The performance of portfolio allocations using robust estimation methods

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Abstract

Real-world stock data can contain outliers which do not follow the pattern of the majority of the data. These outliers can cause dispersion in the estimation of the portfolio weights. This paper looks at identifying these outliers and using robust estimation methods to stabilize the portfolio weights over time. This paper makes the distinction between casewise and cellwise robust methods. The analysis is done for the low-dimensional case as well as for the high-dimensional case. Furthermore, the weight estimation is based on the allocation of the Global Minimum Variance portfolio (GMV) and the Tangency Portfolio. The data used in this paper is daily returns from the S&P 500. The dataset spans from February 2013 until February 2018. This paper found for the low-dimensional case, the robust methods based on the Mahalanobis distances did not improve portfolio performance and weight stability compared to the traditional estimations techniques for the GMV portfolio and the Tangency portfolio. However, this paper found the low-dimensional robust estimation methods based on pairwise covariances improved the performance of the GMV portfolio. Furthermore, this paper also shows that for the high-dimensional portfolios, both the performance and the weight stability of the GMV portfolio and the Tangency portfolio did improve when the robust methods based on pairwise covariances were used.

Contents

1 Introduction

Trading has always been around but lately it seems to be a hot topic, from regular stocks to call options or even cryptocurrencies, it seems that everyone is trading at the moment. But in a world where data is getting more and more important, how does an investor use data to know which assets to choose? This is a difficult question to answer, especially when real-world data do not always follow a certain pattern and so datasets can contain outliers. The outliers can have a significant impact on the estimation procedure and so they can distort the estimation. On the other hand, the outliers can contain valuable information for the investor such as high returns after above-expectation performance of a company or low returns when a dip in the market occurs. Identifying these outliers is therefore crucial for the portfolio performance. This paper investigates which optimal ways there are to identify these outliers and uses robust estimation methods to minimize the impact of these outliers on the estimations to stabilize portfolio weights over time.

The robust methods are used to estimate the mean and covariance matrix which play a big part in the weight distribution. This paper does this for the low-dimensional and high-dimensional case. The robust estimators that are being considered for the low-dimensional case are the Minimum Covariance Determinant (Hubert and Debruyne, 2010) and the Minimum Volume Ellipsoid, which are robust against casewise outliers from the paper (Rousseeuw, 1985). These methods are later reviewed by (Hubert and Debruyne, 2010) and (Van Aelst and Rousseeuw, 2009). Furthermore, this paper looks at robust estimation techniques against cellwise outliers. For this purpose, the methods cellwise Minimum Covariance Determinant (Raymaekers and Rousseeuw, 2023) and the Detection Imputation method (Raymaekers and Rousseeuw, 2019) are considered. Lastly, this paper investigates the Two-Step Generalized method (Agostinelli et al., 2015). This method is robust against both casewise and cellwise outliers. For the high-dimensional case, the robust methods considered are the Orthogonalized Gnanadesikan-Kettenring method (Gnanadesikan and Kettenring, 1972) which is robust against casewise outliers and the Spearman correlation method (Öllerer and Croux, 2015) which is robust against cellwise outliers. The estimation of the mean and the covariance matrix is used in the Global Minimum Variance and the Tangency portfolio. The performance of these portfolios is measured based on return, turnover and value of wealth at the end of the trades and then compared to the benchmark. Which is the sample method. Combining all of this gives us the research question: *How do portfolio allocations perform using robust estimations methods for both casewise and cellwise outliers compared to traditional methods?*

This paper separates this research question into three different sub-questions. First of all, it looks at the robust estimations, why could they work, how they work and most importantly how they compare to the sampling method and each other. The second part this paper investigates is the estimation of the precision matrix. For most of the portfolio allocations, the inverse of the covariance is needed. For low-dimensional cases, this could be as straightforward as taking the inverse but also other methods have been popular to estimate the precision matrix. However, for high-dimensional data this gets more complicated because estimating the inverse normally is not feasible anymore. This paper explores the use of the estimated robust covariance matrix in the Graphical Lasso to overcome this problem.

The last part of this paper investigates the performance of different portfolio allocations. For this, it looks at the Global Minimum Variance portfolio and the Tangency Portfolio. This is done for the unconstrained portfolio but also when short selling is not allowed. The robust portfolio allocations from the paper (DeMiguel and Nogales, 2006) based on the M-estimator and the S-estimator are not considered because here covariance matrix does not need to be estimated and therefore it is outside the scope and purpose of this paper. For the portfolio allocations the main focus is on the stability of the weights. The performance of the weights is measured by the turnover. Low turnover is important for an investor because when a moving time window is used for the estimation the constant changing of the weights can cause extra transaction costs which in turn lowers the value of the portfolio.

This paper has found for the low-dimensional case that most of the robust methods did not improve the performance of the Global Minimum Variance portfolio in terms of weight stability and value of wealth. For the Tangency portfolio the casewise robust method did improve the portfolio based on weight stability and value of wealth compared to the sample method. For the cellwise robust methods this was not the case. Furthermore, the use of constraints did help stabilize the weights as the constraints promote shrinkage also considered in the paper (Jagannathan and Ma, 2003) bringing the turnover down. This resulted in a higher value of wealth for the investor. The use of constraints was particularly helpful for the tangency portfolio where the unconstrained portfolio showed to be too aggressive to be profitable for the investor. For the high-dimensional portfolio the weights were estimated by the OGK method and the Spearman's correlation method. These robust methods are based on pairwise covariances instead of Mahalanobis distances. Both these robust methods were also used for the low-dimensional case and did show improvement. Here the OGK method was the best-performing method. This shows that the robust methods based on the Mahalanobis are not fit to be used when it comes to portfolio allocation and weight distribution for time series data. Lastly in both the low-dimensional case and the high-dimensional case, the casewise robust methods did perform better than the cellwise robust methods. This was independent of whether the methods were based on Mahalanobis or pairwise covariances.

2 Relevance and Motivation

Portfolio strategies are very important in the financial world. Hedge funds, insurance companies but also individual investors all have different ideas about how to invest their money. However, they all have the same goal, making a profit. Using estimations of means and covariance matrices of the returns helps these investors obtain more insight into which stocks are best to invest in. This paper looks at these estimations and tries to investigate whether it is useful to make the estimators more robust.

The first aspect of this paper is the robustness of the location and scatter estimations. An advantage of robust estimations is they are less likely to be driven by shocks in the market. First of all the Minimum Covariance Determinant (MCD) already has a lot of applications for medicine, but also for finance. In the paper (Zaman et al., 2001) they found that the use of the MCD estimator in regression analysis improved the fit and the significance levels of the regressors. The other casewise robust method that this paper looks at is the Minimum Volume Ellipsoid which also has already some interesting applications. One of the applications was done in the paper of (Rousseeuw and Van Zomeren, 1990) where the MVE method was used to detect leverage points which also led to better regression results. This paper however looks at the mean and covariance matrix estimates and so would add another layer to the literature.

The cellwise robust methods are more recently published. For example, one of the methods this paper uses is the cellwise minimum covariance determinant from the paper (Raymaekers and Rousseeuw, 2023) which has only come out last year but offers so many possibilities in terms of dealing with outliers without impacting other data points. The other method that is used in this paper is the Detection Imputation algorithm which also has good possibilities for applications such as they did in the paper Raymaekers and Rousseeuw, 2019 where organic compounds of children were analyzed with estimation of the covariance matrix. These organic compounds are volatile so the DI Method helped with identifying outliers. Lastly, this paper looks at the Two-Step Generalized S-estimator which is a method that is both robust against casewise and cellwise contamination. This estimation method could improve the existing method which the paper (Agostinelli et al., 2015) pointed out that using methods where the focus is on one part of the contamination could lead to poor results and the 2SGS method would overcome this problem. The paper also suggests that the 2SGS has a lot of applications in principal component analysis and linear regression.

Both the casewise and the cellwise robust methods have good applications in the low-dimensional case. However, most papers use robust methods in regressions in combination with very static data. This paper however looks for a robust way of estimating the means and covariance matrix. The data are the daily returns of assets which creates a time series. The added value of this paper is to investigate what happens with these robust methods when they deal with time series data. The hypothesis is that these methods can create robustness in the portfolios which leads to more stability in the weights of the portfolio allocations. This would be beneficial for investors because it creates less risk for them as the magnitude of change in weights would be reduced. Furthermore, this could be helpful for the Global Minimum Variance portfolio but most certainly for the Tangency portfolio as the paper (Kirby and Ostdiek, 2012) found that this portfolio due to its scaling can be very aggressive in terms of weights and can create high turnover.

Secondly, this paper also looks at high-dimensional data. The use of high-dimensional data is very beneficial because of all the information that can be incorporated and the easy access to large amounts of data. However, it does make the estimations more difficult. Big hedge funds can afford to analyze these big data sets, however, for simple investors, this is more unlikely. There is also a lot of literature on high-dimensional robust methods. Some methods are extensions of low dimensional data such as is done for the Minimum Covariance Determinant in the paper (Boudt et al., 2020) where they found that with the new interpretation, the robustness was preserved. For the casewise robust methods this paper however focuses on the Orthogonalized Gnanadesikan-Kettenring method. This method is based on pairwise covariance instead of Mahalanobis distances as is the case for the extended Minimum Covariance Determinant. This method seems to be doing better than the one from the (Boudt et al., 2020) and it is also faster. Another approach for high-dimensional casewise robust estimation is the Stahel-Donoho estimator from the paper (Maronna and Yohai, 1995). The method has similar results to the OGK method according to the paper (Van Aelst et al., 2012) but the paper also points out that the Stahel-Donoho estimator can have difficulties in identifying the outlyingness of the contaminated observations when the majority of the observations are contaminated in at least one of the variables. Because of this, the OGK was favored for the casewise robust estimation. For high-dimensional cellwise robust estimators were also fewer papers because it only recently got some attention. For this paper, we look at a robust estimation method based on Spearman's correlation. This is, similar to the Orthogonalized Gnanadesikan-Kettenring method, based on pairwise estimation. The reason for using pairwise estimation for the high-dimensional case is because of a suggestion in the paper (Pacreau and Lounici, 2024) that methods based on the Mahalanobis distance, which the methods are in the low-dimensional case, have shown to be unstable in the results. They also found that this could also be the case for low dimensions. So one of the goals of this paper is to confirm or deny this claim for the low-dimensional case as the paper from Karim Lounici and Gregoire Pacreau has only been published this year.

Lastly, this paper looks at different portfolio asset selections and robust portfolio allocation. As mentioned in the introduction this does not consider the work done in the paper (DeMiguel and Nogales, 2006) where the portfolios are based on a M-estimator or a S-estimator. They do not estimate the covariance matrix and the means robustly but see the portfolio as one minimization problem and work this problem out in a robust way. So the lack of the estimation of means and covariance is the reason that their work is not discussed in the rest of the paper. The asset selection this paper uses is based on characteristics of the stock such as high means or low variance. In the paper (Van der Hart et al., 2003) they found that stocks with high value for earnings-to-price ratio or the book-to-market ratio perform better than stocks where the value is low. Another paper suggests that investors only look at the characteristics such as returns and volatility of the returns.

To summarize, this paper has both qualitative and quantitative analysis that looks for answers on which financial portfolio strategies to use but also why they work. These findings can also be helpful in other fields where the estimation of the location and scatter points is needed for time series data.

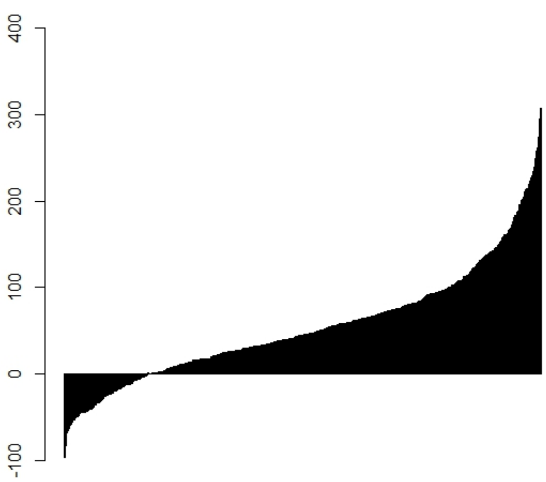
3 Data

The data I have gathered is daily stock data from the S&P 500. The data consists of the date, the open value, the high value, the close value and the volume of the trades from individual stocks between 8 February 2013 and 7 February 2018. In this paper, I am only interested in the daily returns of these stocks. To calculate the daily returns the open value is subtracted from the close value and divided by the open value. This creates daily returns based on percentages. For most of the individual stocks there are 1259 entries because trades are only happening on weekdays. There are however some stocks where the number of entries is lower than 1259 and thus have missing values. In total there are 505 stocks in the dataset where 470 stocks include the full time range of data, that is all the 1259 entries. For 35 stocks there are less than 1259 entries. These stocks were removed from the data set because this paper only wants to focus on the performance of the robust methods when outliers are present without the interference of missing values. Furthermore assigning zero returns to missing values is not a viable solution because this would also interfere with the covariance matrix and it thus would alter the conclusions drawn from this paper.

The S&P500 represents the 500 biggest American firms measured on their market capitalization. I have chosen the stocks of the S&P 500 index because they, as a collective, give a broad overview of the American stock market. Because the stocks are of large companies, the returns are generally less volatile compared to stocks of smaller companies or cryptocurrencies such as Bitcoin or Ethereum. The lower volatility leads to the out-of-sample results being more representative because with high-volatility stocks the performance of a method on a portfolio could be considered more random.

The mean is for 406 out of the 470 stocks positive. This is 86.38%. This however does not give a representative view for an investor because returns can be used cumulatively. Therefore the cumulative return of each of the stocks is more interesting to look at. The results of this are denoted in Figure 1. The number of stocks that have a positive return then is 385 out of the 470 which is 81.91%. The best-performing stock over these 5 years is BBY with a return of 307.25%. This is the stock for Best Buy, a retailer of electronic devices. The worst-performing stock is CHK which stands for Chesapeake. This stock had a return of -96.52 and eventually got delisted. However an investor never chooses just one stock but builds a portfolio, so lastly what would be the return if an investor had an equally weighted portfolio from 8 February 2013 until 7 February 2018 and did not touch the stocks? It turns out that the portfolio would have a return of 54.49%.

Figure 1: Cumulative returns of the 470 stocks



4 Methodology

This section goes over the methodology used in this paper. This section begins by discussing what robustness entails and which versions of robustness there are such as the casewise and cellwise robustness. Next, it covers the low-dimensional case where the estimation setup as well as the robust methods are discussed. This is also done for the high-dimensional case. Furthermore, the portfolio allocations and strategies used to create the portfolios are discussed. This part of the methodology also outlines the portfolio measurements used for method comparison. Lastly, this section dives deeper into the weight stability of the portfolios with the help of constraints.

4.1 What is robustness?

4.1.1 Casewise vs Cellwise outliers

One of the aspects this paper looks at is the difference between casewise outliers and cellwise outliers. It does this by looking at different estimators for location and scale. Before delving deeper into these estimators, we first explore the difference between casewise and cellwise outliers. Casewise outliers are often characterized by data generated from a clean distribution F with a probability of (1 - ε), combined with a contaminating distribution H with the probability of ε. So for the observed dataset X we get X = (1 - ε)Y + εZ. Where Y $\sim$ F and Z $\sim$ H. This representation is also known as the Tukey-Huber contamination model. The goal of this model is to estimate the characteristics of the distribution F while not assuming anything about H. Under this model, it is assumed that a case or row is coming from a perfect draw of the distribution of F or that the case is coming from an arbitrary distribution of H. As a result, the methods and estimators under this model often treat all entries in an individual case as contaminated. This could cause problems when there are situations where some of the entries in a case are contaminated and where other entries are not. In such a case regarding all entries as contaminated could lead to discarding valuable information from uncontaminated entries. This phenomenon has led to the consideration of cellwise outliers. The model for cellwise outliers assumes that the observed random variable X follows X = (1-B)Y + BZ, where B is a diagonal matrix with diagonal entries that can be either 0 or 1. This allows for some of the entries in the vector X to be contaminated while others are not.

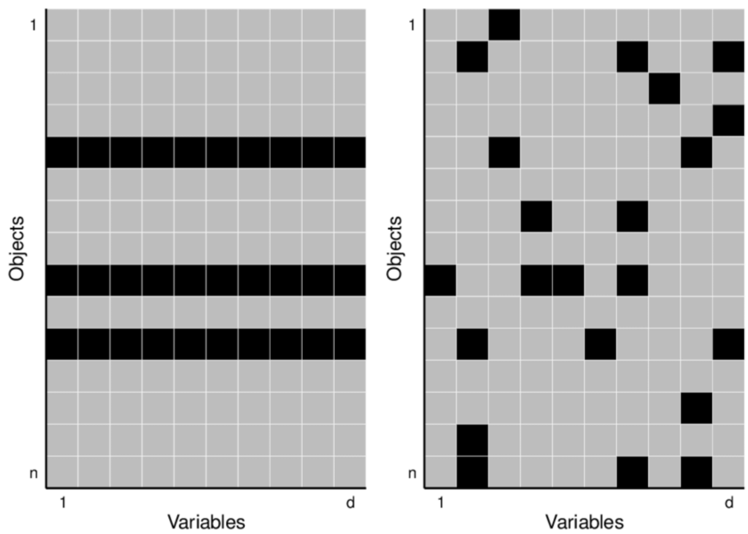


Figure 2: Casewise (left) vs Cellwise (right) outliers

Cellwise outliers are thus a different type of thinking about outliers. The most influential change was the fact that a small percentage of contaminated cells can contaminate a large fraction of cases or rows. The probability that at least one of the d entries within a case is contaminated is equal to

$$P[\text{row is contaminated}] = 1 - (1 - \varepsilon)^d. \quad (1)$$

This grows quickly with the dimension d. If d = 20 and the percentage of having an outlier (ε) is 5 percent then more than 64% of the cases would be contaminated. This leads to the Tukey-Huber contamination model no longer being reliable. In Figure 2 we can see the difference between casewise and cellwise outliers. The black squares indicate an outlier. The casewise outliers are all on the same row while the cellwise outliers are spread out over the matrix. If the casewise outlier method was used on the right figure then almost all rows would have been considered as contaminated. In this paper, we investigate with the help of portfolio allocations the different versions of these models and outliers.

4.1.2 Sensitivity Curve and Influence Function

The sensitivity curve measures the effect of changing one observation in a given dataset. The sensitivity curve is characterized by the following equation.

$$\mathrm{SC}_n(x, T) = \frac{T(X_1, X_2, ..., X_{n-1}, x) - T(X_1, X_2, ..., X_{n-1})}{1/n} \tag{2}$$

An influence function measures the small changes in the assumed distribution on the value of the estimator. The standard equation for the influence function is

$$\mathrm{IF}(z; T, F) = \lim_{\varepsilon \to 0} \frac{T(F_{\varepsilon,z}) - T(F)}{\varepsilon} = \frac{\partial}{\partial \varepsilon} T(F_{\varepsilon,z})|_{\varepsilon=0}. \tag{3}$$

Where the distribution $F_{\varepsilon,z} = (1-\varepsilon)F + \varepsilon\Delta(z)$ and $\Delta(z)$ is the probability for an outlier. One can also see the influence function as a measure of dependency on the value of any of the points in the sample.

To achieve robustness the influence function should often be bounded.

4.1.3 Breakdown point

The breakdown point of a certain method is a measure of robustness. The definition of the breakdown point is the smallest fraction of contaminated values in the sample before the method breaks down. A simple example is the mean of a sample. Let's say that we have 8 entries in the sample, (5,6,3,4,1,5,3 and 1). The mean of this sample is 3.5. However, if the 4 becomes a contaminated value and has a value of 120 then the sample mean increases to 18. This example shows that one contaminated observation can cause the mean to increase drastically. In theory, the mean of the sample including the contaminated value can be any arbitrarily high number. This is the breakdown of the sample mean. In this example, the breakdown occurs when one observation is changed. Thus the finite sample breakdown value is 1/8.

The formal definition of the breakdown value is that the finite sample breakdown of an estimator T_n at X_n is the smallest fraction of contaminated values $\frac{m}{n}$ for which the distance between $T_n(X_n)$ and $T_n(X_{n,m})$ might become arbitrarily large. The distance is measured by an appropriate choice of a distance equation D. Or in more mathematical terms:

$$\varepsilon_n(T_n; X_n) = \frac{1}{n} \min \left\{ m \in \{1, ..., n\} : \sup_m D(T_n(X_n), T_n(X_{n,m})) = +\infty \right\}. \tag{4}$$

If we want a method to be robust then a high breakdown point is preferred. From Equation 3 we can easily see that the maximum breakdown point is 1. However, such an estimator would not make a lot of sense. When we look at the maximum breakdown point of only sensible estimators then we have to look at them in another way. A breakdown point is often associated with a certain equivalence structure. When we have a univariate location, the requirement that the estimators should satisfy is $T_n(X_n + a) = T_n(X_n) + a$ for all $a \in \mathbb{R}$. This requirement is natural because of the desire that when the data is shifted then the shifted estimator of the center of the original dataset should be the estimator of the center of the shifted dataset. In the paper of (Hubert and Debruyne, 2009), it is proven that any location equivalent has a breakdown value of 0.5. Or more in mathematical terms:

$$\varepsilon_n(T_n; X_n) \leq \frac{1}{n} \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{5}$$

For the scale parameter, another requirement is used. This requirement states that $T_n(aX_n) = |a|T_n(X_n)$ for all $a \in \mathbb{R}$. It is also proven that with this requirement the maximum breakdown value is

$$\varepsilon_n(T_n; X_n) \leq \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor. \tag{6}$$

This robustness measurement is used as one of the tools to see whether these methods are robust.

