of the investor is important to declare which framework is preferred when comparing the Mean-Variance framework and the Risk Parity. The Mean-Variance portfolios result in substantially higher expected returns of ranges between 5.71% and 5.88% compared to 1.25% and 2.53%, and larger Sharpe Ratio values. Whereas the Risk Parity portfolios hold lower risk values. The VaR and ES for both confidence rates are significantly lower. The Maximum Drawdown values are closely related. Ultimately we conclude a slightly overall better performance of the Mean-Variance portfolios.

The portfolio weights of traditional portfolios are shown in figure 9, 10 and 11 in Appendix D and the results of the performance measures in Tables 12, 13 and 14 in Appendix D. The weights of the traditional portfolio are divided between stocks and bonds. As both the Mean-Variance and the Risk Parity portfolios outperform the Black-Litterman portfolios significantly, we concentrate on those portfolio results from now on.

We conclude from the performance measures that the expected returns for all portfolios including alternative assets are considerably larger than the expected portfolio returns without alternative assets. The Sharpe Ratios of the portfolios including alternative assets are significantly higher and show more potential, especially in combination with the higher expected portfolio returns. This is risky as the alternative assets also bring more risk to the portfolio, leading to higher values of the VaR and the ES for both confidence rates. On the other hand, the Risk Parity MR portfolio with alternative assets has lower values for VaR and ES for both confidence rates than the Risk Parity portfolios with traditional assets. This shows that the risk of portfolios with alternative assets is comparable to portfolios without alternative assets. Also, the addition of alternative assets to the portfolio lowers the drawdowns substantially for the Risk Parity portfolios and Mean-Variance portfolios, these portfolios are less affected by declines in a specific asset. In conclusion, the addition of alternative assets to the investment portfolios increases the overall return of the portfolio and slightly increases the risk, but lowers the drawdowns.

Table 6: Performance measures Mean-Variance portfolios including alternative assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
GMV (Risk-Averse)	0.0654	0.0688	0.0651	0.0657	-0.1421	1.7088	0.0571
TAN (Risk-Neutral)	0.0656	0.0691	0.0653	0.0660	-0.1411	1.8034	0.0577
MR (Risk-Seeking)	0.0669	0.0704	0.0665	0.0672	-0.1401	1.8159	0.0588

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.
VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown, SR represents the Sharpe Ratio and μ_p represents the expected portfolio return.

Table 7: Performance measures Black-Litterman portfolios including alternative assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
GMV (Risk-Averse)	0.0140	0.0148	0.0136	0.0137	-0.2316	0.0062	0.0016
TAN (Risk-Neutral)	0.0149	0.0157	0.0145	0.0146	-0.1960	0.0400	0.0032
MR (Risk-Seeking)	0.0329	0.0347	0.0324	0.0327	-0.1958	0.3094	0.0187

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.
VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown, SR represents the Sharpe Ratio and μ_p represents the expected portfolio return.

Table 8: Performance measures Risk Parity portfolios including alternative assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
GMV (Risk-Averse)	0.0329	0.0346	0.0326	0.0329	-0.1109	0.8063	0.0253
TAN (Risk-Neutral)	0.0282	0.0297	0.0279	0.0282	-0.1037	0.6821	0.0209
MR (Risk-Seeking)	0.0223	0.0235	0.0221	0.0223	-0.1065	0.6322	0.0163

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.
VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown, SR represents the Sharpe Ratio and μ_p represents the expected portfolio return.

5.4 Comparison to Factor Portfolios

We create a portfolio including only factor returns using the five Fama French factors. The factor portfolios are created for the Mean-Variance framework and Risk Parity framework because these frameworks outperform the Black-Litterman framework. The results of the performance measures of the factor portfolio are presented in Table 9. The Table shows that the risk measures are lower for the factor portfolios in comparison to the portfolios including alternative assets. This makes sense as the basic objective of factors is to restrict equity risk. We find that expected returns and Sharpe Ratios are significantly higher for the portfolios including alternative assets. It can be seen that the Mean-Variance portfolios result in higher risk performance, but also higher returns compared to the Risk Parity portfolios. The performance results of the factor portfolios are in line with the performance measures of the Mean-Variance and Risk Parity portfolios including alternative assets. This means we can validate the robustness of the Mean-Variance framework and Risk Parity framework for the different optimization strategies.

The performance measures of the alternative factor portfolios can be found in Table 10. When comparing the factor portfolio to the alternative factor portfolio, it can be seen that the maximum expected loss on the portfolios is lower for the factor portfolios, but the Maximum Drawdown values are comparable. The addition of alternative assets results in slightly higher expected returns for the alternative factor portfolios than the factor portfolios. However, the Mean-Variance and the Risk Parity portfolios in Table 6, 7 and 8 hold much more appealing expected returns for investors. We can state that the Mean-Variance portfolios including alternative assets outperform the factor portfolios. In turn, the (alternative) factor portfolios outperform the Risk Parity portfolios including alternative assets. Overall, we find very similar outcomes based on risk-return when comparing the factor portfolios to the alternative factor portfolios, just as we do when comparing the standard portfolio to the portfolios that include alternative assets. This means that the robustness check of adding alternative assets to the portfolio has proven to be effective.

Table 9: Performance measures of the Factor portfolio for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
Mean-Variance							
GMV (Risk-Averse)	0.0188	0.0187	0.0198	0.0198	-0.0579	1.0755	0.0155
TAN (Risk-Neutral)	0.0183	0.0181	0.0192	0.0192	-0.0614	0.9538	0.0147
MR (Risk-Seeking)	0.0188	0.0186	0.0198	0.0198	-0.0601	1.0800	0.0155
Risk Parity							
GMV (Risk-Averse)	0.0073	0.0072	0.0077	0.0077	-0.0571	0.2365	0.0042
TAN (Risk-Neutral)	0.0073	0.0072	0.0077	0.0077	-0.0568	0.2363	0.0042
MR (Risk-Seeking)	0.0073	0.0072	0.0077	0.0077	-0.0566	0.2360	0.0042

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.
PE represents Private Equity and VC represents Venture Capital.
VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown,
SR represents the Sharpe Ratio and μ_p represents the expected portfolio return.

Table 10: Performance measures of the Alternative Factor portfolio for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
Mean-Variance							
GMV (Risk-Averse)	0.0314	0.0313	0.0331	0.0331	-0.0679	1.5869	0.0273
TAN (Risk-Neutral)	0.0310	0.0309	0.0326	0.0326	-0.0678	1.7390	0.0272
MR (Risk-Seeking)	0.0313	0.0311	0.0329	0.0329	-0.0678	1.8133	0.0276
Risk Parity							
GMV (Risk-Averse)	0.0316	0.0315	0.0333	0.0333	-0.0716	0.8907	0.0167
TAN (Risk-Neutral)	0.0315	0.0314	0.0332	0.0332	-0.0703	0.8905	0.0164
MR (Risk-Seeking)	0.0318	0.0316	0.0335	0.0335	-0.0687	0.8998	0.0162

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.
PE represents Private Equity and VC represents Venture Capital.
VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown,
SR represents the Sharpe Ratio and μ_p represents the expected portfolio return.

6 Conclusion

Creating a more diversified optimal portfolio is one of the main goals of an investor. Diversification opportunities can be obtained by strategic asset allocation for a portfolio with various asset classes. Due to its different characteristics and low correlation in comparison to traditional assets, investing in alternative assets has become more popular. Including alternative assets in the investment portfolio can lead to higher returns and a risk distribution. PE is an alternative asset class that encompasses a lot of investment opportunities, but as the performance of PE funds is measured differently from other assets due to the investment setup and less available public data, it is more difficult to compare PE to the traditionally composed asset indices. To make PE more comparable to traditional assets, this research created PE and VC indices retrieved from the cash flows of the PE and VC funds, in combination with public tradable factor data. For four different factor models, the Bayesian MCMC is applied to draw the parameters and index returns from the simulated posterior distributions, using the Metropolis-Hasting algorithm and Gibbs Sampling algorithm. Ultimately, the superior PE index and VC index were estimated from the Fama French five-factor model due to the additional explanatory power of the cross-section of the returns of the additional Fama French factors. The estimated PE index and VC index were then added to the investment portfolio.