4.2 Low-dimensional case

Firstly this paper focuses on the low-dimensional case. This means that only a small percentage of the 470 stocks available are used to build a portfolio. The low-dimensional case has its advantages. Firstly the estimation of the covariance matrix is easier and computationally feasible due to the relatively low amount of parameters. Furthermore, another advantage of the low-dimensional case is that the inverse of this covariance matrix is also easy to estimate. However, the estimation of the inverse covariance matrix can be quite difficult in higher dimensions. Keeping the dimensions low is also more appealing for the average investor. The low amount of stocks results in a more clear picture of their investment compared to if they would have invested in a high number of different stocks. Keeping the number of stocks low also comes with its disadvantages. The main disadvantage is the lower level of diversification in the portfolio. Less diversification of the portfolio could lead to larger variance in the portfolio returns which could lead to bigger losses. This consequence is investigated in the Results section.

This section first goes over the estimation setup that is used to get the results and then goes over the different methods that are used for the location and scatter estimates.

4.2.1 The Estimation Setup

The estimation setup for the low-dimensional case has a couple of different components. As mentioned in the data section, there are 470 different stocks with 1259 entries each. This paper uses daily returns for each stock where the daily returns are calculated as seen in Equation 7.

$$r_{it} = \frac{\text{CloseValue}_{it} - \text{OpenValue}_{it}}{\text{OpenValue}_{it}} \cdot 100. \tag{7}$$

Here r_{it} represents the return of stock i at time point t. These are percentage returns. I have chosen to multiply with 100 because it creates fewer values close to zero. This prevents the estimations from being inaccurate and it prevents also numerical problems which can happen if in a matrix a lot of values are close to zero.

The first component of the estimation is choosing the amount of stocks in the portfolio. For the low-dimensional case, I have chosen to include 15 stocks in the portfolio. A portfolio of 15 stocks is not too big so it is still considered a low-dimensional portfolio while also having the benefit of some diversification. Secondly, this paper uses partitioning of the data into subsets to create a window to estimate from. In this case, I have chosen to partition the 1259 data entries into 14 subsets of 90 entries each, where the 14th partition consists of 89 entries. This choice is made for two reasons, first of all, the number of entries in a subset should be large enough for the estimations to be accurate. This is especially true for the robust methods. On the other hand, too many entries in a subset leads to a smaller number of subsets and so the analysis there would be fewer points to analyze the portfolio weights. Furthermore, it is in a lot of papers such as (Hubert and Debruyne, 2010) suggested that the number of entries should be more than 5 times the number of parameters for the robust low-dimensional methods to work. Using 90 entries and 15 assets follows this suggestion while also having enough subsets to do the weight analysis. For the estimation, the out-of-sample results are used. Out-of-sample means that the weights are estimated at time t and used at time t+1. Then weights are estimated at time t+1 and used at time t+2. Lastly, while the location and scatter estimates are done on the whole subset the return of the portfolio is done by using Equation 7 at the beginning and end of the subset. So this essentially calculates the 90-day return of each asset instead of daily.

4.2.2 Heteroskedasticity of the returns

Before we dive deeper into the estimating procedures of the means and covariance matrix, this paper first discusses the heteroskedasticity of the returns and the relation it has with robustness. Returns often possess the characteristic of heteroskedasticity, meaning that the variances of the returns are not constant over time. The paper of (French et al., 1987) found that the daily returns of the S&P500 confirm this characteristic of heteroskedasticity and so the variance of the returns is time-varying. Heteroskedasticity often is caused by outliers or missing values in the dataset. The heteroskedasticity in the returns causes a lot of problems in the estimation of the parameters. This is especially apparent in the regression estimation because the Ordinary Least Squares (OLS) estimator is with the existence of heteroskedasticity, not a minimum variance estimator anymore. To overcome this problem other methods such as the Weighted Least Squares (WLS) are created. Another way to help this problem is to transform the data. For example, taking the natural logarithm can help with heteroskedasticity. Furthermore, a model can be used that takes the time-varying variance into account such as a GARCH model. This paper focuses on the robust estimations method which can also help against heteroskedasticity due to it being able to spot these outliers and minimize their effect. The use of the robust method therefore causes the estimated returns to become closer resulting in less heteroskedasticity.

4.2.3 Estimating the means and the covariance matrix

One of the most important aspects of choosing a portfolio allocation is the estimation of the means and the covariance matrix. This is because the mean and the inverse of the covariance matrix are often used in the portfolio allocation as can be seen in Section 4.4. Estimating these components can be done in multiple ways, and this paper shows different approaches and compares them. This section only focuses on the low-dimensional case. The high-dimensional case is later introduced. For the low-dimensional case, we investigate two casewise estimation methods, two cellwise estimation methods and one method that is suited for both. The performance of these methods is compared to the sample method in the results section.

Minimum Covariance Determinant (MCD)

The first method this paper uses to obtain robust location and scatter estimations is the minimum covariance determinant (MCD) method. The MCD method is one of the first highly robust estimators. This estimation method is robust against casewise outliers. The main idea of the MCD method is to detect these outliers in a data set and disregard them in the estimation procedure. The method uses an eclipse to encompass all the regular data points. The classical approach to encompass an ellipse is to assign each point a Mahalanobis distance value. These values are calculated by

$$MD(x) = \sqrt{(x-\bar{x})^t \Sigma^{-1} (x-\bar{x})}. \tag{8}$$

The Mahalanobis distance gives us an indication of how far the points are from the center. Here $\bar{x}$ is the sample mean and Σ is the sample covariance matrix. This method however tries to include all observations which leads to a bigger eclipse. The MCD method on the other hand does it slightly differently. This version is more robust because it ensures that the eclipse is much smaller and captures only the regular data points. As a result, the equation slightly changes to

$$RD(x) = \sqrt{(x-\hat{\mu}_{MCD})^t \hat{\Sigma}_{MCD}^{-1} (x-\hat{\mu}_{MCD})}. \tag{9}$$

Here $\hat{\mu}_{MCD}$ and $\hat{\Sigma}_{MCD}^{-1}$ are the location and scatter estimates of the MCD. The number of outliers detected in the MCD case usually is greater than when the Mahalanobis distance values are considered. This phenomenon is called the masking effect and it states that classical estimation methods are highly affected by outliers such that normal tools as the Mahalonobis distances can not identify them as outliers anymore. Here is where the robust estimation techniques are crucial.

To compute the robust MCD location and scatter estimates one looks at the subsets with h observations and see which has the lowest determinant. The value of h should be between $\lfloor\frac{n+p+1}{2}\rfloor$ and n. Here n is the number of objects and p is the number of parameters. In this paper, n is the estimation window and p is the number of assets being considered. The choice of h impacts the robustness and the efficiency of the estimates. It is easier to see the differences in choices of h when we look at $\alpha = \lim_{n\to\infty} h(n)/n$. This results in that α can be a value between 0.5 and 1. The estimates of the MCD are most robust when h = $\lfloor\frac{n+p+1}{2}\rfloor$. This translates to $\alpha = 0.5$. However, doing this results in low efficiency. Choosing α to be for example 0.75 can increase the efficiency but then the estimates are less robust to casewise outliers. However, to increase efficiency without impacting on the robustness a slightly different version is presented where the method re-weights the estimates (Lopuhaa, 1999). The estimates then become as seen below, Here c_1 is the consistency factor, d_i the distance value, and the W is a weight function:

$$\hat{\mu}_{MCD} = \frac{\sum_{i=1}^{n} W(d_i^2)x_i}{\sum_{i=1}^{n} W(d_i^2)} \tag{10}$$

$$\hat{\Sigma}_{MCD} = c_1 \frac{1}{n} \sum_{i=1}^{n} W(d_i^2)(x_i - \hat{\mu}_{MCD})(x_i - \hat{\mu}_{MCD})' \tag{11}$$

The distance value $d_i = \sqrt{(x - \hat{\mu_0})'\hat{\Sigma_0}^{-1}(x - \hat{\mu_0})}$ and the weight function is often chosen to be $W(d^2) = I(d^2 \leq \chi^2_{p,0.975})$. This cutoff value is however relatively conservative and so often flags too many observations as outliers. They argue that the true distribution of the distance values can better be approximated by an F-distribution. The singularity of the covariance matrix is often not a problem because it requires h to be greater than p. This is achieved when n $\geq$ 2p. To avoid the curse of dimensionality it is suggested to use n > 5p. In our case, which is also discussed in the estimation set-up, n is 90 and p is 15 and thus adheres to this suggestion.

To exactly estimate the MCD estimates is hard because it requires all the $\binom{n}{h}$ subsets to be evaluated. This is computationally very demanding. Because of this, the choice is often made to look at the FastMCD algorithm from (EStimator, 1999). This algorithm is efficient and uses the C-step. However, this paper uses the DetMCD for the estimation from the paper (Hubert et al., 2012). This is because the FastMCD uses random subsets and so it is not permutation invariant. This caused problems when getting results while coding because identical portfolios had for each iteration different results. Not only is the DetMCD permutational invariant, but it is also faster than the FastMCD. The function *DetMCD* from the R package *DetMCD* is used for the coding where the alpha is equal to 0.75.

Minimum Volume Ellipsoid
Another method to get the location and scatter estimates is the Minimum Volume Ellipsoid (MVE). The MVE method is also a high-breakdown robust estimator similar to the MCD estimator. Where the MCD estimator minimized the determinant of the covariance matrix, the MVE minimizes, as the name suggests, the volume of the ellipsoid. In more mathematical terms, the MVE estimates the location estimator r_n and the scatter estimate C_n of the dataset X_n. It does this by minimizing the volume of the ellipsoid consisting of at least h observations from X_n, where h can be chosen between $\frac{n}{2}+1$ and n. The algorithm minimizes with respect to the following constraint

$$[(r_i - t)'C^{-1}(r_i - t) \leq c^2] \geq h. \tag{12}$$

The value of c determines the magnitude of C_n and is a chosen fixed constant. The value is usually chosen to be $\sqrt{\chi^2_{p,\alpha}}$, where $\alpha = \frac{h}{n}$, because this assumes that C_n is a consistent estimator of the covariance matrix for data coming from multivariate normal. The assumption of normality of stock data comes from the idea that prices follow from an efficient market and investors make rational decisions. In reality, however, this assumption is questionable because investors are not rational and the market is not always efficient. Despite this, I assume that it holds C_n. We go over the algorithm, the breakdown value, and any other properties of this method. This method is from the paper (Van Aelst and Rousseeuw, 2009).

First of all the algorithm for the Minimum Volume Ellipsoid location and scatter estimates is very similar to the MCD method. The MVE also uses subsets that consist of p+1 observations. For each subset the sample mean and sample covariance is estimated.

$$\bar{r} = \frac{1}{p+1}\sum_{i=1}^{p+1} r_i \tag{13}$$

$$S = \frac{1}{p}\sum_{i=1}^{p+1}(r_i - \bar{r})(r_i - \bar{r})' \tag{14}$$

If and only if the subset is in general position, meaning that none of the observations in the subset lie on a hyperplane, then the sample covariance matrix is non-singular. However, if this is not the case then observations are added such that the sample covariance becomes non-singular. The ellipsoid resulting from the estimation is proportional to

$$[\det((D_j^2/c^2)S_j]^{1/2} = (D_j/c)^p \det(S_j)^{1/2} \tag{15}$$

Here $c^2 = \chi^2_{p,a}$ and $D_j = [(r_i - \bar{r}_j)'(S_j)^{-1}(r_i - \bar{r}_j)$. The algorithm finds the solution where Equation 15 is the smallest among the (p+1)-subsets. The number of observations in a subset, h, is often chosen to be $\frac{n+p+1}{2}$.

The MVE method has similar properties compared to the MCD method. First of all, they are both affine equivariant meaning that the estimators behave properly under affine transformation of the data. Furthermore, the finite breakdown value of the MVE denoted in paper (Donoho and Huber, 1983) can be seen in Equation 16

$$\varepsilon_n^*(t_n, X_n) = \varepsilon_n^*(C_n, X_n) = \frac{\min(n-h+1, h-p)}{n}. \tag{16}$$

Here we can see that if n goes to infinity then we get that the breakdown value of the MVE estimator becomes $\min(\alpha, 1-\alpha)$ where $\alpha = \frac{h}{n}$. This is what we have seen above for the square root of the chi-squared value.

This paper uses for the code the R package *rrcov* where the function *CovMve* is used. Here the parameters are $\alpha = 0.75$ and the nsamp is on default 500. The choice of the α is made to line up with the rest of the models and the number of samples stayed at default because it gives a good robust estimate while the computation time is still manageable. Having a higher nsamp does ensure more robustness but this does not always equal better return results.

Cellwise Minimum Covariance Determinant method
The cellwise method of the minimum covariance determinant is a method robust against cellwise outliers. This method, introduced in the paper (Raymaekers and Rousseeuw, 2023) is very new but works well in the presence of these cellwise outliers. In the paper, they found that this method has better efficiency than the Detection-Imputation method with the same robustness as well as having better robustness than the 2SGS method mentioned below. The drawback is that this method has a significantly higher computation time than the casewise minimum covariance determinant. The rest of this section goes broadly over the algorithm and the properties of this method. More details can be found in the paper (Raymaekers and Rousseeuw, 2023).

Like the minimum covariance determinant for casewise outliers this method minimizes an objective function. In this case, the objective function to minimize is:

$$\sum_{i=1}^{n} L(x_i, w_i, \mu, \Sigma) = \sum_{i=1}^{n}(\ln|\Sigma^{w_i}| + d^{w_i}\ln(2\pi) + MD^2(x_i, w_i, \mu, \Sigma)). \tag{17}$$

Here $x_i^{w_i}$ is the vector with only the entries where $w_{ij} = 1$. This goes the same for $\mu_i^{w_i}$. Σ^{w_i} consists of only the columns and rows where $w_{ij} = 1$. Lastly d^{w_i} is the dimension of $x_i^{w_i}$. The MD denotes the

Mahalanobis distance equation :

$$MD(x_i, w_i, \mu, \Sigma) = \sqrt{(x_i^{w_i} - \mu_i^{w_i})'(\Sigma^{w_i})^{-1}(x_i^{w_i} - \mu_i^{w_i})}. \tag{18}$$

The cellMCD method has similar characteristics as the casewise MCD. Just as the casewise MCD the cellwise MCD has high breakdown properties. Using the work of (Donoho and Huber, 1983) the finite-sample cellwise breakdown value of the location estimator is equal to Equation 19

$$\varepsilon_n^*(\mu, X_n) = \min\{\frac{m}{n} : sup_{(X^m)} \, ||\mu(X^m) - \mu(X)|| = \infty\}. \tag{19}$$

The paper (Raymaekers and Rousseeuw, 2023) has found that the breakdown value of the cellwise MCD location estimator is always smaller or equal to the breakdown value of the casewise MCD location estimator. This means that all the upper bounds of the casewise breakdown values also hold for the cellwise breakdown value. For the breakdown value of the scatter estimate from the paper (Raymaekers and Rousseeuw, 2024) sees that if the dataset X is in general position and h $\geq \frac{n}{2} + 1$ then we get that $\varepsilon_n^*(\Sigma, X_n) = \frac{n-h+1}{n}$. So cellMCD is also highly robust.

The cellMCD is implemented using the *cellwise* package in R. The robust location and scatter estimates are estimated using $\alpha = 0.75$ and a cutoff value of 0.99.

Detection Imputation Algorithm
The Detection Imputation algorithm from paper (Raymaekers and Rousseeuw, 2019) constructs robust cellwise location and scatter. The method is based on an iterative two-step process. These are the detecting and the imputation steps. Before these steps can be used, the data is first normalized and initial estimates of location and scatter are chosen. This paper uses the initial estimates of the DDCW estimator. This estimator is derived from the DDC algorithm of (Rousseeuw and Bossche, 2018) and can be found in (Raymaekers and Rousseeuw, 2019) Another option is using the 2SGS method, but this paper did not look at this as an initial estimate because it already looks at this method separately. Furthermore, the DDCW estimator is faster in computation.

In the detection step, all rows are gone through to look for outlying cells. It does this by using the cellHandler method for each of these rows. Doing this with the location and scatter estimates of the previous step. The cellHandler method ranks each cell in its respective row. If any of the cells are missing then they get a value of infinity. Problems could arise when more than too many cells in a column are flagged because then the correlation would be difficult to estimate. Because of this, the DI estimator sets a number of maximum flagged cells per column. This paper uses the default of 25%. The Imputation step uses the Expectation Maximisation algorithm where the flagged cells are set as missing. The E-step of EM does not require extra computation because the flagged cells are one of the sets considered by LAR in the cellHandler. The next iteration of the location and scatter estimates are computed in the M-step. The algorithm stops when the difference between two consecutive locations and scatter estimates is small.

Code is used from the R package *cellwise* where the function *DI* is used with the default parameters.

2SGS Method
The last method this paper uses for the low dimensional estimation is the 2SGS method. This method is currently the best robust method to deal with both the casewise and cellwise outliers. This feature sets it apart from the rest of the methods discussed as they can only deal with one type of outlier. 2SGS stands for two-step generalized S-estimator. The method is first mentioned in the paper (Agostinelli et al., 2015). As the name suggests this method consists of two steps. The section goes broadly through these steps and gives some reasons why the 2SGS could be a good method to use.

The first step of the 2SGS method is identifying the cellwise outliers based on their Mahalanobis distance and setting them to NA, this is called snipping (Farcomeni, 2014). This step makes sure that there are no large robust Mahalanobis distance values in the second step. The method flags the cellwise outliers by using the Gervini-Yohai univariate filter. The filter standardizes a variable based on a distribution.

In a perfect world, this is the actual distribution of the variable but for stock returns, this is not known so the assumption is to take the standard normal distribution. When the variable is standardized then an adaptive cutoff value is introduced to flag the cellwise outliers. In the paper (Agostinelli et al., 2015) they claim that the filter does not wrongly flag outliers, even when the actual distribution is unknown. This is however the case when the tail of the reference distribution, in this case the standard normal, is heavier than the actual distribution. Because stock data is not normal and often has fatter tails this can cause problems that outliers are wrongly flagged. The second step uses the Generalized S-estimator to deal with the casewise outliers. The GSE is a method equipped to overcome missing values in datasets. In this case, the missing values are the outliers set to NA from the first step. The full details of the GSE can be found in the paper (Agostinelli et al., 2015) but what the GSE essentially does is using an auxiliary dataset in combination with the Tukey bisquare loss function $\rho(t)$ to minimize an objective function.

Lastly, why would you use the 2SGS. As said above the two generalized S-estimator is both robust against cellwise and casewise outliers. In the paper (Agostinelli et al., 2015) they also looked at other methods and compared them to the 2SGS method. Methods included MCD, MVE but also other methods that are not used in this paper. What they saw was that the 2SGS method was the best-performing method when it came to cellwise outliers and had also a good performance for casewise outliers while the other methods only had good performance in one of the two sorts of outliers. The 2SGS has thus the advantage that it does not matter what kind of outliers in the dataset are.

4.3 High-dimensional case

4.3.1 The Estimation Setup

This section covers the estimation set-up this paper uses for the high-dimensional case. All the characteristics of the data are the same as discussed in Section 4.2.1. There are still 470 stocks with each having 1259 entries of daily returns. This paper's focus for the high-dimensional case is to look at the out-sample results, also stated in Section 4.2.1 in more detail. Lastly the calculations of the portfolio performance statistics, for example, the portfolio turnover, also stay the same.

For the high-dimensional case, the number of stocks in the portfolio goes up, but the rolling window of 90 entries per stock stays the same. This makes it high-dimensional as now the number of assets in the portfolio is greater than the number of entries in the rolling window. The number of stocks in the portfolio this paper analyses is 94, 188, 282, 376 and the full 470 of the data. These are 1/5, 2/5, 3/5, 4/5 and 5/5 of the dataset. This also gives insights into the behavior of the performance from the portfolios when the number of stocks increases in a high-dimensional portfolio. The assets of the portfolio are again chosen by the three characteristics that were used for the low-dimensional case. For the low-dimensional data, only the location and scatter estimates were enough to get the weights of the portfolio allocations. However, with high-dimensional analysis, some additional estimations have to be done. The estimated covariance matrix cannot easily be inverted and estimation techniques for the inverted covariance matrix have to be used. This is covered in Section 4.3.3.

4.3.2 Estimation of the High Dimensional Means and Covariance Matrix

Orthogonalized Gnanadesikan-Kettenring estimation
One of the ways to estimate a covariance matrix with high dimensional data is the Orthogonalized Gnanadesikan-Kettenring estimation, also known as the OGK estimation. This method, introduced in the paper (Gnanadesikan and Kettenring, 1972), is a casewise robust estimator. The method uses pairwise covariances to create a robust covariance matrix. The OGK estimation consists of 5 steps. The first step is choosing a robust univariate estimator for the location and scale estimate. This paper uses the Qn estimator for estimating location and scale. The second step involves using $y_i = D^{-1}r_i$ where D equals a diagonal matrix with the Qn scale estimator of each r on the diagonal. Here r_i are the returns of individual assets. Step 3 is computing the pairwise covariance matrix from the following equation

$$\Sigma_{x,y} = (\hat{\sigma}(x+y)^2 - \hat{\sigma}(x+y)^2)/4. \tag{20}$$

Where $\hat{\sigma}$ is the Qn scale factor. This pairwise covariance matrix is symmetric however, it does not guarantee positive definiteness. To guarantee positive definiteness the second to last step computes the eigenvectors of Equation 20 and then projects them on the data Y. This results in $\hat{\mu}(Y) = Em$ where E is the matrix of eigenvectors and m equals to a vector of the projections using the location estimate. We also get the positive definite matrix $\hat{\Sigma}(Y) = E\Lambda E^T$ where Λ is a diagonal matrix where on the diagonal the scale estimator squared of the projection is displayed. The last step involves defining the OGK estimates:

$$\hat{\mu}_{OGK} = D\hat{\mu}(Y) \tag{21}$$

$$\hat{\Sigma}_{OGK} = D\hat{\Sigma}(Y)D^T \tag{22}$$

One of the advantages of the OGK estimator is that it is much faster in high dimensions than other robust methods such as the Fast MCD and the Stahel–Donoho method. Furthermore the paper (Maronna and Zamar, 2002) found that the OGK estimator performs significantly better than the FMCD and is at a similar level of performance compared to the Stahel-Donoho method. This paper uses for the coding the already built OGK function in the package *robustbase* in R.

Spearman's Correlation
The other robust method this paper uses to estimate the covariance matrix makes use of Spearman's correlation. This method, from the paper (Spearman, 1961) also employs a pairwise approach, avoiding the need for assumptions about the clean covariance estimate or the tail distribution of the outliers. Contrary to the OGK method discussed in the section above, this method is robust against cellwise

contamination. This makes it a perfect candidate to compare it to the OGK method and the sample method. The pairwise approach has several advantages. Firstly, the computation complexity is relatively low and secondly, it has a high breakdown value for the covariance estimate. The covariance is estimated following Equation 23.

$$\Sigma_{ij} = \hat{\sigma}_i \hat{\sigma}_j \hat{\rho}_{ij}. \tag{23}$$

The sigma is estimated using the MAD. The MAD is great to use because it also has a high breakdown. To also make the estimator consistent for normal distribution the MAD is multiplied by the constant $(\Phi_{-1}(0.75))^{-1}$. This brings the full estimate of to $\hat{\sigma}_i = (\Phi_{-1}(0.75))^{-1} \cdot \hat{m}_i$, where $\hat{m}_i$ is the MAD of the respective asset. The $\hat{\rho}_{ij}$ is estimated by the Spearman's rho correlation. We know from the papers (Kendall, 1948) and (Kruskal, 1958) that when there is no contamination the following statements holds: $\rho_{ij} = 2sin(\frac{\pi}{6}\rho_{ij}^S)$. So our robust estimation of the covariance matrix is defined as

$$\hat{\Sigma}_{ij}^S = \hat{\sigma}_i \hat{\sigma}_j 2\sin\left(\frac{\pi}{6}\hat{\rho}_{ij}^S\right). \tag{24}$$

The only drawback to this approach is that it does not specify how to achieve a robust estimate of the location. The paper (Loh and Tan, 2018) suggests using a coordinate-wise approach however this paper uses the medians of the assets to get a robust location estimate. This assumption is valid because when the sample size is greater than 25 the median can be used as an estimate for the means (Hozo et al., 2005). Since the sample size for each asset is 90 the median can be used.

4.3.3 Estimation of the Precision Matrix

For many portfolio weight allocations the inverse of the covariance matrix, also known as the precision matrix, is required instead of the covariance matrix itself. In the low-dimensional case, taking the inverse is not a problem because it can be done computationally. However, it becomes a problem in the high-dimensional case.

One of the ways to solve this problem is by using the graphical lasso. The graphical lasso is a penalized maximum likelihood estimator that promotes shrinkage. This is done by the following minimization problem:

$$\Sigma^{-1} = \Omega = \text{arg min}_{\Omega}(-\log(\det(\Omega)) + \text{tr}(S\Omega) + \lambda||\Omega^{-\text{diag}}||_1 \quad \text{s.t.} \quad \Omega = \Omega^T, \Omega \succ 0). \tag{25}$$

Here S denotes the sample covariance matrix, $\Omega \succ 0$ means that we only consider positive definite matrices and $\lambda > 0$ is a penalty term that controls the degree of sparsity. This method has been shown to perform poorly in the presence of outliers because it relies on the sensitive estimation of the sample covariance matrix. However, in the paper by (Louvet et al., 2023), it was demonstrated, with the help of the influence function, that the Graphical Lasso can be used when the covariance matrix estimation is robust. In other words, the influence function of the Graphical Lasso was shown to be bounded when a robust plug-in estimate for the covariance was used. This means that the sparsity of the solution is not affected by the contamination and so the covariance matrices of Section 4.3.2 can be used.

The Graphical Lasso does however rely on a penalty term λ which has to be chosen by the researcher. Selecting this λ comes with a trade-off between higher likelihood values for the likelihood and the shrinkage of the precision matrix. There are different methods to come up with the optimal value of this parameter. One of the most common ones is the use of the Bayesian Information Criterion (BIC). The BIC for a precision matrix has been given in Equation 26, where S is a robust covariance matrix. This could be cellwise or casewise. Furthermore if $(\hat{\Omega}_\lambda)_{ij} \neq 0$ then $\hat{e}_{ij} = 1$ and 0 otherwise. This paper computes the value of the BIC for different values of λ and chooses the λ where the BIC is the lowest.

$$BIC(\lambda) = -\log\,\det\hat{\Omega}_\lambda + \text{tr}(\hat{\Omega}_\lambda S) + \frac{\log n}{n}\sum_{i\leq j}\hat{e}_{ij}(\lambda) \tag{26}$$

This paper uses the built in R package *glasso* to estimate the Graphical Lasso precision matrices.

4.4 Portfolio allocations & Strategies

This section discusses various portfolio allocations and strategies. Most of the allocations use the covariance or precision matrix estimated by the methods discussed earlier. The portfolio allocations and strategies discussed in this section are both applied to the low-dimensional and the high-dimensional case. The casewise outliers are compared to the cellwise outliers based on returns, standard deviation but also weight stability and turnover. As a benchmark, the sample techniques for the location and scatter estimates are used. This is to examine whether and how the techniques impact returns and turnover.

This paper looks at the Global Minimum Variance portfolio and the Tangency portfolio as the portfolio allocations. Furthermore, it investigates the use of constraints in the portfolio. In particular the use of no short selling constraints. The performance of the portfolios is captured by the characteristics of the returns such as the mean, standard deviation, median as well the 4 quantiles. Additionally, the performance of the weights is measured by turnover.

4.4.1 Asset Choice Strategy

The first step for any investor is to choose the assets for the portfolio. Investors often consider the mean and volatility of returns when making this choice, where high returns and low volatility are most desired. This paper suggests four different ways based on these characteristics to choose the assets in the portfolio. The first method is the most widely known for assessing the performance of an asset, which is the Sharpe ratio. The Sharpe ratio, first introduced in the paper (Sharpe, 1966), is the mean of the returns divided by the standard deviation. The second way of choosing the asset is the Sortino ratio. The Sortino ratio is similar to the Sharpe ratio but only takes the deviation of the returns below the target return into account. In this case, the target return is set to zero so this ratio only takes the deviation in the negative returns into account. The Sortino Ratio has been shown to be a good performance measure of an asset. They found in the paper (Rollinger and Hoffman, 2013) that this is true especially when the distribution is positively skewed. In addition to these two ratios, this paper also explores another method for selecting assets for the portfolio. This method uses high means as a way of selecting assets for the portfolio. This comes from the fact that momentum plays a role in the asset market. In the paper (Grundy and Martin, 2001) they showed that momentum profits are stable and predictable. The use of assets with the highest means hopefully can capture this momentum factor to have higher returns. Once the assets have been chosen, this paper investigates two strategies.

Choose and hold strategy
The first strategy is choosing the assets for the portfolio based on one of the three characteristics in the first time period and holding these assets until the end. This would mean that only the weights of the assets would change for each time period instead of the assets themselves. This strategy has some advantages such as that the turnover is likely to be lower because the assets in the portfolio do not change over time. However, not changing the assets can also impact the performance negatively because the choice of assets for the entire time period is based only on the first time interval. It is highly unlikely that the best assets in the first time period are also the best throughout the whole period. This could likely result in lower returns and so the question would be whether the low turnovers are more beneficial than the the low returns are harmful.

For the low-dimensional case, the portfolio contains 15 assets. This portfolio is constructed with 15 assets that performed the best out of one of the three characteristics. This creates 1 portfolio. Furthermore, portfolios are created by taking the 8 assets that performed the best and taking every combination of 7 assets out of the 9th until the 19th best-performing assets. This creates 330 different portfolios. The performance of these portfolios is also measured. For the high-dimensional case only the five portfolios where the number of assets is 94, 188, 282, 376 or 470 are investigated.

Choosing each time period stategy
The second strategy this paper is considering is the strategy where the portfolio changes throughout the time periods. The assets for each time period are ordered based on one of the three characteristics. For example in t=4 the assets are ordered based on the highest Sharpe ratio and then the portfolio weights

are estimated which are used in t=5. The advantage of this strategy is that the information on which the portfolio is based is more recent. Recent information should lead to better returns and thus higher portfolio performance. However, the constant changing of the assets in the portfolio could also lead to higher turnovers. If this happens then the performance of wealth can be influenced.

Just as for the first strategy, in the low-dimensional case different portfolios are created. The first portfolio is the one where each time period the 15 best-performing assets based on the characteristics are chosen. This gives one portfolio. But also for this strategy, 330 portfolios are created consisting of the 8 best-performing assets and all combinations of 7 assets out of the 9th to 19th best-performing ones. For the high-dimensional case, similar to the *choose and hold* strategy only the five portfolios are investigated.

4.4.2 Standard Portfolio Allocations

Global Minimum Variance portfolio

The Global Minimum Variance (GMV) portfolio is the portfolio that is formed when we solve the objective function: min $w'\Sigma w$ with the constraint of $w'\iota = 1$. Using Lagrange to solve this problem, we obtain the weights of the GMV portfolio, $w_{GMV} = \frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota} = \frac{\Omega\iota}{\iota'\Omega\iota}$ where Ω is the precision matrix. The GMV portfolio thus minimizes the variance without considering the portfolio return.

The advantage of using the GMV portfolio is that it is resilient during market turbulence due to the low volatility of the portfolio. However, the downside is that the returns can be low because portfolio minimization only considers risk. Another advantage of this portfolio allocation is only the use of the inverse covariance matrix. Because of this, the estimation is reduced compared to if the mean estimate was also to be used. A disadvantage is that as mentioned above this allocation only cares about minimizing the risk without looking at the return. This can cause a portfolio to be sub-optimal for an investor if the returns are low or even negative whereas if a little bit more variance was allowed could have led to much better portfolio gains.

To compute the global minimum variance portfolio, the function *globalMin.portfolio* of R package *IntroCompFinR* was used. The arguments are the location and scatter estimate of the method in question and the last argument is *TRUE* when shorts are allowed and *FALSE* if not.

Tangency portfolio

The second portfolio this paper investigates is the Tangency portfolio. This portfolio uses excess returns and so in this paper, the risk-free rate is set to zero. The weights are computed by the following equation:

$$w_{TAN} = \frac{\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\mu}. \tag{27}$$

The tangency portfolio has shown to be less favorable due to its high turnover. This is also shown in the paper (Kirby and Ostdiek, 2012). This paper suggests that this is the case because of the scaling with $\iota'\Sigma^{-1}\mu$. Additionally, incorporating μ in the weights introduces more estimation error. The tangency portfolio is an efficient portfolio that is fully invested in risky assets. The objective function of the problem stays the same as does the constraint that the sum of the weights should be equal to one. However the constraint for the target return changes to $w'\tilde{\mu} = \tilde{\mu}_p$ where $\tilde{\mu}$ represents the excess returns.

4.4.3 Portfolio Performance

This section discusses the performance characteristics that this paper uses to evaluate portfolios. Firstly for the lower dimensional case, there are two different portfolio approaches. For the approach where only the 15 best-performing assets are chosen the characteristics that are looked into are the return of the portfolio, the turnover of the portfolio and the wealth after using the portfolio. For the approach where 330 portfolios are created, the analysis is much deeper. Here the median, median absolute deviation of the returns as well as the turnover and the value of wealth are considered. Furthermore, to get a better understanding of the density of the value of wealth the quantiles are also considered. Additionally, this paper looks at how many portfolios of a robust method beat the portfolios of the sampling method and lastly the number of portfolios where money is made. This paper uses the median as a measurement because it is robust and more reliable than the mean. The value of the mean can be significantly impacted by one extreme positive or negative value and this can cause wrong conclusions. Furthermore, it looks at the wealth of the portfolio, because a strategy with lower returns but also lower turnover could have better net performance than a strategy where the returns and the turnover as higher. The value of wealth is calculated using Equation 28, also used in the paper (DeMiguel et al., 2009). Here the start capital is equal to 100 and then the wealth is updated for each time period where in the end at t=13 the final wealth is displayed. The percentage of transaction costs is denoted by c. This is set at 1 percent because of the article of (Downey, 2024) where it is stated that 1% often is the commission of stock trading. Since S&P500 companies are often traded, 1 percent seems like a good choice. Equation has however a drawback and that is when the the returns become extremely negative and go below -100% the wealth equation has a harder time getting accurate results. This occurs due to short selling, where there is no limit on the maximum loss. For the GMV portfolio, this is probably not a problem because the weights are generally not that extreme but for the tangency portfolio, this could pose some problems if no short selling is allowed. In the results section, we see if this is the case. It's important to note that these issues affect the performance and value of wealth but do not impact the weights or the turnover. So conclusions taken from these portfolios based on the turnover are still valid.

$$W_{t,k} = W_{t-1,k} \cdot (1 + R_{t,p,k}) \cdot (1 - c \cdot \sum_{j=1}^{N} (|w_{k,j,t+1} - w_{k,j,t}|) \tag{28}$$

The returns are calculated by using the weights obtained in t and using them in the portfolio for t+1. The returns for each time period are the difference between the close value of the asset from the last element in a certain time period and the open value from the asset at the beginning of the same time period. This is also shown in Equation 29 where $R_{p,k}$ is the return of the portfolio for combination k.

$$R_{p,k} = \sum_{t=1}^{T-1} w_{t,k} r_{t+1,k} \tag{29}$$

Furthermore, to compare the portfolios for transaction costs this paper uses turnover as a measurement. Turnover is calculated using the following equation:

$$\frac{1}{T-M} \sum_{t=1}^{T-M} \sum_{j=1}^{N} (|w_{k,j,t+1} - w_{k,j,t}|). \tag{30}$$

The equation measures the weight stability of the portfolio. It is also mentioned in the paper (DeMiguel et al., 2009). The $\frac{M}{T-M}$ is disregarded in the Results section because it is the same for all the portfolios and because the differences in methods become less clear when dividing the weight changes by a big number, which is as T is far greater than M.

4.5 Weight stability

Weight stability refers to the change of weights from time point to time point within a strategy. This is an important aspect in portfolio strategies because disregarding it could lead to extra transaction costs which ultimately harms the value of wealth. A strategy could theoretically be optimal based on the covariance matrix and mean return estimates but could perform terribly when it comes to weight stability. This section investigates the portfolio weights from different angles to this part of the portfolio strategy. Furthermore, this section analyzes the sensitivity curve of the portfolio turnover to better understand the fluctuations of the weights.

4.5.1 Sensitivity Curve

First of all, I investigate the portfolio measure of turnover, denoted in Equation 30. To achieve this, the sensitivity curve is used. The sensitivity curve measures the effect of changing one data point. For the turnover, this means that another data interval where weights are derived from is added. The derivation of the sensitivity curve of the turnover for one stock can be seen below:

$$\text{SC} = \frac{\frac{1}{T}\sum_{t=1}^{T}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}| - \frac{1}{T-M}\sum_{t=1}^{T-M}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}|}{1/T} \tag{31}$$

$$= \frac{\frac{T-M}{T^2-TM}\sum_{t=1}^{T}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}| - \frac{T}{T^2-TM}\sum_{t=1}^{T-M}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}|}{1/T} \tag{32}$$

$$= \frac{\frac{T}{T^2-TM}|\hat{w}_{k,T} - \hat{w}_{k,T^+}| - \frac{M}{T^2-TM}\sum_{t=1}^{T}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}|}{1/T} \tag{33}$$

$$= \frac{T}{T-M}|\hat{w}_{k,T} - \hat{w}_{k,T^+}| - \frac{M}{T-M}\sum_{t=1}^{T}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}| \tag{34}$$

$$= |\hat{w}_{k,T} - \hat{w}_{k,T^+}| - \frac{M}{T-M}\sum_{t=1}^{T-M}|\hat{w}_{k,t+1} - \hat{w}_{k,t^+}| \tag{35}$$

We conclude that the value of M plays a crucial role in the sensitivity curve. The higher M is, the lower the sensitivity curve is. However, in many cases, the full time period (T) is significantly longer than the individual time window (M). If this is the case then the left side is close to the mean of the turnover up until time T. To see if this performance measure is robust we have to look at what happens when $T \to \infty$. As mentioned in Section 4.1.2, a bounded sensitivity curve indicates robustness in the measurement, whereas an unbounded curve signifies a lack of robustness. In theory, if there are no constraints on the weights then the sensitivity curve could go to infinity as the difference of the weights could be infinite meaning that the performance measure would not be robust. However, in reality, this could never be the case as there are a finite number of stocks for a company and so the difference between the weights from one point to another also always has to be finite. This leads to the sensitivity curve being bounded when $T \to \infty$, which means that for real-life problems this performance measurement of turnover is robust.

4.5.2 Portfolio Constraints

One of the ways to make the weights more stable is by imposing constraints. In this section, the impact of constraints is considered. Constraints are often used to mimic the real world better or to push the optimization problem in a certain direction. This is often done in logistical problems, however they are also very useful in the financial world.

One of the most important works in the field of using constraints to make portfolio weights more stable was done by (Jagannathan and Ma, 2003). They showed that solving the global minimum variance optimization with constraints was equivalent to using shrinkage estimation on the covariance matrix. The constraint of the sum of weights being equal to one is standard, but the additional constraint would be that all the weights are greater than or equal to zero (no short-selling). The paper of (DeMiguel et al., 2009) shows that using constraints can not only decrease turnover for more stable weights but also increase returns.

The no short-selling constraint could benefit investors and the stability of the weights. Short selling is borrowing shares of a stock from a dealer to be bought back at a later point in time. If, during this period, the stock value decreases then the investor makes a profit. Short-selling is a risky part of portfolio strategies and has its drawbacks. One of them is that there is no limit to the amount of losses you can make on a short stock. This makes it very risky for inexperienced investors. Using the no short-selling constraint is useful to avoid having allocations where short-selling is necessary. Additionally, the use of this constraint can also increase returns. This happens because the constraint can limit the impact the robust methods have on the portfolio weights as the constraint already introduces shrinkage to the covariance matrix. This is investigated and analyzed in the Results section.

5 Results

In this section we look at the results from the methodology. The Results section is subdivided into low-dimensional and high-dimensional subsections with each subsection examining at the two portfolio allocations. The Global Minimum Variance and the Tangency Portfolio. The last section goes over concluding remarks and takeaways from the results

5.1 Low dimensional Portfolios

In the low-dimensional case this paper looks at three different characteristics. For each characteristic, Sharpe Ratio, Sortino Ratio or High Mean, the 15 best assets are selected and investigated and as mentioned in the Methodology section this paper also looks at the 330 portfolios that use the 8 best assets with a combination of the 9th until 19th best assets for each characteristic. The portfolios with the 15 best assets are discussed in a broad sense but this paper also dives into some examples and intricacies that are worth pointing out or could add some explanation. For the 330 portfolios, a broader analysis is provided and the focus specifically lies on the overall distribution of the portfolios.

5.1.1 Global Minimum Variance Portfolio

First, this paper investigates the Global Minimum Variance portfolio. The benchmark for all the strategies is the sample method. This paper looks at the characteristics individually where both the *choose and hold* strategy and *choosing each time period* strategy are considered.

Sharpe Ratio
In Table 1 the return, turnover and wealth are shown for both strategies using the 15 assets with highest Sharpe ratio. Here *NS* means that short selling is not allowed and so is referred to as the constrained portfolio. Immediately it becomes clear that the sample method is the optimal choice when considering Sharpe ratios. When we look at the *choose and hold* strategy we can see that the turnover for both the constrained and unconstrained portfolio are lower than for the robust methods. For the unconstrained case the only two robust methods that come close to the sample methods is the MVE and the 2SGS. The DI method has a higher return than the sample method but because of the turnover the final wealth is considerably lower. On closer inspection, the high turnover in the DI method is attributed to the constant fluctuations in asset importance. Where the sample method often chooses for different time intervals the same 2 or 3 assets to put most of the money in, the DI changes constantly which results in higher turnover. Additionally, we observe that the determinant of the inverse covariance matrix in the DI method is notably higher compared to the sample method, resulting in higher weight allocations. However there is improvement across the board for the constrained case. This is in line with the conclusions drawn in the study by (Jagannathan and Ma, 2003), where using constraints can help with the returns as well as the turnover. Interesting to note is that the cellwise methods significantly improve.