Multiple optimal portfolios were created using three different frameworks: the Mean-Variance of MPT, Black-Litterman, and Risk Parity, in combination with the GMV, TAN, and MR optimization strategies. The short-term mixed asset portfolios are constructed using traditional and alternative assets. The alternative assets included are hedge funds, gold, oil, and the self-constructed PE index and VC index. Furthermore, we estimated portfolios including factor returns and portfolios including factor returns and assets to validate the robustness of the addition of alternative assets to the portfolio. The sample period of all used data ranges from 01-01-2000 until 31-12-2020.

By creating the portfolios with alternative assets we answer the following research question: how does the addition of alternative assets to a traditional portfolio affect the overall risk-return of the portfolio and what are the best techniques for allocating alternative assets? Diversified optimal portfolios are created from the three optimization frameworks. It can be concluded that the Mean-Variance framework and Risk Parity framework outperform the Black-Litterman framework based on the used performance measures. To declare the optimal portfolio based on the risk-return preference of the investor is of great importance. The Risk Parity portfolios are less risky, but based on the other performance measures such as the closely related Maximum Drawdown values and the higher expected returns and Sharpe Ratios, the Mean-Variance optimal portfolios outperform the Risk Parity portfolios. When looking at the weights, it can be declared that a substantial amount of weight is allocated to the PE index and VC index. Therefore, the addition of the PE index and VC index adds value to the investment portfolio.

When adding alternative assets to a traditional portfolio, the characteristics of a portfolio change and modify the performance of the portfolio. For the used frameworks, the addition of alternative assets to the portfolio leads to higher overall returns of the portfolio, but also an elevated risk. However, because the Maximum Drawdown is much lower for the portfolios including alternative assets, the trade-off is still highly positive. This is also seen in the high value of the Sharpe Ratio. Overall, the addition of alternative assets leads to more optimal portfolios with higher returns and a decent trade-off between risk-return.

When comparing the alternative portfolios to the factor portfolio and the alternative factor portfolios, the following conclusions can be drawn. The factor portfolios result in relatively safe performance measures with low risks and moderate expected returns. The comparison to the alternative asset portfolios shows that the optimal framework Mean-Variance outperforms the factor portfolio. However, the factor portfolio and the alternative factor portfolios outperform the Risk Parity portfolios. Furthermore, the addition of alternative assets in the factor portfolio results in the same conclusion as adding alternatives to the traditional portfolios. This validates the robustness of the Mean-Variance framework and Risk Parity framework for the different optimization strategies and the addition of adding alternative assets to the portfolio.

A possible extension of this research could be to increase the amount of additional alternative assets to the ARP factor portfolio, so it is possible to evaluate the performance of factor portfolios holding multiple assets. Another extension could be to create portfolios with the ERI, as this method minimizes significant losses and tends to outperform minimum variance portfolios such as our GMV portfolio. In addition, other optimization tools such as a copula could be beneficial to add as it looks at the dependence between the assets. Furthermore, we have included a PE index and VC index based on different types of strategies, but it might be beneficial to evaluate which PE strategies lead to higher returns for the created indices, lower risks, and possibly bring additional diversification opportunities. An obstacle in our research was the essence of an alternative value for the market capitalization vector which represents the size of the asset in Black-Litterman as most alternative assets do not hold a clear market capitalization value. Lastly, this research is limited in creating the PE index and the

VC index as the indices are very dependent on choosing the right initial values and priors for the factor loadings, making the outcomes of the posterior distribution less plausible. Hence, the performance of the indices for the four different factor models varied widely.

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A Assumptions Private Equity Index and Venture Capital Index

We follow Ang et al (2014, 2018) and display various assumptions in order to construct the PE index and VC index.