For the constrained case both methods that performed the worst when short selling was allowed score now the best compared to the other robust methods. The DI method is now the best performing method, earning the most amount of wealth at the end of the time period. This is because of the significantly decrease in turnover due to the shrinkage introduced in the covariance matrix which led to lower weight allocation. It is interesting to note that this did not harm the returns and thus the DI method slightly outperforms the sample method. Furthermore the turnover of the sample method is still the lowest and so the early indication is that these robust methods based on Mahalanobis distances are in general not better for stabilizing the weights.

Table 1: characteristics of 15 highest sharpe ratios

choose and hold	Sample	MCD	MVE	cellMCD	DI	2SGS
Return GMV	95.99%	58.01%	94.69%	69.57%	97.16%	84.97%
Turnover GMV	20.786	25.809	23.345	36.671	53.339	22.497
Wealth GMV	158.93	121.74	153.81	116.86	114.28	147.40
Return GMV NS	97.38%	74.45%	87.73%	94.79%	106.24%	88.72%
Turnover GMV NS	12.290	14.223	13.323	13.690	14.211	13.222
Wealth GMV NS	174.44	151.20	164.20	169.73	178.77	165.23
choosing each time period	**Sample**	**MCD**	**MVE**	**cellMCD**	**DI**	**2SGS**
Return GMV	49.38%	43.46%	46.81%	44.51%	2.49%	43.48%
Turnover GMV	38.454	44.357	43.046	52.967	65.131	40.089
Wealth GMV	101.08	91.33	94.73	84.10	52.51	95.47
Return GMV NS	49.15%	42.99%	55.42%	37.63%	28.77%	37.44%
Turnover GMV NS	24.778	24.625	24.691	24.987	25.000	24.855
Wealth GMV NS	116.13	111.51	121.12	106.94	100.04	106.93

The *choosing each time period* strategy does significantly worse than the *choose and hold* strategy. None of the returns or values of wealth are higher. Also as hypothesized in the Methodology section the turnover is also higher due to changing of the assets and therefore a lot of transactions are need to be made. There are however similarities between the different strategies. For the unconstrained portfolios the sample method is still the best performing as well as the only method that makes money. All the robust methods lose money. What is also interesting is that where the DI method had the highest returns for the *choose and hold* strategy, the returns of DI for the *choosing each time period* strategy are by far the worst. Where are the reasons for this? First of all when we look into the time periods individually we see that the DI method compared to the sample method has really extreme returns ranging from -13% to 10% where the sample method lowest return is only -3%. These extreme results are a consequence of having more extreme weights in the portfolio. These extreme weights, although not as extreme as those of the tangency portfolio, are significantly higher compared to those of the sample method. The reason why the DI method has such extreme weights is because of that the determinant of the covariance is very low. This is expected as the values of the datasets come closer together as outliers are minimized. However, due to the low determinant of the data point matrix, the inverse of this matrix has a high determinant. This results in high values in the inverse, leading to extreme weights as the inverse matrix is used for weight allocation. Furthermore we also see this for the casewise robust methods but in less extend, this could be because the outlier detection is row based and so the determinant is than higher in the original matrix.

For the constrained portfolio the wealth value are all above 100. This is mainly due to the fact that all the turnovers are significantly decreased. The difference in returns on the other hand is mixed. For some methods the returns have increased when the no short selling constraint is used but for others it has decreased. Furthermore it is interesting to see that with so little assets in the portfolio and thus constantly changing the weights does not only increase the turnover but actively decreases the return compared to the *choose and hold* strategy. This shows that having recent information does not always equal higher returns.

Secondly, the 330 portfolios based on the Sharpe ratio are considered. These portfolios give a more broad view of the effectiveness of this characteristic and how the robust methods compare to the sample method. The results for the *choose and hold* strategy are denoted in Table 2. For the unconstrained portfolio, we see similar results as in Table 1. It does however show that the MCD method and the 2SGS method perform considerably better in terms of median value of wealth compared to previous results. The sample method on the other hand has a median which is similar to the one in Table 1. Comparing the methods we can see that the sample method has one of the highest returns and looking at the MAD it is also very stable across the portfolios. Furthermore, this also holds for the turnover. Both these factors contribute to the sample method being the best method to use. Comparing the robust method itself we see the combination of casewise robustness and cellwise robustness yields the best results. This is different from the conclusion of Table 1. The casewise robust methods are better at stabilizing the weights than the cellwise robust methods are. This high value for turnover is also the reason why the cellwise robust methods have very low values of wealth. With both of them even losing money on some portfolios. For the constrained portfolio we see the turnovers as expected significantly decrease due to the shrinkage created in the covariance matrix. This also results in the returns for most methods to increase. It also shows in general that the portfolios are more stable as shown by a lower MAD for return and turnover. However, we still observe the sample method remaining the best-performing method, with none of the other methods coming close.

Table 2: sharpe ratios GMV portfolios (choose and hold strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	94.18%	77.61%	84.97%	79.60%	102.10%	95.54%	103.27%	88.55%	93.68%	93.19%	87.84%	89.54%
MAD Return	0.007	0.149	0.138	0.182	0.255	0.101	0.058	0.080	0.086	0.104	0.085	0.052
Median Turnover	20.617	26.395	24.324	37.571	51.557	22.501	12.514	13.900	13.518	13.856	15.518	13.402
MAD Turnover	1.079	1.420	1.555	2.873	4.628	0.966	0.491	0.897	0.796	0.788	0.964	0.385
Median Wealth	157.95	135.76	145.07	122.91	118.37	155.53	179.29	163.85	169.09	168.03	160.43	165.60
MAD Wealth	5.84	11.69	11.66	13.22	16.25	8.03	5.44	7.15	8.03	10.29	7.87	4.92
1q Wealth	145.34	109.03	118.92	90.40	71.44	138.60	168.11	144.24	148.79	146.27	141.98	155.55
2q Wealth	153.61	128.08	137.29	114.55	108.18	150.72	176.02	158.44	164.07	161.46	155.53	163.15
3q Wealth	161.58	143.69	152.96	132.33	130.23	161.43	183.81	168.29	174.71	175.10	166.03	169.81
4q Wealth	176.74	166.59	176.35	173.43	198.69	177.91	196.11	182.85	191.49	190.61	182.51	177.60
Beat Sample	n/a	0.00%	1.21%	1.21%	14.54%	4.24%	n/a	0.30%	4.24%	7.27%	1.21%	0.00%

For the *choose every time* strategy we see different results compared to Table 2. For the unconstrained portfolios, the sample method is not the best method anymore. Looking at the median value of wealth we see the 2SGS method having the best median wealth. Additionally, the casewise robust methods perform considerably better than the sample method, and they also have higher potential earnings. However, the turnover of the sample method is still the lowest indicating the weights are not more stabilized by using the robust methods. Although the robust methods performance is more positive the overall results are as expected worse compared to the *choose and hold* strategy. The reasoning behind this is mentioned above and is not portfolio-specific. Finally, the cellwise methods continue to underperform, and at least for the Sharpe ratio are not the methods to use to get good performance. For the constrained we see the casewise robust methods having a lower turnover but this difference is so minimal that we conclude that the weights are not significantly more stabilized. Furthermore, we once again see the casewise robust methods performing better than the cellwise robust method and having a higher potential in earnings. But overall the sample method is still the best method to use, due to the shrinkage that the constraint brings.

Table 3: Sharpe Ratio GMV portfolios (choose every time strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	40.37%	42.49%	42.05%	50.45%	41.37%	50.50%	45.96%	40.72%	44.39%	45.45%	43.51%	39.26%
MAD Return	0.101	0.166	0.121	0.216	0.275	0.104	0.082	0.085	0.092	0.087	0.106	0.081
Median Turnover	48.331	54.623	53.078	67.017	84.343	49.531	30.837	30.622	30.753	31.413	30.919	31.451
MAD Turnover	1.276	1.912	1.993	3.428	3.912	0.882	1.166	1.231	1.218	1.152	1.154	0.968
Median Wealth	85.99	82.04	83.14	76.11	59.52	91.33	106.72	103.00	105.84	105.67	104.67	101.75
MAD Wealth	5.67	10.39	7.54	9.59	10.65	6.05	5.93	6.35	6.87	6.18	7.22	5.60
1q Wealth	73.05	63.21	64.72	50.26	27.97	77.59	93.92	92.79	91.75	91.07	86.27	88.37
2q Wealth	82.27	75.11	78.49	69.46	52.36	87.36	102.81	99.42	101.24	101.75	100.04	97.27
3q Wealth	90.10	89.11	88.48	82.29	66.72	95.56	110.79	108.22	110.52	109.96	110.08	104.90
4q Wealth	104.15	108.48	106.82	98.78	99.68	107.48	123.88	120.79	126.03	120.02	123.19	115.59
Beat Sample	n/a	32.73%	32.73%	11.21%	2.42%	82.12%	n/a	30.91%	45.45%	18.18%	19.40%	12.73%

Sortino Ratio
For the second characteristic the portfolios are based on is the Sortino Ratio. This ratio only takes the deviation of returns below the target return in consideration. The target return in this case is 0. For the *choose and hold* strategy the returns are slightly higher for the unconstrained case comparing it to the portfolios based on the Sharpe ratio. The sample method performs the best, having the highest returns, lowest turnover and highest wealth compared to the robust methods. When we compare it the the worst performing method, the DI method, two reasons come up why the sample method works so well. First of all the turnover is a big factor. This is especially true in the first 6 time periods where the turnover of the DI method was often more than 3 times the turnover of the sample method. So even though the returns of the DI method were significantly higher in the first 6 time periods than the sample method the value of wealth difference was minimal. Secondly in the 8th time period the DI method chose the wrong asset to go long in due to the negative outlier being disregarded and so the weight on the asset was too high compared to the sample method who took the negative outlier into account. This was not only the case for the DI method but also other methods had a negative return in this time period due to this. Thus this is a disadvantage of using the robust methods.

For the constrained portfolio the same is true. The sample method performance the best in returns and wealth and the difference between the sample method and the best-performing robust methods remained consistent. The constrained portfolio again reduced the turnover and therefore increased the wealth even if the returns were lower. For example the return for the MCD in the unconstrained version was 101.34% and for the constrained portfolio 94.36% but due to the turnover being lower for the constrained portfolio the value of wealth was greater than for the unconstrained portfolio. The main reason why in the constrained case the sample method does so well compared to the other is the fact that the sample method has no negative return.

Table 4: characteristics 15 highest sortino ratios

One Time	Sample	MCD	MVE	cellMCD	DI	2SGS
Return GMV	109.76%	101.34%	85.13%	81.56%	77.22%	84.11%
Turnover GMV	18.776	23.547	24.403	30.999	43.961	21.962
Wealth GMV	173.61	158.74	144.68	132.62	113.27	147.53
Return GMV NS	109.72%	94.36%	92.88%	87.51%	63.56%	78.23%
Turnover GMV NS	12.354	13.565	13.964	13.286	15.870	14.479
Wealth GMV NS	185.24	169.58	167.61	164.06	139.41	154.07
Every Time	**Sample**	**MCD**	**MVE**	**cellMCD**	**DI**	**2SGS**
Return GMV	64.72%	45.68%	69.72%	94.26%	57.04%	83.29%
Turnover GMV	37.169	40.026	39.392	48.555	70.657	39.651
Wealth GMV	112.95	96.99	113.75	118.34	75.88	122.51
Return GMV NS	63.44%	63.34%	67.72%	60.27%	52.89%	64.84%
Turnover GMV NS	24.673	24.648	24.749	25.000	24.863	24.859
Wealth GMV NS	127.39	127.37	130.63	124.51	118.94	128.24

Looking at the *choosing each time period* strategy we see again that the returns in general are all lower. This is true for the unconstrained and the constrained version. Also, just as in Table 1, the turnovers are higher, which results in lower wealth. We also see that for the unconstrained version three out the five robust methods outperform the sample method. This is mainly due to the higher returns; however, the sample method, with the lowest turnover, indicates that in this setting, the Sortino Ratio is preferable to the standard Sharpe Ratio. The other two robust methods lose money where especially the performance of the DI method is bad compared to the rest of the methods. This was also the case for the Sharpe Ratio. Interesting to see is that the casewise robust methods as well as the cellwise robust methods give different results for different methods. Comparing the MCD with the MVE method we see that the turnover is almost the same but the returns are not. So the question is, why does the MCD method not get as high returns as the MVE method. When we compare the returns for each individual time period we see that the MCD and the MVE method do not differ too much from each other except from one time period where the MCD method has a negative return of 13% where the MVE had only a negative return of -3%. This could be luck but if we look closer to the weights in this time period we do see some differences in two of the stock. First of all the stock AKAM, which we also had seen was a talking point for the Sharpe Ratio, where the MCD method puts to much weight. This is due to the fact that the inverse covariance matrix of the MCD method has a higher value at point (4,4) compared to the MVE estimator. One of the reasons could be that the MCD and the MVE estimator do not agree on 10% whether a row is contaminated. Furthermore the MVE flags more rows than the MCD method does. This could also be the reason the methods differ on the second stock which is the TSS stock. This difference is more significant as the MVE methods takes a long position but the MCD method takes a short one. The values on the diagonal of the inverse of the MCD method are very high as well as having a lot of negative covariances with the other stocks which led to the stock having a negative weight.

For the constrained the turnovers are considerably lower and really close to each other. The highest turnover is 25, while the lowest is 24.648. The returns are also relatively close to each other which leads to no to little distinction in the value of wealth across the different methods. Compared to the unconstrained version the values for wealth have increased. This is, just as for the *choose and hold* strategy, mainly due to the decrease of turnover. However the DI method still shows to be the weaker method to use, mainly due to the lower returns. When compared to the MVE method some interesting things come to light. First of all the DI method, as seen before, loses a lot of value in the 7th time period and in this case it is no different with a -11% return. But this is not the only time the MVE method had a significant advantages in terms of returns. For the 9th time period for example the MVE method gets a return of almost 6% where the DI method only gets a 1.8% return. The first thing we notice is that the weights of the DI method are not diversified with 1 asset holding 67% of the wealth. The reason for this is that the value on the diagonal of the inverse is very high with low covariances. For the MVE method the values on the diagonal are much closer to each other leading to more diversification.

Secondly, the performance of the 330 portfolios based on the Sortino ratio are considered. For the *choose and hold* strategy the results are denoted in Table 5. We see that the returns are roughly the same when compared to the performance of the Sharpe ratio. For the unconstrained portfolios we can see that the returns in Table 4 slightly overestimated the performance. This is especially true for the MCD method but also the returns of the sample method are lower. When we look at the turnover we can also see that the sample method has the lowest turnover and that the cellwise robust methods have again the highest. This shows that this is the case independent of the portfolio used. What is also interesting is that the 2SGS method, similar to the performance of the Sharpe ratio has a very low MAD, for turnover. This means that despite using different asset combinations the turnover does not differ that much, making the case that the choice of assets the methods have low weight allocations. When it comes to value of wealth we can clearly see in Table 5 and in Figure 3 that the sample method dominates over the other methods with the highest *beat sample* percentage is 4.24% for the MVE method. This was also the case for the Sharpe ratio and it gives the sense that the robust methods in this strategy with the unconstrained portfolios are unable to outperform the classical sample approach.

For the constrained case the sample method also is better than the robust methods. This can also be seen in Figure 4 where the density of the sample method is clearly shown to be on the right of the robust methods. Furthermore, as expected, the use of constraint helps to lower the turnover and reduces the difference between the methods. I also conclude that the constraints help reduce the spread of the turnover because the MAD for all the methods is lower. This was also the case for the Sharpe ratio. The value of wealth is therefore also higher but as mentioned above none of the robust methods beat the sample method. This is another indication that these robust methods based on the Mahalanobis distance do not improve the stability of the weights and so the conclusion when using these ratios for asset selection is to stick with the sample method for weight estimations. Lastly, it is also evident from Figure 4 that the casewise robust methods are far better than the cellwise robust methods in terms of values of wealth.

Table 5: Sortino Ratio GMV 330 portfolios (choose and hold strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	96.92%	72.29%	83.77%	87.39%	93.76%	80.57%	104.08%	85.57%	94.66%	92.80%	70.36%	81.64%
MAD Return	0.099	0.181	0.185	0.188	0.332	0.117	0.088	0.131	0.128	0.089	0.096	0.078
Median Turnover	18.445	24.221	22.421	32.527	44.874	21.374	12.042	13.214	12.729	13.423	15.307	13.653
MAD Turnover	0.641	1.254	1.690	2.603	3.232	0.664	0.367	0.772	0.667	0.586	0.830	0.604
Median Wealth	163.49	134.66	146.279	134.35	121.91	145.48	180.91	162.17	171.34	169.09	145.72	158.37
MAD Wealth	8.54	13.90	15.83	14.76	20.03	9.37	8.30	11.47	12.20	8.33	8.76	7.44
1q Wealth	146.20	104.82	110.73	95.10	74.34	124.20	166.04	134.24	140.89	149.30	121.46	144.75
2q Wealth	158.53	125.95	135.67	125.46	109.39	139.61	175.00	154.80	162.70	162.97	140.88	153.76
3q Wealth	169.80	144.37	157.42	144.82	136.90	152.27	186.03	170.70	178.78	174.02	152.29	163.54
4q Wealth	184.82	173.42	186.03	187.49	223.54	176.00	208.01	195.64	210.18	192.49	171.03	174.47
Beat Sample	n/a	0.00%	4.24%	1.52%	3.64%	0.61%	n/a	0.61%	13.33%	3.64%	0.00%	0.00%

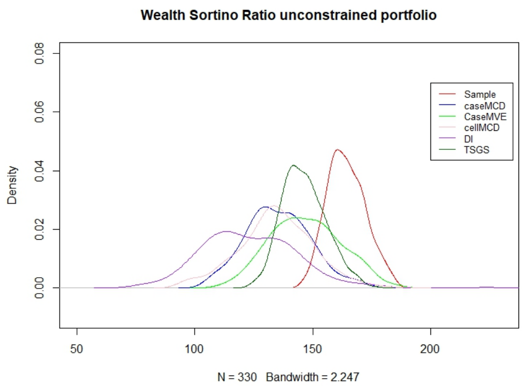


Figure 3: Unc port choose and hold strategy

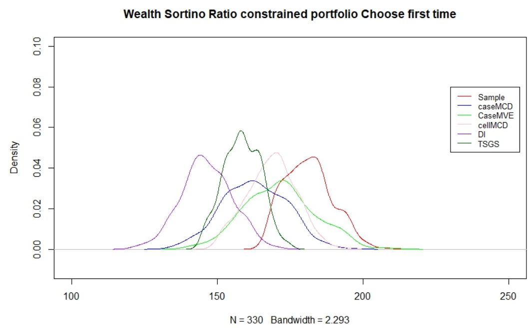


Figure 4: Con port choose and hold strategy

Looking at the performance of the *choosing each time period* strategy denoted in Table 6 we see a lot of interesting things happening similar to the Sharpe ratios. For the unconstrained portfolios, we can see that the returns, for almost all the robust methods, are higher compared to the sample method. However with this high return also comes a high MAD for the return with the MAD for the MVE being as high as 35.8%. This combined with a very high turnover for the MVE method makes it one of the worst performing methods in this table. Comparing it to the turnover denoted in Table 4 gives an interesting read as now for some methods the median is more than twice as big. This big difference is also shown in the MAD for the turnover. Furthermore the cellMCD and the DI method once again stand out as the methods with the highest turnover. This shows that for both the ratios the cellwise robust method is not a good method to use when considering a GMV portfolio. The method that is performing well for the unconstrained portfolios with the *choosing each time period* is the 2SGS method. It has the second highest return but also a low MAD which means that the returns are less scattered. Furthermore, it has the second-best turnover behind the sample method but the difference is not as big as for the other methods and it has the highest median value of wealth which again is backed with a low MAD. This good performance is also shown in the fact that is beats the sample method for 90.61% of the portfolios. That the 2SGS performs well compared to the sample method can also be seen in Figure 5 where the line of the 2SGS method is clearly to the right of the sample method line.

For the constrained portfolios we see again that the performance improves compared to the unconstrained case. This is mainly due to the decrease in turnover because the returns stay the same or even drop. What we also see is, just like for the Sharpe ratios, the casewise robust methods have a lower turnover than the sample method. However, for the Sortino ratio, these methods also have higher returns. This translates to a higher value of wealth. Furthermore, the median of the value of wealth is higher for the casewise robust methods and also just under 75% of the portfolios outperform the sample method. For the cellwise robust methods, we see the same results. Both methods do better than the sample method with the DI method even having a lower turnover than the sample method. This is compared to the unconstrained portfolio a great improvement. Lastly, the 2SGS method does not do as well compared to the other robust methods and only barely beats the sample method in terms of the value of wealth. It is however the only method that makes money on all the 330 portfolio and this is for an investor also very important.

In conclusion, similar to the Sharpe Ratio, when the Sortino ratio is used as the characteristic to choose the assets for the *choose and hold* strategy it is better to use the sample method than the robust methods for unconstrained portfolios as well as the constrained ones. This is because it does better in return, turnover, and therefore also the value of wealth. For the *choose every time* strategy this is not true for the unconstrained portfolios as the 2SGS does considerably better when it comes to value of wealth and return. Furthermore for the constrained portfolios all the robust methods beat the sample method in terms of value of wealth more times than not with most of them beating the sample method more than 70% of the time.

Table 6: sortino ratios GMV 330 portfolios (choose every time)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	54.93%	52.83%	65.89%	80.05%	63.38%	68.95%	56.23%	58.98%	58.71%	56.45%	56.10%	55.18%
MAD Return	0.129	0.126	0.358	0.201	0.244	0.133	0.098	0.106	0.101	0.100	0.108	0.092
Median Turnover	46.018	52.190	92.675	64.981	80.755	49.047	31.054	30.906	30.843	31.376	30.482	31.052
MAD Turnover	1.209	1.660	7.202	3.859	4.877	1.153	0.984	1.040	1.061	1.033	1.181	0.733
Median Wealth	97.48	90.32	63.52	92.62	71.87	102.76	114.30	116.60	116.52	114.25	114.59	113.60
MAD Wealth	8.69	8.43	14.04	9.03	11.11	8.16	7.01	8.40	7.99	7.79	8.41	7.17
1q Wealth	78.79	72.67	6.48	61.11	36.34	84.45	98.49	99.75	97.89	97.47	94.50	100.37
2q Wealth	90.98	84.85	54.30	86.36	64.06	97.42	109.19	110.95	111.16	109.29	109.51	108.94
3q Wealth	102.31	96.04	73.72	98.54	78.66	108.44	118.82	122.32	122.00	119.66	120.65	118.60
4q Wealth	116.97	113.24	108.98	121.90	102.40	122.63	131.44	135.14	138.32	135.13	140.70	128.82
Beat Sample	n/a	17.58%	23.33%	29.70%	1.21%	90.61%	n/a	74.24%	73.64%	70.30%	56.36%	50.30%

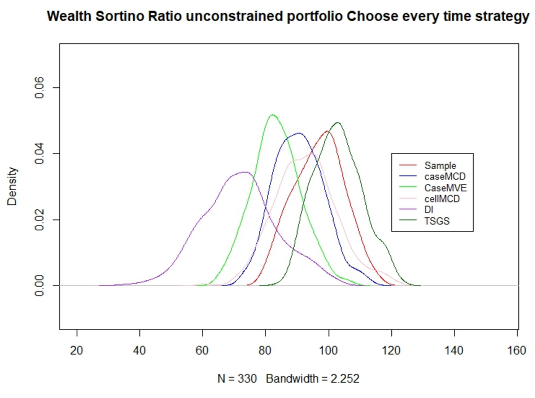


Figure 5: A figure

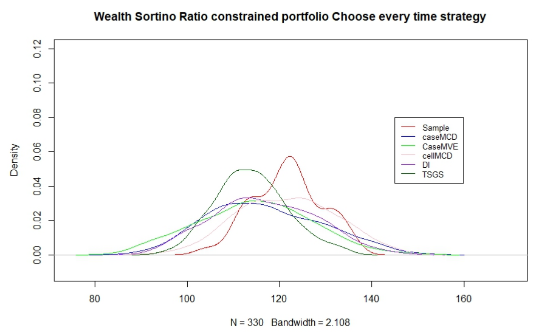


Figure 6: Another figure

High Mean

The third characteristic is the assets with the highest means. The results are shown in Table ??. Firstly we look at the *choose and hold* strategy. The performance of the portfolios is much higher for the high means than the two characteristics previously discussed. What is also interesting is for the unconstrained portfolio the returns are greater than for the sample method at 4 out of the 5 robust methods. This is noteworthy because for the Sharpe ratio and Sortino ratio, the returns for the sample were always one of the highest. Not only the returns but also the wealth is higher when using the high mean portfolio. All the methods for the unconstrained portfolio report wealth above 200 and 3 out of the 5 robust methods outperform the sample method in this regard. However, when looking at the turnover the sample method still performs the best. The constrained version is less promising. Although the turnover has again decreased, none of the returns or wealth have resulted in an increase. Still compared to other characteristics the constrained portfolio performs the best overall with all the returns above 100% and also all the values for wealth close to the 200 mark. This however shows the ability to short sell is not beneficial for the high means characteristic. If we dive deeper into the reasons why this is then we conclude that the assets with high means that have low covariance with the other assets or even a negative covariance are mostly shorted and these assets often do not move with the overall market so they are shorted which leads to higher returns when the market is up.

Table 7: characteristics 15 highest means

	One-Time					
	Sample	MCD	MVE	cellMCD	DI	2SGS
Return GMV	155.48%	202.95%	154.54%	199.75%	199.25%	167.39%
Turnover GMV	16.179	18.942	18.120	25.337	37.134	18.970
Wealth GMV	217.09	250.31	212.06	232.02	205.22	220.85
Return GMV NS	122.24%	130.50%	118.50%	121.90%	142.40%	127.84%
Turnover GMV NS	11.793	13.001	12.363	14.854	16.915	13.552
Wealth GMV NS	197.41	202.26	192.97	191.10	204.44	198.81
	Every-Time					
	Sample	MCD	MVE	cellMCD	DI	2SGS
Return GMV	78.32%	119.93%	92.46%	89.54%	77.24%	69.23%
Turnover GMV	34.399	38.194	37.309	51.662	64.554	36.365
Wealth GMV	125.81	149.23	131.78	111.79	91.27	117.01
Return GMV NS	62.84%	72.98%	72.12%	71.29%	46.00%	55.29%
Turnover GMV NS	23.924	24.400	24.277	24.862	24.771	24.635
Wealth GMV NS	127.90	135.21	134.70	133.26	113.69	121.09

When we look at the *choosing every time* strategy we can see, just as with the other characteristics, a decrease in performance. However, just as for the *choose and hold* strategy, these performances are all still the best out of the characteristics. For the unconstrained portfolio the sample method is based on returns beaten by 3 out of the 5 methods but due to higher turnover only 2 out of these 3 also beat the sample by value of wealth. These two methods are the casewise outlier methods. Both the MCD and the MVE method beat the sample in the unconstrained portfolio based on wealth. The only method that does not make money is the DI method. This is the third time that the DI method is the worst performing for the *choosing every time* strategy while also often having the highest turnover. The problem with the DI method is that the positions in the portfolio are very extreme going long for high value and also shorting high values. This not only leads to higher turnover as can be seen but also that the returns are more extreme. However, with such high turnovers even the extremely positive returns are set back by transaction costs resulting in less increasing values of wealth. So the values of wealth do not increase that much but when the returns a negative and for DI they could be extremely negative then the effect counts double with losing value of wealth due to high negative return but also transaction costs. This is why the DI method does the worst. The MCD on the other hand does a better job at keeping the turnover low but what the MCD and MVE do well is identifying the good stocks to pick.

For the constrained portfolio we see the turnover has again decreased and has a value of around 24. This is similar to that of the other characteristics. Furthermore, the sample method is beaten by both the casewise methods and the cellMCD method when it comes to returns and wealth but the sample method still has the highest turnover. It is also interesting to see that the 2SGS method performs better than the sample method for the *choose and hold* strategy in the unconstrained as well as the constrained portfolio but using the *choose every time* strategy performs worse in both portfolios. We see that the DI method once again performs the worst. Comparing it to the MVE method and the sample method gives us good insight into why this is the case. Immediately we see differences in the returns. For example, in the 5th time period both the sample method and the MVE have a positive return while the DI method has a negative one. Firstly all three methods had roughly 50% of their wealth in a stock that had a return of -7.5%. However, the DI method also had a 25% share in a stock that had a return of -15% while the others did not. How come this? When investigating the inverse of the covariance matrix we see that the DI method still has very high values on the diagonal compared to the MVE method and the sample method. Furthermore, the covariances of these high values are low which leads to less diversification. The DI method flagged a lot of values for the stock where it had a 25% share so it created less covariance with the other stock which led to a higher stake. The MVE method does not have this problem as it sees the whole row as contaminated and so the covariances are less hindered.

Table 8: high mean GMV portfolios (choose and hold strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	175.13%	161.16%	178.03%	142.96%	170.11%	147.08%	146.74%	132.31%	141.29%	127.40%	141.15%	126.17%
MAD Return	0.246	0.358	0.321	0.337	0.753	0.315	0.194	0.179	0.170	0.118	0.284	0.178
Median Turnover	17.517	22.145	20.131	31.475	40.705	20.958	11.935	13.482	12.629	14.565	16.152	13.610
MAD Turnover	0.725	1.815	1.582	3.956	4.818	1.944	0.422	0.919	0.868	0.984	0.899	0.798
Median Wealth	233.87	209.24	228.39	177.94	177.91	200.64	219.13	203.11	211.43	196.44	205.17	197.36
MAD Wealth	19.90	29.73	23.38	26.40	47.28	25.40	16.60	15.94	14.65	10.10	23.81	16.48
1q Wealth	187.80	140.78	164.39	100.12	82.51	137.62	190.59	169.28	177.29	164.61	142.54	162.49
2q Wealth	219.13	190.25	212.13	157.97	151.74	183.87	208.18	191.45	203.07	190.23	189.67	186.54
3q Wealth	246.35	230.01	243.84	194.20	215.30	218.39	230.56	212.78	222.21	204.30	221.99	208.51
4q Wealth	277.21	309.91	305.22	275.38	388.70	265.76	256.36	254.19	264.42	229.43	262.98	235.48
Beat Sample	n/a	21.52%	42.42%	5.45%	17.58%	10.61%	n/a	13.64%	28.48%	26.06%	34.85%	6.36%

Now we look at the high mean GMV portfolio for the *choose and hold* strategy. The results are shown in Table 8. When we look at this table and compare it to Table 7 we see some similarities and some differences. Starting with the unconstrained portfolio the similarity is the turnover for the sample method stayed the lowest of all the methods. We also see the order of highest to lowest turnover for the robust methods also stayed roughly the same. However, there are big differences when it comes to the returns and the value of wealth. Firstly when it comes to returns the interesting part is that the MVE and the DI methods outperform the sample method on returns. The other three methods also have a high return percentage but not higher than the sample method. When we look at the value of wealth the sample method stands out when it comes to median wealth and a low MAD. Although the sample method has the best median wealth, the other robust methods have higher ceilings. All the robust methods except for 2SGS have a higher maximum than the sample method, where the MVE method and the DI method even surpass the sample method before the 3rd quantile. In terms of beating the sample portfolios, the casewise robust methods are the best performing. The MVE even outperforms the sample method for over 42% of the time. This is interesting as for the Sharpe Ratio and Sortino ratio the robust methods were performing significantly worse compared to the sample method.

For the constrained version the differences we see a couple of interesting things happen. Firstly when comparing Table 8 and Table 7 we see for the returns all the median returns in Table 7 are higher than the returns in Table 8. Furthermore introducing the no short selling constraint usually increases the returns but in this case, all the returns have decreased. Looking at the value of wealth for the median we see that the sample method as well as the casewise robust methods and the 2SGS method all have lower median where the cellwise robust methods have increased median compared to the unconstrained portfolios. This is because the cellwise robust methods have a high turnover and so the constraints have more impact on their turnover than for the other methods. This results in the DI method beating the sample method in almost 35% of the portfolios. What we also see is just like for the unconstrained portfolios the MVE method and the DI method have long tails and thus their earning potential is higher than for the sample method.

Table 9: high mean GMV portfolios (choose every time strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	89.54%	73.03%	74.25%	59.37%	47.53%	59.75%	78.68%	70.10%	71.87%	55.68%	43.72%	54.00%
MAD Return	0.197	0.276	0.248	0.257	0.338	0.165	0.144	0.126	0.140	0.129	0.126	0.097
Median Turnover	46.208	49.123	48.201	63.593	73.827	46.651	32.672	32.221	32.586	32.934	32.388	32.921
MAD Turnover	1.291	2.583	2.204	2.844	3.846	1.499	1.362	1.643	1.463	1.406	1.864	1.494
Median Wealth	118.68	105.71	107.47	83.63	69.80	99.78	128.78	122.62	124.21	111.97	103.99	110.46
MAD Wealth	12.83	17.24	15.05	13.81	16.67	10.21	10.17	9.44	10.20	9.06	9.05	6.94
1q Wealth	87.98	71.92	69.44	44.99	33.17	76.20	108.87	98.07	101.84	94.55	84.62	94.77
2q Wealth	111.29	94.51	97.24	74.35	58.40	93.13	122.99	116.70	117.40	106.72	97.43	106.37
3q Wealth	127.83	118.05	117.61	93.16	80.80	107.47	136.55	129.34	131.11	119.04	109.56	115.64
4q Wealth	148.97	149.23	152.31	154.04	125.03	129.02	155.56	149.17	155.87	147.22	142.47	134.65
Beat Sample	n/a	17.27%	15.76%	0.00%	0.61%	0.00%	n/a	21.52%	26.06%	42.73%	0.00%	0.00%

In conclusion for the global minimum variance portfolio, the sample method is still the best method to use if you look at these characteristics. The only characteristic where the sample method did worse than the robust methods is when the characteristic of high means was used. Also, the turnovers for the sample method are always lower than those for the robust methods. This indicates that the estimation error of the inverse covariance matrix is not a big problem for the weight, as the weights of the global minimum variance portfolio are only based on the inverse of the covariance matrix. The introduction of the estimation error in the means could be the reason why for the characteristic of high mean the robust methods outperformed the sample methods. Looking at only the robust methods the casewise outlier methods often performed better than the cellwise outlier methods. This is especially true for the unconstrained portfolios where the casewise methods performed considerably better than the cellwise outliers in terms of wealth. This difference could be because the cellwise outliers often had higher turnovers than the casewise outliers.

5.1.2 Tangency Portfolio

Sharpe Ratio

The first characteristic is again the Sharpe ratio characteristic. The results are denoted in Table 10. First, we look at the *choose and hold* strategy and for the unconstrained, we see the worry of Equation 28 which is also mentioned in the Methodology section. The methods affected by this are all the methods except for the MVE method, so these results are marked with a star. The turnover however is not affected by this and so we can conclude that the turnovers for the cellwise robust methods, similar to when the Global Minimum Variance portfolio was used, are extremely high. When comparing the sample method to the robust casewise methods we see that the turnover is lower for the robust casewise methods and so here there is more stability to the weights. This does not however help a lot with the value of wealth because both the sample method and the robust casewise method lose money, due to the aggressiveness of the tangency portfolio. For example for the MCD method weights could be as high as 29.6 for one asset and as low as -12.2 for the other asset within the same time period. The extreme positions in assets combined with the constant changing of the 'important' assets leads to high turnover.

For the constrained portfolio the picture looks a lot better. All the methods make money and have low turnover. Just as for the GMV portfolio, the turnover of the sample method is the lowest, slightly lower than the MVE method. The turnover of the other method is also not that different from the sample method. So the turnover is quite equal for the different methods but when we look at the returns and the value of wealth the MVE method outperforms the rest, followed by the sample method and the DI method. What makes these methods do so well compared to for example caseMCD method? To answer this question we look deeper into the values of wealth over time and compare the two highest (MVE and sample) with the two lowest (MCD and 2SGS). The values of wealth of these methods are in the beginning very close to each other but the MCD method starts with a negative trend of returns from the fourth time period where the 2SGS has the highest value of wealth at the 8th time point. The big difference happens at the 10th time point where the MCD and the 2SGS both have high negative returns of -6.06% and -5.45%. This came due to a lack of diversification in combination with choosing both assets with the lowest return. Meanwhile, the MVE and the sample method have positive returns because they diversified more, and thus their weights were not as high for the asset with negative returns.

Table 10: performance of 15 highest Sharpe Ratio

	Choose and Hold					
	Sample*	MCD*	MVE	cellMCD*	DI*	2SGS*
Return Tan	-111.18%	-138.70%	-50.01%	343.12%	634.25%	4899.33%
Turnover Tan	638.413	451.643	239.551	2018.310	7204.192	2139.211
Wealth Tan	-0.03	-1.11	2.52	13625.39	2124319.00	71649.30
Return Tan NS	85.95%	66.53%	95.10%	77.64%	81.90%	73.43%
Turnover Tan NS	21.846	22.384	21.864	22.950	23.057	22.923
Wealth Tan NS	149.17	132.87	156.49	140.92	144.13	137.62
	Every Time					
	Sample	MCD	MVE	cellMCD	DI	2SGS
Return Tan	87.62%	59.14%	72.25%	202.79%	-19.97%	132.39%
Turnover Tan	49.252	69.107	62.560	109.924	142.066	56.672
Wealth Tan	113.48	78.13	90.60	95.10	17.51	130.06
Return Tan NS	66.91%	59.07%	51.18%	42.38%	61.64%	52.71%
Turnover Tan NS	24.428	24.675	24.595	25.000	24.848	24.961
Wealth Tan NS	130.42	123.98	117.93	110.61	125.77	118.68

For the *choose every time strategy* we see something interesting that did not happen for the Global Minimum variance portfolio when it comes to the unconstrained portfolios. Where for the GMV portfolios the turnovers increased compared to the *choose and hold* strategy do they decrease when the tangency portfolio is used. This indicates that recent information is more important for the unconstrained tangency portfolio and that it ensures less extreme positions in assets compared to the *choose and hold* strategy. Furthermore, none of the methods has the Equation 28 problems that occurred for the *choose and hold* strategy. When we compare the turnovers to each other we see that here the sample method is the lowest of all the methods. Furthermore, the cellwise robust methods have again the highest turnover. For the value of wealth, the 2SGS outperforms the rest and is the only robust method that makes money on the portfolio. The sample method is also doing not too bad with also having a value of wealth above 100. What these methods do well and what is lacking for example with the DI method is first of all keeping the turnover low by not having such extreme positions. For the GMV the determinant of the matrix was a good indicator of the extreme positions in the portfolio and this is still the case but for the tangency portfolios also the mean plays a role in the weight allocation. Comparing the sample method, the 2SGS method and the DI method for each time period give us more detailed insight into why the 2SGS does so well and the DI method does so badly. First of all the 2SGS has slightly higher weight allocation than the sample method does but nothing compared to the weight allocations of the DI method, as we have seen for GMV and confirmed in Table 10. Furthermore what is also interesting is that in the 7th time period both the sample method and the DI method made a loss but the 2SGS method made a positive return. What does the 2SGS method do that the others do not? The difference is in multiple stocks but here we highlight one of them. The stock we highlight is the AKAM stock where the 2SGS shorts for -0.166 but the sample method and the DI method go long for 0.086 and 0.275. One of the reasons for the 2SGS shorts is the fact that the stock in the 7th time period has some big returns but overall has a negative trend. Furthermore, the 2SGS method does a better job in re-estimating the detected outliers compared to the DI method because the DI method often makes from a high positive return a negative return and this does not give the desired effect of more stable weights as it harms the estimations of means and the covariance matrix too much.

For the constrained portfolios the turnovers contrary to the unconstrained portfolios have than the *choose and hold* strategy meaning that the influence of incorporating recent information is limited due to the shrinkage that the constraint already introduces in the portfolio. The turnovers across the methods are almost equal and so the magnitude of the value of wealth is mainly based on the returns. Here we see the sample method having the highest returns followed by the DI method. What is also interesting is the MVE method, which performed the best, loses most of its returns and thus has the lowest value of wealth. These lower returns and mainly because the MVE method diversifies the portfolio too much which leads to less return. The sample method on the other hand chooses fewer assets so their return is higher.

Furthermore, we look at the 330 portfolios where the asset choice is explained in Section 4.4. The results when using the *choose and hold* strategy are denoted in Table **??**. In this table, one row is added compared to the analysis on the global minimum variance portfolio and that is the percent > 100 row. This indicates how much percent of the portfolios tested have a value of wealth above 100. So in other terms how many portfolios are profitable. When we look at the unconstrained portfolios, we immediately see the aggressiveness of the tangency portfolio which was also mentioned in the paper (Kirby and Ostdiek, 2012) because the spread in the returns, turnover and the value of wealth is significant. For the returns, we see that all the methods have high negative returns as well as a very high MAD for the returns. The magnitude of the MAD is put in perspective when we compare these to the ones for the global minimum variance portfolios where the MAD for returns was 30 times as small. The turnovers are also interesting because for the global minimum variance portfolio the turnover of the sample was always smaller than the robust methods but we see that for the tangency portfolio, this is not the case. All methods apart from the DI method have a lower median turnover as well as a lower MAD. This means that the turnovers are not only smaller but also the spread is smaller. The effect of this can be seen in the results of the value of wealth because the spread between minimum and maximum for the sample method is the biggest of all the methods. The robust methods perform better than the sample method. The beat sample row shows that most of the robust methods outperform the sample more times than not with the percentage for the 2SGS being more than 60%. Furthermore, we can see for the robust methods the minimum value of wealth is not that bad compared to the sample method and the maximum is on level or even above the one from the sample method. This is best visible for the MCD and the cellMCD. Lastly, we look at the percent > 100 row. Here the sample method performs the best with 13.63% of the portfolios making money. This is good compared to the robust methods and it shows that the right tail end of the density function is thicker for the sample method.

For the constrained portfolios we see a completely different picture. The returns are positive and some are even higher than the global minimum variance portfolios. Also, the spread in returns is not as big anymore. The turnover improved significantly as well. The casewise robust methods have again the lowest turnover but the sample method outperforms the cellwise robust methods and the 2SGS method now. The high returns and the low turnover for the sample method translate to a high value of wealth. The returned sample outperforms the other robust methods. Looking at the beat sample row we see that the robust method that beats the sample method the most amount of time is the MVE method with only 5.76%. Lastly, all the portfolios apart from one portfolio in the cellMCD make money. This is a big improvement compared to the unconstrained portfolio. In conclusion for the *choose and hold* strategy, the robust methods perform better than the sample method in the unconstrained case but because of the aggressiveness of the tangency portfolio, this is not a viable, or at least very risky option to invest in. For the constrained portfolio the sample method is best to use.