Assumption 1:

$$lnR_{i,t} = lnR_{i,t} + \epsilon_{i,t}, \tag{55}$$

where $\epsilon_{i,t}$ is normally independent identical distributed (i.i.d). Here $Var(\epsilon_{i,t}) = s^2$, and $E(\epsilon_{i,t}) = \mu$, set μ = -$\phi\frac{1}{2}s^2$. E(.) is the expectation. Combining assumption 1 with equation 1 gives the following:

$$\frac{I_i}{R_{i,t_1}R_{i,t_2}...R_{i,T}}\exp(\epsilon_{i,t+1} + ... + \epsilon_{T_j}) = \frac{D_{T_j}}{R_{D,t_1}R_{j,t_2}...R_{j,T}}. \tag{56}$$

For the error terms we have,

$$U^i_{t_i,T_j} = exp(\epsilon^i_{t_i+1} + ... + \epsilon^i_{T,j)}. \tag{57}$$

When we take the log, $U^i_{t_i,T_j}$ is normally distributed, due to the characteristics of ϵ_t. The sum over the exponential returns for the amount of investments for each PE fund can be written as:

$$\sum_{i=1}^{N} \frac{I_i}{R_{t1}...R_{t_i}} U^i_{t_i,T_j} = \sum_{j=1}^{N} \frac{D_{T_j}}{R_{t1}...R_{Tj}}. \tag{58}$$

We can now calculate PV_D, PV_I and w_i as follows:

$$i.PV_D = \sum_{j=1}^{N} \frac{D_{T,j}}{R_{t,1}...R_{T,j}}, \tag{59}$$

$$ii.PV_I = \sum_{i=1}^{N} \frac{I_i}{R_{t,1}...R_{t,i}} U^i_{t,i,T,j}, \tag{60}$$

$$iii.w_i = \frac{\frac{I_i}{R_{t,1}...R_{t,i}}}{PV_I}, \tag{61}$$

which we can rewrite as,

$$\sum_{i=1}^{N} w_i(U^i_{t_i,T_j})\frac{PV_D}{PV_I}. \tag{62}$$

Assumption 2: Not one investment should overrule when the amount of investments is increasing to infinity, where κ is the number of investments.

i) $\kappa \rightarrow \propto$, $w_i \rightarrow 0$, meaning when the number of investments goes to infinity, the weight of one investment should not overrule the other weights..

ii) $\frac{T_j - t_i}{\sum_j^{\kappa}(T_j - t_j)} \rightarrow 0$, meaning no outstanding long investments.

From assumption 2, we can get the Present Value Ratio (PVR):

$$PVR_h = ln\frac{PV_{D,h}}{PV_{I,h}} = lnu_h, \tag{63}$$

$$lnu_h \sim N(-\frac{1}{2}\sigma_h^2, \sigma_h^2). \tag{64}$$

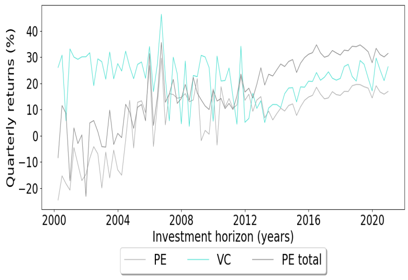
Assumption 3: The volatility of the PVR is equal for each included fund: $\sigma_h^2 = \sigma^2$.

Assumption 4: State equation dynamics are equal to:

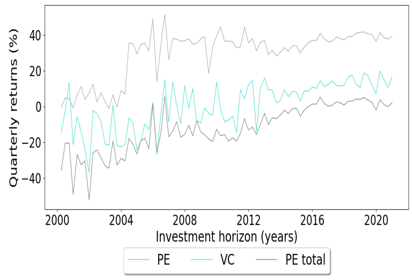
$$R_{i,t} = \alpha + \beta' F_t + f_t + r_{f_t}. \tag{65}$$

Following the same explanation for the variables as in section 3.1.

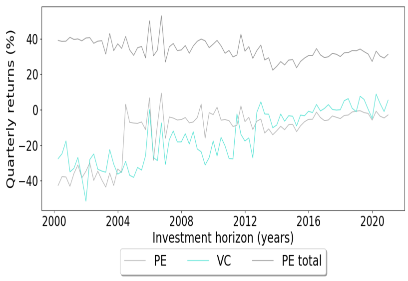
B Bayesian Markov Chain Monte Carlo Simulation