Table 11: sharpe ratios Tangency portfolios (choose and hold strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	-99.99%	-82.19%	-91.63%	-82.06%	-97.09%	-70.60%	128.08%	71.80%	87.13%	72.81%	79.41%	73.81%
MAD Return	2.146	1.398	1.080	0.524	2.147	1.412	0.154	0.162	0.170	0.236	0.193	0.208
Median Turnover	551.045	414.035	441.744	336.731	599.397	318.243	21.669	21.364	21.337	23.046	23.064	22.759
MAD Turnover	339.267	229.821	286.276	157.083	366.786	160.26	0.677	0.746	0.744	0.666	0.713	0.382
Median Wealth	0.00	0.15	0.03	0.30	0.00	0.41	182.42	138.38	150.62	137.16	142.51	137.91
MAD Wealth	2.57	2.19	2.39	2.04	1.56	6.46	13.35	13.01	14.10	19.60	15.57	17.16
1q Wealth	-1.11e+11	-2.34e+4	-6.28e+7	-4.26e+5	-5.42e8	-1.00e+6	141.16	102.35	110.30	94.11	100.85	103.28
2q Wealth	-2.96	-0.09	-0.61	-0.07	-0.87	-0.58	174.08	129.65	142.19	122.92	131.97	124.76
3q Wealth	1.35	4.54	2.23	2.88	1.33	8.40	191.59	147.10	161.44	149.10	152.98	147.66
4q Wealth	2.12e+8	1.14e+9	1.32e+6	3.19e+9	2.59e+8	2.00e+7	220.78	184.46	198.06	184.39	190.35	171.76
Beat Sample	n/a	57.88%	53.03%	58.48%	49.09%	60.12%	n/a	0.91%	5.76%	0.91%	2.42%	0.91%
percent > 100	13.63%	8.79%	7.58%	5.45%	5.45%	8.18%	100.00%	100.00%	100.00%	99.09%	100.00%	100.00%

When using the *choose every time* strategy for the unconstrained case we see something interesting happening that we did not see when the global minimum variance portfolio was used. The turnovers are lower compared to the *choose and hold* strategy. This indicates the hypothesis of more recent data working better to select a portfolio than only choosing the first time period looks to be true for the tangency portfolio. For the returns, we see the cellwise robust methods as well as the 2SGS method doing well compared to the sample method. However, this good performance cancels out when looking at the difference in turnovers. The cellwise robust methods have such a high turnover and MAD of the turnover compared to the other methods that they have the lowest median value of wealth. The 2SGS method on the other hand has a relatively low turnover and so with the high return this results in the highest median value of wealth. Even higher than the sample method. This also results in the fact that for almost 59% the portfolios of the 2SGS method outperform the sample method. The two methods that perform the worst are the casewise robust methods with barely any portfolios beating the sample method or making money. Lastly what is interesting to note is that due to the high spread in returns and turnovers for the DI method, the minimum and maximum are still extreme.

For the constrained portfolios we see a decrease in performance compared to the *choose and hold* strategy. The returns are lower and, similar to the global minimum variance portfolio, the turnover is higher. The difference in value of wealth between the sample method and the robust methods is not as big as it was before but it is clear that the sample method is probably still the best. The big difference is for the constrained case the beat sample is significantly higher than for the *choose and hold* strategy. The two main robust methods that perform well are the casewise robust method with both beating the sample method around 20% of the time. This is mainly due to the long tails, which we also saw for the global minimum variance portfolios. In conclusion when using the Sharpe ratios combined with the *choose every time* strategy then for the unconstrained case the robust method 2SGS is the best method to use and for the constrained case it is the sample method.

Table 12: sharpe ratios Tangency portfolios (choose every time)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	72.79%	66.33%	61.57%	159.96%	92.48%	104.31%	62.81%	56.22%	52.82%	53.26%	51.74%	48.38%
MAD Return	0.179	0.412	0.316	0.867	1.131	0.266	0.038	0.135	0.127	0.106	0.113	0.095
Median Turnover	59.328	87.517	76.582	130.450	176.320	71.551	27.563	29.399	28.815	28.986	29.253	30.276
MAD Turnover	1.254	3.689	4.669	17.933	33.977	1.608	0.729	0.726	0.745	0.822	1.136	0.724
Median Wealth	94.10	67.36	73.33	66.19	27.60	97.94	123.304	116.14	114.04	114.66	112.68	109.34
MAD Wealth	9.69	15.99	16.07	17.46	20.02	13.40	3.10	9.80	9.74	8.19	8.78	7.12
1q Wealth	76.78	41.52	41.26	0.983	-1.19e+4	67.28	113.69	89.39	83.67	92.52	91.60	92.23
2q Wealth	88.23	57.11	62.74	54.30	15.36	89.60	121.24	108.96	108.16	108.97	106.77	104.14
3q Wealth	100.79	79.11	84.68	77.94	42.93	107.40	125.38	122.32	121.17	119.96	118.58	113.79
4q Wealth	122.77	124.63	120.84	662.37	5.47e+6	131.64	130.42	155.85	150.70	135.92	138.13	126.31
Beat Sample	n/a	5.76%	5.15%	3.03%	2.73%	58.48%	n/a	20.61%	18.48%	13.94%	13.64%	2.42%
percent > 100	29.09%	3.94%	4.85%	3.03%	3.03%	43.64%	100.00%	95.15%	92.42%	98.18%	92.73%	90.00%

Sortino Ratio
The second characteristic is the Sortino Ratio. The results are denoted in Table 13. For the *choose and hold* strategy we see very different results for the unconstrained portfolio compared to the Sharpe Ratio. This was also observed when the Global Minimum Variance allocation was used but for the Tangency allocation, this difference was much bigger. The only method that did have accuracy issues was the MVE method. The big difference is especially apparent for the cellwise robust methods including the 2SGS method. Furthermore, all the robust methods show to have lower turnover indicating more stable weights. If we compare the sample method with the 2SGS method we see big differences when it comes to weight distribution and extreme positions. The weights of the sample method are often very extreme with positions of +11 or -14 not uncommon where as the weights for the 2SGS method are almost never above 2 which is also rarely seen. This makes a big difference in the value of wealth because the sample method loses a lot of value with constantly changing extreme positions. Furthermore, these extreme positions not only lead to high transaction costs but also carry a great risk with them. This is apparent when the MCD and the DI are compared to the 2SGS. The MCD and the DI are much more susceptible to big losses when the extreme positions do not pay out as estimated. For example, in the 8th time period both the MCD and the DI method had multiple positive and negative positions exceeding 2 but because they picked the wrong assets their returns were -64% and -75% respectively. The 2SGS method on the other hand chose mostly the same stock to buy or short but did this with less extreme positions and so only had a return of -23%. These big losses can take a hit on a portfolio value because getting back up to the original value is even harder.

For the constrained portfolio we see that similar to the Sharpe Ratio the returns and turnover improve. The turnover is also very close to each other with the both casewise robust method and the cellwise minimum covariance determinant method having a lower turnover than the sample method. In terms of returns, the casewise robust methods do not perform so well compared to the cellwise robust methods. Comparing the sample method, the MVE method, the cellMCD method and the 2SGS method in more detail on why the casewise methods perform poorly. First of all, when it comes to turnover the differences as also seen in Table 13 are not that big, and the difference in the value of wealth is mainly due to the returns. The reason why the MVE does not do so well, and this is also the case for the MCD method, is first that the MVE method did have a big loss in the 8th time period which we also had seen for the MCD method and the DI method in the unconstrained case. The other robust method such as the cellMCD and the 2SGS had more conservative losses. Another and I think more important reason is the fact that the MVE does not have a constant return and thus the returns between periods fluctuate. Whereas the cellMCD method has more stable returns. Besides that, the cellMCD often also has higher returns. This constant return is also a reason why the returns for cellMCD are so much higher compared to the MVE method. For example, a portfolio could have returns 4 times returns of 2% or 1%, 1%, 4%, 2% where the percentage points are the same but the eventual returns are 8.24% and 8.21%. Doing this for more time periods and higher returns only increases this difference.

Table 13: characteristics 15 highest sortino ratios

	Sample	MCD	MVE*	cellMCD	DI	2SGS
Return Tan	-11.53%	-77.12%	-111.10%	-61.40%	-92.90%	-26.48%
Turnover Tan	524.862	367.045	294.533	287.143	623.816	140.611
Wealth Tan	-0.01	0.18	-0.23	1.34	0.24	16.02
Return Tan NS	129.91%	78.57%	87.28%	137.49%	88.03%	110.05%
Turnover Tan NS	21.292	21.143	20.318	21.025	22.576	22.799
Wealth Tan NS	185.48	144.28	152.59	192.11	149.72	166.88
	Every Time					
	Sample	MCD	MVE	cellMCD	DI	2SGS
Return Tan	95.55%	100.42%	93.46%	206.08%	98.19%	142.22%
Turnover Tan	46.905	85.032	84.147	137.142	155.932	66.665
Wealth Tan	121.20	82.84	80.29	70.95	37.30	122.02
Return Tan NS	73.88%	77.40%	80.56%	70.15%	81.53%	70.90%
Turnover Tan NS	24.423	24.894	24.646	25.000	24.883	24.920
Wealth Tan NS	135.88	137.97	140.78	132.18	141.20	132.88

When the *choose every time* strategy is used we see, similar to the Sharpe Ratio, that the methods are not affected by having returns below the 100%. Furthermore, we see the sample method is once again the method with the lowest turnover and is only barely beaten by the 2SGS method in terms of the value of wealth. The turnover for the casewise robust methods is also much higher compared to the Sharpe Ratio causing the value of wealth to be lower. Furthermore, the cellwise robust methods have once again the highest turnover but also high return, especially the cellMCD method. Comparing the sample method, cellMCD method and the 2SGS gives us insight into why the turnovers for the cellwise robust methods are so high and how the 2SGS and cellMCD get such high returns. First of all, looking at the returns we see in the 3rd time period a good example of why the cellwise robust methods have such high returns. The first sign is the weight allocations, where the sample method never goes long or short for more than 25% of the wealth, the robust methods easily go above that. This is especially true for the cellMCD where there are stocks with weights of 1.38 or -0.70. This is also true for the positive returns where the stock that increased the most had a portfolio weight of 0.54 for the cellMCD while the sample method only 0.12 of their wealth in this portfolio. However, due to these extreme positions, a good positive return can still cause the value of wealth to go down due to transactions costs. Thus the cellMCD would only be valuable if the rate of transaction would be significantly below 1%

Looking at the 330 portfolios for the Sortino ratio we see the same happening as for the Sharpe ratios. The median returns for the unconstrained portfolios are also negative and close to -100% but here the MAD of the returns is slightly lower than for the Sharpe ratio. This is especially the case for the sample method. The only method where the spread in returns has increased is the DI method. We also see the turnover for the sample method being lower than it was when using the Sharpe Ratio. Furthermore, only two methods have lower turnovers than the sample method in the MCD method and the 2SGS method. The turnover of the 2SGS method is especially low as the MAD is also significantly lower than the rest. Finally, the value of wealth for the portfolios is as scattered as it was for the Sharpe ratio but the minimum value of wealth of the robust methods is more extreme. This is the case for all the robust methods except the 2SGS method. What is also interesting is to see is the same robust method as for the Sharpe Ratio does not beat the sample method in more than 50% of the portfolio where the others do. Lastly of all the methods, 2SGS looks to be performing the best with the highest median return, lowest median turnover and highest median value of wealth but when it comes to portfolios that make money the 2SGS is the worst-performing method with only 2.12% of the portfolios having a value of wealth above 100. This is only 7 out of the 330 portfolios.

For the constrained case the picture is also very similar. The sample method has a median return of over 100% whereas the robust methods are just under the 100%. This was also the case for the Sharpe Ratio. The turnover for the sample is also the lowest and it has, apart from the 2SGS, the lowest MAD for the turnovers. These high returns and low turnover lead to the best value of wealth in terms of the median value of wealth but also a low MAD and the highest minimum and maximum. The robust method with the best results is the MVE. However, it only outperforms the sample method for 7.27% of the portfolio. So to conclude when using the Sortino Ratio as the characteristic in tangency portfolios with the *choose and hold* strategy then for the unconstrained case, it is best to use one of the robust methods. However, as was the case for the Sharpe Ratio, due to the aggressiveness of the tangency portfolio the likelihood of earning money with this setup is extremely low. In contrast, the constrained portfolio is a viable option to invest in with the sample method being the best method to choose.

Table 14: Sortino Tangency portfolios (choose and hold strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	-99.80%	-89.77%	-91.83%	-91.03%	-94.06%	-67.60%	135.68%	80.05%	95.88%	82.52%	83.51%	83.69%
MAD Return	0.807	0.868	0.941	0.446	3.811	1.118	0.211	0.182	0.213	0.253	0.239	0.232
Median Turnover	334.122	312.457	351.477	355.364	620.459	215.15	20.936	21.433	21.043	22.782	22.992	22.717
MAD Turnover	182.744	167.092	212.824	166.380	355.818	69.089	0.500	0.711	0.714	0.808	0.651	0.348
Median Wealth	0.00	0.17	0.09	0.06	0.00	1.67	190.75	144.78	158.26	144.61	145.37	146.45
MAD Wealth	1.22	3.81	3.35	0.68	2.05	8.90	17.57	14.99	18.27	21.46	19.64	18.38
1q Wealth	-1.07e+7	-3.09e+9	-2.98e+8	-3.92e+7	-1.80e+9	-5.25e+5	139.60	106.11	111.66	91.16	98.43	108.98
2q Wealth	-1.33	-0.55	-0.46	-0.02	-0.60	-0.94	178.00	135.07	146.09	129.35	131.91	133.61
3q Wealth	0.47	5.92	5.98	2.22	2.14	15.96	201.83	155.35	170.80	157.99	158.54	158.48
4q Wealth	7.40e+8	2.97e+6	9.30e+8	5.52e+7	1.87e+9	3.56e+6	231.32	204.17	213.06	207.44	185.82	180.96
Beat Sample	n/a	61.21%	58.79%	58.18%	49.09%	63.64%	n/a	2.12%	7.27%	3.94%	3.03%	3.03%
percent > 100	7.27%	6.97%	7.58%	4.55%	7.88%	2.12%	100.00%	100.00%	100.00%	99.70%	99.70%	100.00%

When it comes to the unconstrained portfolios for the tangency portfolios the *choose every time* strategy seems to do much better than the choose and hold strategy. Looking at the characteristic Sortino ratio denoted in Table 15 we see again that all the returns are positive and just as for the Sharpe Ratio the cellwise robust methods and the 2SGS method have the highest returns. Although these results look promising, due to the high turnover of these cellwise robust methods the median value of wealth is extremely low compared to the other methods. This is especially true when it is compared to the sample method. Even the 2SGS method, which had the best median value of wealth for the Sharpe ratio is now beaten by the sample method because the turnover of the 2SGS is too high. Looking at the casewise robust methods it becomes clear that they are not suitable for a tangency portfolio combined with this strategy. This is mostly due to the increase in turnover compared to the Sharpe Ratio without the returns to bring the value of wealth up.

For the constrained portfolios the returns of the methods are very close together and the same holds for the turnover. Furthermore, the MAD of the returns and the turnover are also very similar to each other but the sample method performs overall slightly better than the robust methods. For the median value of wealth, the same story goes. The two best methods are the sample method and the cellMCD method with both having median value of wealth close to 121. However, all the other methods are also performing well with a median value of wealth. Lastly, we see the use of constraints reduced the returns for the methods but also the uncertainty in the returns due to the shrinkage the constraints offer.

Table 15: sortino ratios Tangency portfolios (choose every time)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	73.18%	63.48%	61.57%	146.72%	156.14%	128.62%	60.94%	54.69%	52.68%	64.87%	57.19%	53.75%
MAD Return	0.122	0.422	0.316	1.102	1.838	0.307	0.094	0.171	0.154	0.145	0.162	0.100
Median Turnover	59.565	105.135	76.582	188.422	248.276	90.312	27.568	28.622	28.680	30.037	30.045	30.211
MAD Turnover	1.600	6.471	4.669	25.156	76.381	3.524	0.880	0.985	1.096	1.096	0.976	0.693
Median Wealth	94.35	55.44	73.33	31.13	13.80	90.79	121.91	115.88	114.62	121.65	116.11	113.43
MAD Wealth	7.362	16.46	16.07	17.06	18.04	11.94	7.62	13.34	12.30	11.60	11.75	7.77
1q Wealth	75.86	26.15	41.26	-5.13e+4	-1.87e+4	61.45	102.74	89.25	86.23	91.96	92.73	94.42
2q Wealth	89.25	46.18	62.74	19.96	3.16	83.03	116.29	107.74	106.08	113.80	108.93	108.85
3q Wealth	99.05	69.41	84.68	42.82	27.89	99.15	126.46	125.68	122.78	128.61	124.81	118.88
4q Wealth	121.20	114.58	120.84	7.56e+4	7.53e+4	134.65	137.52	149.67	148.69	146.35	146.49	134.90
Beat Sample	n/a	0.91%	13.94%	0.91%	3.33%	36.97%	n/a	27.27%	19.39%	47.27%	32.73%	1.21%
percent > 100	21.52%	0.61%	3.94%	0.91%	3.33%	23.33%	100.00%	93.33%	88.18%	98.79%	93.64%	97.58%

High Mean

The third characteristic is the high mean where the results are denoted in Table **??**. For the unconstrained case we see again three methods that had returns lower than -100% within the time period. These are marked with a star and these methods would because of the shorting have the investors in debt. This is especially the case for the 2SGS method. For the methods that had more accurate results the cellMCD method does really well compared to the other two in terms of value of wealth. This is mainly due to the lower turnover. This high turnover comes from a lot of extreme positions which is very risky. It can pay off when the right assets are chosen which in this case there are but it can also backfire and have the investor lose a large portion of his capital. This is not the case for cellMCD. For example at the 8th time period all the methods had a negative return but because of the extreme postions the sample method had a negative return of -74% where the cellMCD only lost 5%. Combined with high turnovers for the sample method meant the this portfolio lost 85% of its value in one trade where the portfolio of the cellMCD only lost 14%. Furthermore the high turnovers also played a big role in neglecting big wins. For example in the 7th time period the returns for both the MVE method and the cellMCD method were respectively 50% and 49% but because the MVE method had to trade a lot with turnover for that time period being 33.02 and so the portfolio value only went from 201 to 202, an increase of lower than 0.5%. The cellMCD on the other hand only had a turnover of 4.33 and so the portfolio value went from 107 to 143, an increase 33.6%. This shows once again that turnover is very important and that returns can vanish if the investor has to trade a lot.

For the constrained portfolios we see the turnover drops and this helps the portfolios in their value of wealth. The turnovers do not differ too much from each other with the only method with a slightly higher turnover being the DI method. The high mean characteristic has a much higher value of wealth compared to both the ratios. This is similar to the GMV portfolio. The returns are also very high with all the returns apart from the cellMCD method having a returns higher than 100% with the sample method and the 2SGS method almost touching the 200% mark. Why does the cellMCD perform so much worse than the other robust methods? When we dive deeper into the results we see that the biggest difference happens at the 6th time window. Here the cellMCD method has a return of -12.46% where all the other methods have a positive return close to or even higher than 10%. The weight distribution at this time point shows that all the methods apart from the cellMCD have 2 assets in common. These were also the assets with a high return, but the 3 assets that were chosen by the cellMCD did not occur in the portfolio of any of other methods. The results show comparing it the other cellwise robust method that the number of cells flagged as outlying is significantly lower for the cellMCD than for the DI method. This could be one reason that the asset selection was so different. Furthermore the cells that the cellMCD did flag were mostly of asset that had high returns meaning that these were nullified which led to not choosing them in the asset allocation.

For the *choose every time* strategy we see that for the unconstrained portfolio the turnovers decreasing compared to the *choose and hold* strategy. The sample method does well even though it does not have the highest returns, but the value of wealth is the highest due to such low turnover. The cellMCD method and the 2SGS have extremely high returns but this does not lead to higher values of wealth due to high turnover. If we compare these three methods we see that the returns for the cellMCD and the 2SGS are mostly extreme returns. For the 2SGS method we see in the first 3 time periods already a cumulative return of more than 65% but there are also multiple time periods where more than 10% is lost. These extreme returns are also apparant for the cellMCD method where in three individual time period returns are recorded of more than 30% where the highest being 49% in one time period. These extreme returns are of course the result of having extreme positions in the portfolio allocations. This is also the explanation for the high turnovers. On the other hand the returns for the sample method are less extreme which leads to lower returns but also lower turnover.

For the constrained portfolios we see that the turnovers have increased which was expected looking at the rest of the characteristics. Furthermore we see that the differences in turnover are minimal and so the performance of the value of wealth is mostly based on the returns. The sample method as well as the casewise robust methods have the best values of wealth and the cellwise robust methods are lacking behind. When we compare the sample method with the best performing method, the MVE method, and the worst performing method, the DI method we see a big difference in choice of weights.