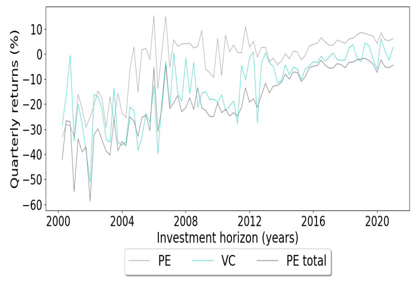
(a) Five-factor model



(b) Four-factor model



(c) Three-factor model



(d) CAPM model

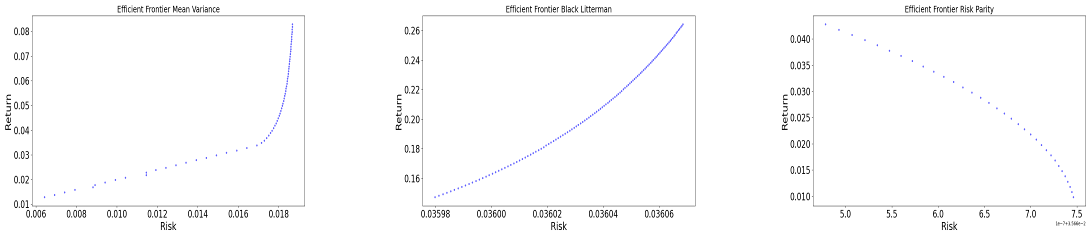
Figure 7: The fluctuations of the total index, the Private Equity index (PE), and the Venture Capital index (VC) for the full sample period from 01-01-2000 until 31-12-2020 using the average $\bar{\theta}$.

Table 11: Results of the Bayesian Markov Chain Monte Carlo Simulation of the parameter set average $\bar{\theta}$ per factor model for a burn-in period of 2,500 draws and 10,000 sample draws for the full sample period 01-01-2000 until 31-12-2020.

$\bar{\theta}$	α	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	$\beta_{profitability}$	$\beta_{investments}$	ψ	σ_R	σ
5-factor model										
Total	0.0002	0.5825	0.2465	0.0224		0.2196	0.0717	0.2016	0.4547	0.1141
PE	0.0004	0.5259	0.2225	0.0202		0.1982	0.0647	0.1820	0.5055	0.1280
VC	0.0004	0.5612	0.2374	0.0216		0.2115	0.0691	0.1943	0.4991	0.3455
4-factor model										
Total	0.0004	0.5801	0.2454	0.0223	0.2231			0.2008	0.4563	0.1140
PE	0.0004	0.5259	0.2225	0.0202	0.2023			0.1820	0.5081	0.1280
VC	0.0004	0.5616	0.2376	0.0216	0.2160			0.1944	0.4984	0.3455
3-factor model										
Total	0.0004	0.5808	0.2457	0.0223				0.2010	0.4557	0.1140
PE	0.0004	0.5259	0.2225	0.0202				0.1820	0.5083	0.1280
VC	0.0005	0.5933	0.2510	0.0228				0.2054	0.4543	0.3484
CAPM										
Total	0.0004	0.5849						0.2025	0.4463	0.1142
PE	0.0004	0.5259						0.1820	0.5010	0.1280
VC	0.0004	0.5653						0.1957	0.4879	0.3459

Notes: PE represents Private Equity and VC represents Venture Capital.

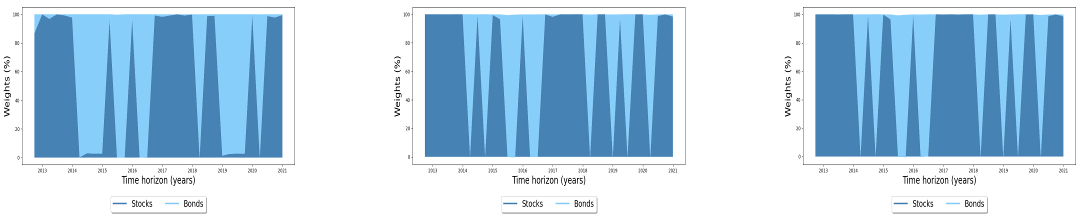
C Efficient frontiers



(a) Mean-Variance **(b)** Black-Litterman **(c)** Risk Parity

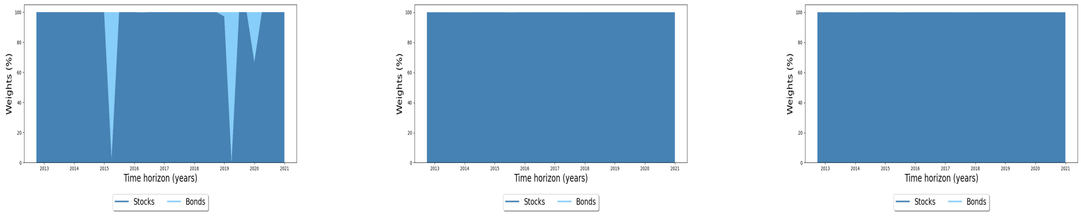
Figure 8: The efficient frontiers for the three optimization frameworks for the in-sample period ranging from 01-01-2020 until 29-06-2012.

D Traditional Assets Portfolios



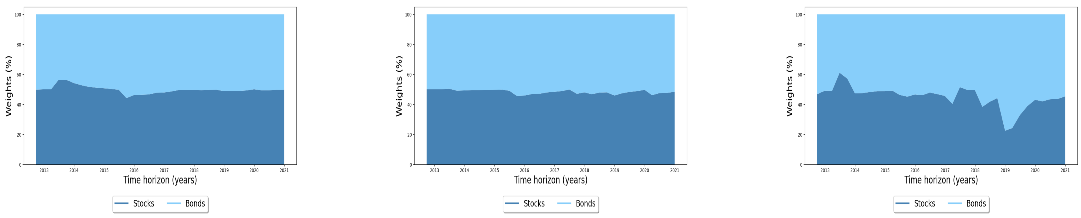
(a) GMV **(b)** TAN **(c)** MR

Figure 9: Weight allocation of the Mean-Variance portfolios with traditional assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.



(a) GMV **(b)** TAN **(c)** MR

Figure 10: Weight allocation of the Black-Litterman portfolios with traditional assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020.The weights are re-balanced every quarter.



(a) GMV **(b)** TAN **(c)** MR

Figure 11: Weight allocation of the Risk Parity portfolios with traditional assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020.The weights are re-balanced every quarter.

Table 12: Performance measures Mean-Variance portfolios with traditional assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
GMV (Risk-Averse)	0.0456	0.0480	0.0453	0.0457	-0.1676	0.9592	0.0364
TAN (Risk-Neutral)	0.0504	0.0530	0.0500	0.0505	-0.1675	1.0001	0.0405
MR (Risk-Seeking)	0.0505	0.0531	0.0501	0.0506	-0.1675	1.0012	0.0405

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.
VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown, SR represents the Sharpe Ratio and μ_p represents the expected portfolio return.

Table 13: Performance measures Black-Litterman portfolios with traditional assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
GMV (Risk-Averse)	0.0394	0.0415	0.0389	0.0393	-0.2608	0.4170	0.0250
TAN (Risk-Neutral)	0.0426	0.0448	0.0420	0.0425	-0.2609	0.4434	0.0276
MR (Risk-Seeking)	0.0426	0.0448	0.0420	0.0425	-0.2609	0.4434	0.0276

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.

VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown, SR represents the Sharpe Ratio and μ_p represents the expected portfolio return

Table 14: Performance measures Risk Parity portfolios with traditional assets for the out-of-sample period ranging from 28-09-2012 until 31-12-2020. The weights are re-balanced every quarter.

	VaR_{α_1}	ES_{α_1}	VaR_{α_2}	ES_{α_2}	MD	SR	μ_p
GMV (Risk-Averse)	0.0280	0.0295	0.0276	0.0279	-0.1672	0.4411	0.0182
TAN (Risk-Neutral)	0.0276	0.0290	0.0272	0.0275	-0.1624	0.4400	0.0180
MR (Risk-Seeking)	0.0270	0.0284	0.0266	0.0269	-0.1565	0.4690	0.0180

Notes: $\alpha_1 = 0.05$, $\alpha_2 = 0.01$.

VaR represents the Value at Risk, ES represents the Expected Shortfall, MD represents the Maximum Drawdown, SR represents the Sharpe Ratio and μ_p represents the expected portfolio return