Table 16: characteristics 15 highest means

	Sample	MCD*	MVE	cellMCD	DI*	2SGS*
Return Tan	2546.90%	-270.59%	2383.52%	413.10%	-286.19%	-5736.61%
Turnover Tan	553.344	402.503	278.419	87.133	265.170	1599.453
Wealth Tan	0.11	-0.69	88.37	206.73	-6.33	-10847.32
Return Tan NS	197.25%	137.04%	127.48%	47.08%	189.76%	197.45%
Turnover Tan NS	19.685	19.915	19.945	20.055	22.386	20.991
Wealth Tan NS	243.76	193.92	186.05	120.15	231.17	240.69
	Every Time					
	Sample	MCD	MVE	cellMCD	DI	2SGS
Return Tan	65.24%	86.78%	52.97%	168.98%	58.72%	93.44%
Turnover Tan	40.195	105.705	56.980	95.937	135.120	62.850
Wealth Tan	109.75	59.34	85.30	98.75	35.10	101.23
Return Tan NS	52.68%	51.38%	65.44%	43.31%	13.82%	52.53%
Turnover Tan NS	24.001	24.667	24.150	24.905	24.906	24.701
Wealth Tan NS	119.83	118.00	129.64	111.44	88.51	118.86

Furthermore, we analyze the 330 portfolios of the high mean characteristic. The results of the *choose and hold* strategy are displayed in Table 17. For the unconstrained portfolios, we immediately see in the returns a difference between the sample method and the robust methods. The returns of the robust methods are, similar to the Sharpe ratio and the Sortino ratio, negative. For the high mean the returns are even more negative with some even having lower returns than -100%. This is however not the case for the sample method where the returns are positive. However, the positive returns come with a very high MAD. This high MAD is also true for the robust methods, but the MAD for the sample is the highest at 41.390. This is equal to 4139%. Looking at the turnovers of the methods we see that the robust methods are performing well with casewise methods again having the lowest value, but on the other hand the cellwise methods and the 2SGS method have higher turnovers than the sample method. Lastly when considering the value of wealth we see for the median that the sample method performs the best but only slightly. If we however consider the quantiles we can see that similar to the results of the Sharpe ratios and Sortino ratios the sample method has the lowest minimum and therefore has the highest downward potential. The difference in this aspect is significant with an order of 10^7 compared it the robust methods. For the maximum, we can see that the cellwise robust methods have the highest earning potential. Finally, the robust methods and the sample method are close to each other when looking at the beat sample statistic. Here we see that all the values are close to 50%. Furthermore, the percent $>$ 100 is also very close to each other with all the values around 20%.

For the constrained case we again see more steady portfolios. For the returns, we see that now all the methods have a value above 100% instead of just the sample method. Furthermore, the MAD of the returns is low so the returns are also very stable. The turnovers are performing also very well with the help of the constraint and are even lower than the ones from the Sharpe ratio and Sortino ratio. Furthermore also here the MAD is very low. Lastly, when looking at the value of wealth we see that the 2SGS has the highest minimum so it has the biggest potential loss. This method beat the sample 43% of the time but this happens mostly in the left part of the density function. The two other robust methods that are interesting are the MVE and DI methods. These methods perform worse than the sample method at around 75% of the time but they have very high earning potential compared to the sample method. This also has the downside that their minimum is also significantly lower.

In conclusion, the unconstrained portfolio does again show the aggressiveness of the tangency portfolio. Here the 2SGS method is the best method to choose. For the constrained portfolios the results are the best up until now with very stable returns and thus stable value of wealth. Here the best method is choose from for risk-averse investors is the 2SGS method. But in more general terms the sample method just outperforms the other methods.

Table 17: High Mean Tangency portfolios (choose and hold strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	5.88%	-105.03%	-173.94%	-96.27%	-93.58%	-99.86%	193.71%	143.36%	161.33%	129.78%	154.46%	193.59%
MAD Return	41.390	8.570	33.766	31.722	30.644	23.521	0.368	0.326	0.384	0.313	0.489	0.292
Median Turnover	450.361	433.960	389.458	714.403	564.393	482.082	20.022	20.748	20.441	21.775	21.624	21.202
MAD Turnover	256.506	290.761	214.847	468.188	374.325	312.773	0.444	0.599	0.745	0.907	1.044	0.342
Median Wealth	0.11	0.00	-0.32	0.00	0.00	0.00	239.95	197.68	213.18	183.97	204.55	237.07
MAD Wealth	35.69	18.26	62.45	11.67	33.61	44.38	31.00	26.05	31.36	24.77	38.64	23.81
1q Wealth	-1.24e+18	-2.43e+9	-1.48e+11	-2.22e+13	-8.32e+10	-2.13e+8	189.34	104.25	130.51	123.40	107.41	193.51
2q Wealth	-22.25	-14.05	-43.68	-2.43	-15.70	-28.73	214.20	177.14	193.15	169.56	178.73	217.21
3q Wealth	25.37	9.74	35.10	26.53	28.36	29.28	256.99	213.70	235.81	202.86	230.69	249.82
4q Wealth	7.78e+10	1.06e+10	2.70e+11	2.31e+12	6.22e+14	7.96e+8	295.32	271.70	338.76	259.19	377.03	277.91
Beat Sample	n/a	51.52%	47.58%	50.10%	47.88%	47.88%	n/a	13.94%	23.03%	2.73%	22.12%	43.03%
percent > 100	17.88%	12.42%	20.00%	19.09%	19.93%	18.48%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%

The *choose every time* strategy for the high mean characteristic also shows some interesting insights. The results are denoted in Table 18 and here we see for the unconstrained portfolios the first difference between the Sharpe and Sortino ratio and the high mean. Where for the Sharpe and Sortino ratio the cellMCD and the 2SGS method had the highest returns, is this not true of the high mean characteristic where the 2SGS is still the highest but the cellMCD has a median return close to that of the lowest. Looking at the turnover we do see similarities with both the cellwise robust methods again having the highest turnover and the sample method with the lowest. This high turnover of the robust methods also translates to bad performance for the value of wealth where only the sample method has a median value of wealth that makes money. The only method that comes somewhat close is the 2SGS method with a median value of wealth of 76.38. That the sample method clearly outperforms the robust methods is also visible in the quantile and the beat sample statistic. The only time the robust methods have a higher value of wealth in a quantile is for the maximum of the casewise or cellwise robust methods. But this is for the rest of the portfolios not true with the highest beat sample statistic of 3.03%. Lastly, the sample method is also the only viable option when looking at the percentage of portfolios that make money with 76.06% of the sample portfolios above 100.

Table 18: high mean Tangency portfolios (choose every time strategy)

	Unconstrained						Constrained					
	Sample	MCD	MVE	cellMCD	DI	2SGS	Sample	MCD	MVE	cellMCD	DI	2SGS
Median Return	77.50%	46.67%	53.75%	3.15%	3.13%	94.51%	63.60%	60.55%	59.89%	43.25%	37.08%	64.70%
MAD Return	0.178	0.797	0.580	0.523	1.234	0.379	0.104	0.175	0.179	0.127	0.145	0.148
Median Turnover	49.695	131.436	89.324	121.84	219.003	88.131	29.902	29.995	30.086	31.380	31.696	30.964
MAD Turnover	1.623	38.449	14.233	16.325	75.761	7.003	1.197	1.812	1.590	1.985	2.464	1.757
Median Wealth	107.02	34.24	58.59	27.23	9.35	76.38	121.39	118.53	117.96	104.57	99.46	120.60
MAD Wealth	10.35	30.52	22.95	18.80	13.63	15.15	7.44	13.52	14.11	10.13	10.25	9.39
1q Wealth	82.74	-3.52e+4	-5.18e+4	-2.69e+5	-7.00e+5	10.45	103.23	80.82	81.90	71.03	72.43	100.71
2q Wealth	100.17	13.79	44.40	15.80	0.87	66.95	116.48	109.04	109.05	97.64	93.03	114.44
3q Wealth	114.30	55.39	77.05	41.07	23.48	88.11	126.67	126.88	127.87	110.72	106.66	127.24
4q Wealth	136.89	1201.53	1.06e+5	98.75	1.30e+6	120.31	140.98	155.46	153.59	135.05	133.95	154.40
Beat Sample	n/a	3.03%	2.12%	0.00%	2.73%	0.61%	n/a	40.30%	37.27%	3.64%	6.36%	50.91%
percent > 100	76.06%	3.94%	7.58%	0.00%	2.73%	8.48%	100.00%	92.12%	92.73%	68.18%	47.88%	100.00%

For the constrained portfolios we that the returns are not as high as they were for the *choose and hold* strategy. But this is also the case for the Sharpe ratio and the Sortino ratio. What is interesting though is the fact that compared to the returns of *choose every time* strategies of the other characteristics the high mean produces the lowest returns. This is noteworthy because for the *choose and hold* strategy they were the highest among all the other characteristics. Furthermore, the turnover is also slightly higher, where the sample method is the lowest. Looking at the value of wealth we see that the 2SGS does very well and even beats the sample method more times than not. This is especially the case for the high value of wealth portfolios. Apart from the 2SGS, does the casewise robust method also pretty well and also outperforms the sample method consistently with a 40.30% and 37.27% *beat sample* statistic for MCD and MVE method respectfully. The cellwise robust methods on the other hand are not performing well with the DI method not even making money on more than half of the portfolios.
In conclusion, when using the high mean characteristic combined with the *choose every time* strategy for an unconstrained portfolio the sample method is by far the best method to use which is different from the other characteristics, and for a constrained portfolio the 2SGS method is best to use.

5.2 High dimensional Portfolios

5.2.1 Global Minimum Variance Portfolio

Sharpe Ratio

The first characteristic this paper focuses on is the Sharpe Ratio denoted in Table 19. When we look at the *choose and hold* strategy we immediately see the difference between the sample method and the robust methods. For the unconstrained portfolios, the turnover of the robust methods is always lower, indicating that the weights are more stable. This observation is interesting because the robust methods based on the Mahalanobis distance did not exhibit this behavior in lower dimensions. Also, the use of more assets reduces the value of wealth for the robust methods because of the diversification. This is however not true for the sample method. Furthermore, it is interesting to note that for the sample method the turnover decreases when the number of assets increases. This is however not true for the robust methods, where especially for Spearman's method we can see the turnover increasing. Similar to the lower-dimensional case, the cellwise robust method, in this instance the Spearman's method, exhibits a higher turnover than the casewise robust method. This indicates the trend continuing regardless of the number of assets. The portfolio where the number of assets (N) equals 188 is interesting because it shows the OGK method is clearly better than Spearman's correlation method due to having lower turnover and slightly higher returns. This difference is shown for example in the 11th time period where all the methods have a very diversified portfolio but the sample method shorts too much on stocks going up in value. This is mainly due to one or two negative returns in the previous time period letting the mean estimations, as well as the covariance estimation, to be skewed toward these data entries making the portfolio short these assets. The main difference between the robust methods is not in individual time periods but in the changing of weights throughout. This is mainly because the cellwise robust methods identify single cells and so the weight distribution per time period is more random which leads to more changing of the weights over time.

For the constrained portfolios the turnover, similar to the low dimensional case, decreases and due to this the value of wealth increases. The sample method performs significantly better but when N = 94 or N = 188 is still beaten by the robust methods. Moreover, the turnovers of the robust methods remain lower compared to the sample method, although this difference has significantly reduced. When the number of assets increases the opposite happens to the value of wealth for the robust methods but this is not the case for the sample method where the number of assets has a positive effect on return and value of wealth. Looking at the portfolio where N = 376 we can see the sample method having higher returns. This is mainly because of less diversification in the weights, leading to slightly higher returns in each individual time period. And because this happens for 13 time periods, the cumulative return is much higher for the sample method compared to the robust methods. The biggest difference is between the sample method and the OGK method. If we look at per time period how many weights in the portfolio are not equal to zero, or in other words how many assets are used per time period we see over the 13 time periods the mean number of assets used for the sample method is roughly 27 but for the OGK method this is 63. Thus the OGK method uses on average more than twice the number of assets than the sample method. For Spearman's correlation method the average is 36 meaning this method has also problems of overdiversification but not as bad as the OGK method. This is also why Spearman's correlation method has a better value of wealth for the portfolio where the number of assets exceeds 282.

Table 19: Sharpe Ratio GMV

(choose and hold)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-13.17%	81.47%	78.11%	72.56%	85.34%	72.55%	74.56%	64.75%	70.04%	59.38%	57.58%	57.34%
Turnover	313.742	21.178	35.926	88.413	25.190	39.318	74.467	25.517	42.866	69.553	23.597	43.638
Wealth	2.29	146.54	123.71	69.02	137.47	115.72	80.99	127.30	109.93	77.94	124.17	100.91
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	57.24%	68.70%	65.05%	62.62%	63.51%	63.89%	70.49%	54.07%	62.14%	73.80%	51.33%	61.18%
Turnover NS	18.841	13.634	17.699	19.410	15.102	18.075	20.931	15.965	19.821	20.860	16.099	19.958
Wealth NS	130.06	147.09	138.11	133.73	140.46	136.62	138.05	131.21	132.78	140.83	128.69	131.81
Beat Sample NS	n/a	Yes	Yes	n/a	Yes	Yes	n/a	No	No	n/a	No	No

(choose every time)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	102.38%	45.73%	54.10%	36.11%	62.55%	61.71%	27.96%	49.20%	55.22%	51.95%	56.26%	57.82%
Turnover	428.908	43.030	57.576	120.030	40.355	57.413	91.985	34.749	52.645	77.672	27.689	48.519
Wealth	0.82	94.03	85.49	38.53	107.83	89.87	49.23	104.89	90.65	68.18	118.10	96.23
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	60.57%	57.46%	64.14%	60.79%	62.38%	75.09%	53.68%	51.34%	58.22%	58.42%	47.87%	51.75%
Turnover NS	23.023	22.505	23.051	22.184	20.424	21.580	21.677	18.729	20.774	21.468	17.575	20.563
Wealth NS	127.28	125.48	130.07	128.54	132.16	140.84	123.50	125.31	128.32	127.58	123.89	123.34
Beat Sample NS	n/a	No	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	No	No

For the *choose every time* strategy we see similarities but also differences in the results compared to the *choose and hold* strategy. For the unconstrained portfolio, we see the robust methods again outperforming the sample method This holds for all the different sizes of portfolios considered in this analysis. The turnover is also for the *choose every time* strategy the cause of this. The turnover has increased compared to the *choose and hold* strategy, as was the case for the low-dimensional portfolios, but it decreases when more assets are added to the portfolio. This is because when more assets are chosen, the chance an asset is in the portfolio at time t as well as at time t+1 is increased. We also see the returns having decreased compared to the *choose and hold* strategy, especially when the number of assets is small. This decrease in returns and increase in turnover leads to a decrease in the value of wealth. This decrease results also in the loss of money for the robust methods. None of the portfolios using Spearman's correlation method is profitable except for the OGK method when N = 94. Looking at the portfolio with 94 assets, we notice a similar pattern as observed in the *choose and hold* strategy. The sample method has more extreme positions which leads to higher returns in a market that is going up but the robust methods have the higher value of wealth due to the lower turnover. We also see the determinant of the sample method inverse covariance matrix is extremely high compared to the robust methods. Furthermore, due to these high weight allocations, the turnovers are also very high. These high turnovers were even so high that a return of 40% still led to a decrease in the value of wealth due to transaction costs.

For the constrained portfolios we immediately see that the turnovers are significantly lower. The sample method has the most advantage using the no short-selling constraint. Furthermore, we see the returns in most cases also have increased due to the use of the constraint. This is interesting because for the *choose and hold* strategy the complete opposite happened. Another difference is the robust methods performing better up until N = 282. But similar to the *choose and hold* strategy the differences in value of wealth are minimal. When analysing the portfolios where the number of assets is 94 we see the returns of all the individual time periods are almost identical to each other, but the OGK is often the lowest so this explains the lower return when taken cumulatively. There is however still a big difference in diversification. For example, on average the sample method has a position in 16 of out 94 assets, for the OGK this is 31 out of 94 and for Spearman's correlation this is 21 out of 94. So the OGK is more diversified and this leads to lower turnover because the chance of having a big weight in an asset that is not in the a choice in the next period is smaller.

Sortino Ratio

The second characteristic we look at is the Sortino Ratio. In the case of unconstrained portfolios with N = 94 assets, the performance of the sample method is notably worse compared to the Sharpe Ratio, evidenced by a high turnover of 333.454. The robust methods on the other hand do perform really well with high returns and low turnover for the N = 94. This is mainly due to the lack of diversification of the sample method in combination with high portfolio weights. This combination can lead to low returns as it is more dependent on choosing the right assets. When we look at the 2nd time period we see this is the case. The sample method makes first of all a lot of trades which leads to high transaction costs and second of all when the sample method picks the wrong assets, which it does frequently, it causes high negative returns. It is worth noting the determinant of the inverse covariance matrix can indicate extreme weight allocations. In this case, the determinant of the robust methods is notably lower than that of the sample method, highlighting the difference in portfolio diversification. However, adding assets to the portfolio does diversify the portfolio of the sample method which leads to higher returns and lower turnover. This is not observed in the robust methods, as adding more assets tends to over-diversify the portfolio, ultimately resulting in lower values of wealth. Lastly, the lower turnover observed in casewise robust methods compared to cellwise robust methods suggests the casewise robust methods may be more effective due to their ability to maintain stable weight allocations over time.

For the constrained portfolios we see similar results compared to the Sharpe Ratio characteristic. Initially, the robust methods outperform the sample method for N = 94 and N = 188. However, the sample method performs better as the number of assets in the portfolio increases. The robust methods have a higher return and a lower turnover. Due to the use of constraints, the difference is not as big as when short selling is allowed. It is also interesting to see the importance of low turnover when you compare the results of the unconstrained portfolios to the constrained portfolios for N = 188. Although the returns of for example the OGK method are much higher for the unconstrained portfolio but because of the low turnover for the constrained portfolio the value of wealth is still higher. When N = 282 or N = 376 we see the sample method has a higher value of wealth. This is mainly because of the robust diversifying the portfolio too much. This shows that the robust methods have a more conservative approach and so get a lower return. The reasoning for this is similar to the Sharpe Ratio.

Table 20: Sortino Ratio GMV

(choose and hold)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-46.81%	77.40%	81.73%	80.57%	79.00%	73.84%	70.45%	63.54%	67.18%	59.38%	57.58%	57.34%
Turnover	333.454	22.686	35.892	86.595	24.604	38.731	74.507	26.344	42.866	69.553	23.597	43.638
Wealth	1.06	141.09	126.26	73.64	139.59	117.29	79.10	125.30	108.08	77.94	124.17	100.92
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	60.22%	61.06%	68.72%	64.67%	66.50%	72.35%	67.55%	51.65%	61.08%	73.80%	51.33%	61.18%
Turnover NS	19.164	14.171	17.360	19.117	15.110	18.112	20.720	16.203	19.806	20.860	16.099	19.958
Wealth NS	132.08	139.67	141.66	135.82	143.02	143.61	135.96	128.83	131.94	140.83	128.69	131.81
Beat Sample NS	n/a	Yes	Yes	n/a	Yes	Yes	n/a	No	No	n/a	No	No
(choose every time)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	65.36%	42.77%	61.01%	53.97%	64.35%	64.87%	31.01%	52.09%	56.89%	50.35%	56.19%	56.32%
Turnover	420.898	42.834	57.991	118.513	40.090	57.195	92.218	35.386	51.783	77.825	27.860	49.013
Wealth	0.91	92.33	88.93	44.33	109.32	91.83	50.27	106.22	92.46	67.35	117.84	94.83
Beat Sample	n/a	Yes	Yes	n/a								
Return NS	62.36%	55.98%	61.55%	63.89%	65.75%	74.98%	56.33%	47.48%	58.34%	59.08%	46.95%	52.09%
Turnover NS	23.410	23.011	23.433	22.180	20.341	21.649	21.623	18.814	20.594	21.409	18.358	20.568
Wealth NS	128.20	123.66	127.53	131.03	135.02	140.65	125.70	122.01	128.66	128.19	122.14	123.61
Beat Sample NS	n/a	No	No	n/a	Yes	Yes	n/a	No	Yes	n/a	No	No

When we look at the *choose every time* strategy we see similar results. The turnover is understandably higher just as in the low-dimensional case. The returns however did not increase so the value of wealth decreased. This is especially true for the robust methods because all the robust unconstrained portfolios in the *choose and hold* strategy made money but when the *choose every time* strategy is used only the OGK method makes a profit. Also interesting to note is that with increasing the number of assets in the portfolio the turnover decreases. This is true for the sample method but also for the robust methods. This comes, as mentioned above due to the probability of an asset being in the portfolio for time period t and in time period t+1. However when an asset is selected in time t but then is left out in time t+1 then everything the investors hold needs to be sold. Or in the case of a short position, needs to be bought. This leads to a lot of transactions and so when the number of assets in such a portfolio is high the chance that an asset stays in the portfolio is higher which leads to lower turnover.

For the constrained portfolios the sample methods do a lot better. But also the robust methods improved slightly due to having lower turnover. Comparing it to the *choose and hold* strategy we see that the returns are lower and the turnover is higher which leads to a lower value of wealth. This difference is however smaller than for the unconstrained portfolio. The sample method performs so well mainly due to the high returns compared to the robust methods. Furthermore, we also see here the turnover decreases when the number of assets increases. This is similar to the Sharpe Ratio. Moreover, the OGK method once again has a more diversified portfolio which leads to lower turnover. This effect looks to be stronger when more assets are added to the portfolio. However, due to this the sample method is more profitable in 3 out of the 4 portfolios.

To conclude, the *choose every time* strategy is inferior to the *choose and hold* strategy but when an investor uses the *choose every time* strategy and uses the Sortino ratio as his characteristic then for the unconstrained the casewise robust method OGK is the best to use. For the constrained portfolios the sample method is slightly better but the difference is not as significant so using the robust methods could also be a viable choice, especially when the number of assets is not too high.

High Means
The third and last characteristic where the selection of assets is based, is the highest means. The results are denoted in Table 21. This paper looks first at the *choose and hold* strategy where we see similar results for the unconstrained portfolios compared to when the Sharpe ratio and the Sortino ratio were used as a characteristic. The robust methods perform the best due to having more diversification and lower wealth allocations. Furthermore, this leads to the robust methods having higher returns and lower turnover. Interestingly to note is that the turnover of the sample method is also for the high means characteristic decreasing when the number of assets goes up. This shows the phenomenon is independent of the characteristics used. The opposite happens to the robust method where in general the turnover goes up when the number of assets goes up. Note that this is only the case for unconstrained portfolios. Furthermore, we also see when using an unconstrained portfolio with the *choose and hold* strategy that only the robust methods make money, as also mentioned in the sections above. The sample method does however get more and more value of wealth and the opposite happens for the robust method so it would be interesting to see what happens when the whole portfolio is used. This is discussed later in this paper.

For the constrained portfolios we see the turnover going down and now all the methods make money. Looking at the returns they are generally lower compared to the unconstrained portfolio. This is especially true for the OGK method. Similar to the Sharpe and Sortino ratio, when N = 94 or N = 188 the robust methods outperform the sample method but with more assets the sample method performs better. This is mainly because the returns of the sample method stay high while the robust methods have lower returns because of over-diversification of the portfolio. The differences for the high mean characteristic are however smaller compared to the ratio characteristics. Looking at the N = 94 portfolio the differences between returns are almost none although we do see the returns of the sample method being a bit more extreme but not compared to the unconstrained portfolios. The robust method, and then the OGK method especially, are more diversified and also the weights for assets are low which eventually leads to lower turnover. The Spearman's correlation method on the other hand does have more problems with the return as it finds it more difficult to choose which assets to pick. We have also seen this for the other characteristic that due to specific dealing with contamination, the weights and thus the returns are a bit unstable. However, the turnover is still low due to the weights still not being extreme.

Table 21: High Mean GMV

(choose and hold)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-82.09%	80.26%	65.09%	70.54%	82.22%	73.58%	67.83%	69.60%	68.27%	60.46%	57.56%	56.09%
Turnover	395.120	20.067	31.931	88.201	24.878	38.969	73.773	25.669	41.013	69.288	23.458	42.519
Wealth	0.14	147.24	119.48	68.36	141.72	116.83	78.49	130.85	110.89	78.69	124.34	101.27
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	70.22%	75.33%	64.32%	65.89%	70.90%	75.48%	69.41%	58.77%	65.38%	72.52%	53.69%	60.64%
Turnover NS	18.460	13.172	16.038	20.368	15.601	18.340	20.494	16.012	19.453	20.913	15.999	20.187
Wealth NS	141.34	153.59	139.83	135.10	146.07	145.88	137.79	135.14	135.94	139.72	130.83	131.06
Beat Sample NS	n/a	Yes	No	n/a	Yes	Yes	n/a	No	No	n/a	No	No
(choose every time)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-11.71%	63.19%	64.41%	21.68%	52.47%	54.83%	34.70%	55.70%	61.05%	51.94%	52.20%	57.05%
Turnover	420.545	41.127	53.559	115.939	39.567	56.835	92.569	35.239	53.436	77.711	29.363	48.331
Wealth	0.51	107.43	95.13	36.04	101.99	86.56	51.49	108.90	93.29	68.14	113.07	95.946
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	76.04%	72.35%	71.91%	55.38%	58.91%	73.79%	55.40%	52.51%	60.73%	61.40%	45.19%	53.47%
Turnover NS	24.279	23.954	24.659	22.361	21.561	22.284	21.763	19.356	20.909	21.206	17.707	20.239
Wealth NS	137.77	135.33	134.02	124.00	127.85	138.80	124.77	125.49	130.18	130.32	121.48	125.15
Beat Sample NS	n/a	No	No	n/a	Yes	Yes	n/a	Yes	Yes	n/a	No	No

When we look at the *choose every time* strategy we see that the performance is much worse. Similar to the *choose and hold* strategy the robust methods outperform the sample method for the unconstrained portfolios but Spearman's correlation method also loses money. We see the returns are lower compared to the *choose and hold* strategy. This is similar to the low-dimensional case. The returns of the sample method go up when more assets are added to the portfolio. For the robust methods, the returns are quite stable. Furthermore, the turnover of the robust methods is significantly lower compared to the turnover of the sample method. Looking into more detail we see why this is the case. Analyzing the portfolio where the number of assets is 94 we see in the first time period the sample method had a return of -55% while the robust methods had a positive return. This was because in the first period, there was a lot of volatility and some assets had negative means due to some high negative returns. This led to extreme negative positions in a market that was on the up. The robust methods on the other hand negated these high negative returns and thus did not have so many short selling positions in the portfolio. This is also what happened in the portfolio with 94 assets when the *choose and hold* strategy was used. However, this also happened in the other direction where the returns of the sample method were far higher than the returns of the robust methods due to the robust methods making the high outliers less important and thus putting less weight on it. Furthermore, the performance of the robust methods was also hindered by the turnover due to having to change assets so much.

For the constrained portfolios we see an improvement but the performance is still worse than the *choose and hold* strategy. We do however see something interesting that we have also seen in the tables above, the turnovers of the sample method are not in general the highest of the methods and when they are there is not a significant difference. This is also true for the returns and so for the value of wealth. Furthermore, the problems the sample method had, with for example the portfolio of N=94, are with the introduction of the constraint almost completely gone. The constraint as mentioned before gives a certain shrinkage to the covariance estimate and thus there are not a lot of extreme positions. Furthermore, the different sizes of portfolios also give different results when it comes to whether the robust methods are performing better than the sample method so there are not a lot of conclusions to take from this strategy with the constrained portfolios.

470 asset portfolio

Lastly, we look at the portfolio where all the 470 assets are used. Here there is no distinction between the two different strategies. For the unconstrained portfolio, we see the value of wealth of the sample method did not increase compared to the *choose and hold* strategy and thus this trend does not continue. Furthermore, we see that this is also the case for the robust methods. In terms of turnover, the turnover for the sample method keeps decreasing with adding more assets to the portfolio. The turnover of the robust methods methods stayed roughly the same. As seen before, the reason why the robust methods are much better in terms of the value of wealth is because they diversify more equally over the assets which leads to lower turnover. So it is interesting to note that the 54.36% is almost equal to the return the equally weighted portfolio had mentioned in Section 3. For the constrained portfolios, however, we see a different picture with the sample method having the higher value of wealth. This difference is mainly because of the higher returns as the robust methods still have lower turnover. This difference is contributed to the sample method already benefiting from the shrinkage from the constraint and so using additionally the robust methods creates over-diversification which harm the returns.

Table 22: 470 asset portfolio GMV

	N=470		
	Sample	OGK	Spearman
Return	50.21%	54.36%	53.12%
Turnover	65.011	21.802	41.291
Wealth	77.06	123.88	100.63
Beat Sample	n/a	Yes	Yes
Return NS	61.59%	47.83%	54.74%
Turnover NS	21.285	15.948	20.119
Wealth NS	130.37	125.91	126.34
Beat Sample NS	n/a	No	No

Concluding remarks high dimensional GMV portfolios
To conclude this section we have seen for the GMV portfolios both the casewise robust method and the cellwise robust methods often outperform the sample method. This is especially the case for the unconstrained portfolios where short selling is allowed. This is independent of which strategy is used but the sample method did perform better for the *choose and hold* strategy. This is similar to the low-dimensional case. The reason why the robust methods were so effective in the unconstrained case is the fact that they were able to keep the weight allocations low and make a well-diversified portfolio, even with 94 assets. The sample method was not able to do this because the inverse covariance matrix had extremely high values on the diagonal which led to a high determinant for this matrix as well as extreme weight positions in the portfolio. This was not the case for the robust methods. The high-value determinant showed that linear combinations were harder to make so hedging of assets was not easy for the sample method. This also caused these high-weight allocations. Adding assets to the portfolio helped this somewhat with the sample method being more able to diversify with lower weight allocation but the robust methods were still better. With the introduction of the no short-selling constraint, the issue of the sample method was mostly fixed and so the turnover for the sample method was almost the same as for the robust methods. The robust methods often performed better in the *choose and hold* strategy but this was only where the number of assets was 94 of 188. Adding more assets to the portfolio led to the robust methods over-diversifying, leading to lower returns which meant that the sample method had higher values of wealth. For the *choose every time* strategy there were not any trends to see and so there was no way to make an informative choice on when to use the sample method or a robust method.

5.2.2 Tangency Portfolios

Sharpe Ratio

The first characteristic is the Sharpe ratio, where the results are denoted in Table 23. For the unconstrained portfolios where the *choose and hold* strategy is used, we see very clearly the robust methods performing better in stabilizing the weights. Furthermore, both the robust methods do not have a problem with the value of wealth calculation whereas the sample method in all the portfolios does except for the portfolio with 376 assets. The good performance of the robust methods compared to the sample method does however not lead to profitable returns confirming again that the *choose and hold* strategy does not work for tangency portfolios. Although the robust methods both perform well the OGK method is considerably better in stabilizing the weights. This is especially evident when the number of assets is relatively low. When we compare the performance for the individual time period for the N = 188 portfolios we see why the robust methods particular are so good in keeping the turnover low. For example, in the 8th time period the weights of the sample method are so extreme compared to the robust methods and so the sample method has the risk of choosing the wrong assets which happens in this case with a return of -69% where the robust methods only had a return of -3% and -4%. When we look at the mean estimates of the methods we see the differences are not that significant but the differences in the inverse of the covariance matrix are very significant and this is also what leads to the extreme portfolio positions. Where the diagonal values of the robust methods are all between 0 and 2.5 are the diagonals of the sample method all between 4000 and 18000. This shows that the robust methods are very good in minimizing the risk for the investor.

For the constrained portfolios, however, the turnover once again drops significantly. This leads to the sample method performing better than the robust methods due to shrinkage created in the portfolio weights. Furthermore, we see when the number of assets increases the bigger this difference between the sample method and the robust methods becomes. This is mainly because with the increasing number of assets the sample method is almost forced to diversify which helps with the returns but the OGK and Spearman's correlation method almost overdo it on this and so the portfolio becomes overdiversified losing returns in the process. The big difference here is that the sample method better knows which stocks to pick and the robust methods due to the minimizing of outliers are almost left with no information. We can see this when the portfolio of N=376 is considered. Similar to the GMV portfolio both the robust methods suffer from overdiversification compared to the sample method. On average the OGK uses 36 assets per time period and Spearman's correlation method uses considerably less with an average of 19 assets per time period. This is compared to the GMV portfolio, but the sample method only uses 17 assets per time period. So overdiversification is mostly a problem for the OGK method. This shows in the 10th time period where the OGK method uses 33 assets in the portfolio and Spearman's correlation method, as well as the sample method, uses only 20 assets. Furthermore, both the robust methods have the weights very equally divided, almost like it behaves as an equally weighted portfolio while the sample method has assets that hold 15-20% of the wealth. Due to these assets, the returns are also higher and so in this time period only the sample method manages to get a positive return. This is however mostly the case because the mean estimation is taken into account for the weight allocation of the tangency portfolio and so the sample method gets better information which leads the returns to increase. This is not true for the GMV portfolio where the mean estimations are not taken into account. Furthermore, the fewer assets there are to choose from the better the robust methods perform compared to the sample method.

Table 23: Sharpe Ratio Tangency

(choose and hold)	N=94			N=188			N=282			N=376		
	Sample*	OGK	Spearman	Sample*	OGK	Spearman	Sample*	OGK	Spearman	Sample	OGK	Spearman
Return	-104.02%	101.44%	-18.84%	-621.44%	41.21%	53.69%	-83.79%	179.09%	125.76%	196.12%	69.47%	83.79%
Turnover	4443.650	192.358	1069.22	5549.325	298.834	419.780	1963.640	186.233	412.388	971.428	260.797	399.478
Wealth	-0.33	19.34	1.42	-8389.38	2.63	0.39	-0.27	34.78	1.02	0.00	4.14	0.83
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	yes
Return NS	87.88%	81.51%	59.45%	72.34%	50.88%	49.82%	112.53%	76.70%	52.04%	75.43%	49.80%	42.66%
Turnover NS	23.497	22.863	24.020	22.849	23.130	23.327	22.865	22.457	23.779	23.451	23.049	24.000
Wealth NS	148.22	144.11	125.12	136.86	119.47	118.39	168.74	140.87	119.60	138.46	118.71	111.97
Beat Sample NS	n/a	No	No	n/a	No	No	n/a	No	No	n/a	No	No
(choose every time)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK*	Spearman
Return	560.86%	77.79%	101.87%	-9.83%	418.36%	102.01%	101.41%	65.82%	557.13%	221.00%	848.32%	147.31%
Turnover	1526.993	110.760	185.161	497.812	241.47	248.114	396.518	151.920	663.815	480.186	2953.028	956.307
Wealth	0.00	55.62	26.88	-0.12	23.46	11.94	1.25	31.53	100.70	0.48	67986.33	0.22
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	No
Return NS	75.91%	71.19%	46.37%	79.38%	72.11%	46.44%	80.58%	43.62%	45.54%	80.54%	38.55%	42.26%
Turnover NS	24.162	23.667	24.520	24.030	23.654	24.348	23.760	23.236	23.768	23.761	22.989	23.809
Wealth NS	137.83	134.81	114.27	140.74	135.56	114.52	142.07	113.60	114.50	142.03	109.87	111.87
Beat Sample NS	n/a	No	No	n/a	No	No	n/a	No	No	n/a	No	No

For the *choose every time* strategy in Table 23 we see for the unconstrained portfolios that the value of wealth is a little bit better but the high turnover still makes all the portfolios lose money. This is different from the GMV where the robust portfolio was either profitable or close to 100 euro. Furthermore, it is also different from the Tangency portfolios for the low-dimensional case where the estimations based on recent information would also mean that the portfolios are close to 100 or profitable. This is especially true for the sample method. The turnover of the sample method is however lower compared to the *choose and hold* strategy but this is not generally the case for the robust methods. Looking at the portfolio with 94 assets more closely we see why the returns of the sample method are so high. First of all, the turnover of the sample method suggests the portfolio has extreme positions and this is also the case when looking at the returns with three time periods having more than 100% return and one even having as high as 540% return. However, not all returns are positive with four time periods having returns close to -50%. In the second time period, the sample method had a return of 540% while the robust methods had 4% and 28% returns. Where is here the difference? As expected the sample method had way more extreme positions than the robust methods but choosing the assets that were extreme also paid off. The OGK method was so diversified that all the weights were between 0.08 and -0.05. This is because the mean estimates for every asset were very close to each other. The Spearman's correlation method was in between these methods where some assets had relatively high weights but most of them were just as low as the OGK method. This choice is for investors of course not sustainable and so the robust methods seem to be the better choice.

Sortino Ratio

The second characteristic is the Sortino Ratio, denoted in Table 24. First of all, we look at the *choose and hold* strategy. The portfolios are, similar to the Sharpe Ratio, very extreme and so in general are not a good option for an investor to use. Comparing the methods gives us that the turnover for the robust methods is significantly lower than for the sample method. This is however in vain because they are still too high to result in a profitable portfolio for the investor. Here the cellwise robust method is often higher than the casewise robust method except for the case when the number of assets is 188. Going into more detail we see the biggest reason for the higher turnover is the big differences in values of the center. Some assets get big wins in a time period and others big losses. This difference and the fact that the tangency portfolio uses the mean estimate in the calculation of the weights leads to the weights being very extreme when no robust method is used. Furthermore, this is aggregated when the order of highest means and losses is changing all the time which leads to extreme positions going from for example -4 in one period to +5 in another. This is mainly the case for N = 188 and N = 282. For the constrained portfolio the turnovers are again back to a normal value and thus the portfolio makes money again. We see, similar to the Sharpe Ratio, that the sample method does well compared to the robust methods. This is because the sample method now benefits from the shrinkage that comes with the constraint and the robust methods under-appreciated the stocks that do very well and over-appreciate stocks that do badly. In other words, because the mean estimate is used in the portfolio allocation, the robust methods are less good at selecting the stocks that do very well and make a more diversified portfolio. This diversification harms the returns and so the robust methods end up with lower values of wealth.

Table 24: Sortino Ratio Tangency

(choose and hold)	N=94			N=188			N=282			N=376		
	Sample*	OGK	Spearman	Sample*	OGK*	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-408.72%	12.92%	37.70%	-2431.97%	-186.11%	18.16%	-43.80%	137.58%	157.62%	196.12%	69.47%	83.79%
Turnover	1691.992	131.537	340.842	6198.072	738.170	562.558	6737.661	430.606	450.338	971.428	260.799	399.476
Wealth	-132.35	27.66	1.96	-5.58e+8	-0.19	0.01	-0.81	5.40	0.60	0.00	4.14	0.83
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	89.74%	64.40%	89.83%	82.19%	72.62%	61.25%	112.52%	56.29%	53.28%	75.43%	49.80%	42.66%
Turnover NS	23.507	22.976	23.459	22.874	23.038	23.444	22.865	22.601	23.777	23.451	23.049	24.000
Wealth NS	149.66	130.38	149.81	144.64	136.81	127.27	168.73	124.42	120.57	138.46	118.71	111.97
Beat Sample NS	n/a	No	Yes	n/a	No	No	n/a	No	No	n/a		
(choose every time)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	270.38%	33.60%	148.27%	-11.32%	203.83%	111.99%	133.88%	77.30%	499.71%	168.91%	41.24%	127.87%
Turnover	1102.604	90.253	183.902	489.657	166.360	245.923	378.826	173.070	595.002	459.035	142.942	1309.211
Wealth	0.00	52.04	33.77	-0.02	47.45	12.91	2.20	25.12	61.83	0.69	30.53	0.04
Beat Sample	n/a	Yes	Yes	n/a								
Return NS	76.45%	60.42%	46.33%	79.31%	43.12%	46.85%	80.58%	41.96%	45.79%	80.54%	44.79%	41.19%
Turnover NS	24.167	23.676	24.743	24.044	23.261	24.346	23.760	23.270	23.789	23.761	23.231	23.829
Wealth NS	138.25	126.32	113.98	140.67	113.17	114.85	142.07	112.25	114.66	142.03	114.53	111.00
Beat Sample NS	n/a	No	No	n/a	No	No	n/a	No	No	n/a	No	No

For the *choose every time* strategy we see the same as for the low dimensional case. For the unconstrained portfolios, we see the turnover decreases and so the value of wealth increases. We also see, similar to the *choose and hold* strategy, the turnover difference between the sample and the robust method being very large for N=94 but this slowly declines over time with the sample turnover even getting lower than the turnover for the Spearman's correlation method. When we look at the value of wealth the robust methods perform the best due to high returns as well as often having the lower turnover. This is however not enough to make the portfolios profitable with the highest value of wealth being 61.83. This indicates that this is not a viable strategy to use. For the constrained portfolio we see a better picture with all the portfolios making money. The performance is however slightly lower in terms of the value of wealth compared to the *choose and hold* strategy. This is due to lower returns as well as the turnover being marginally higher. Furthermore, the difference between the sample method and the robust methods is low with the sample method even having a lower turnover than Spearman's correlation method. The OGK does have a lower turnover compared to the sample method but as mentioned this difference is minimal. Lastly, when looking at the value of wealth the sample method is superior, beating the robust methods in all the portfolio allocations. This difference is mainly due to the higher returns the sample method generates. The sample method can do this because when an asset suddenly has a high return the sample method can put more weight on this asset. The robust method minimizes this outlier and so has a more conservative approach which leads to lower returns.

To conclude, the unconstrained portfolios perform slightly better with the *choose every time* strategy but the combination of tangency and unconstrained portfolios is still not a good strategy. For the constrained portfolios the *choose every time*is a good strategy but the *choose and hold* strategy is slightly better but both are profitable.

High Mean

The third characteristic is the high mean where the results are shown in Table 25. Firstly we look at the *choose and hold* strategy where the unconstrained portfolios give an interesting read. In general, the results show that the tangency portfolio, due to its aggressiveness is not a good portfolio to use when short selling is allowed. When we compare the methods we see the turnover of the robust methods is significantly lower than the sample method. Looking at the returns we see big differences between methods but also within the same method, we see big differences. This shows the unpredictability of the out-of-sample tangency portfolio. This unpredictability can also be seen in the sample method when N = 282. The value of wealth is the highest of the high-dimensional portfolios we have seen so far and it is very different from the performance of this portfolio when Sharpe Ratio or Sortino Ratio is used. Lastly what is also interesting is that this is not only the case for this portfolio but also the other portfolios are different to both the ratios. For example when N=94 the method that has the highest value of wealth is the OGK method while for the Sharpe and Sortino Ratio the Spearman's correlation method. For the constrained portfolios we see much better performance and even the best performance compared to the other characteristics. The returns are higher and the turnover is significantly lower compared to the unconstrained portfolios. Similar to what we have seen before, the turnover of the sample method is not always higher than the robust methods. This gives an extra indication that due to the shrinkage of the constraint, the weights are automatically more stable. This is also a reason why the sample method does so well due to the extra stability you achieve by using the robust methods is small while the trade-off with the return is high.

Table 25: High Mean Tangency

(choose and hold)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-2.06%	-94.52%	325.60%	-65.19%	73.89%	173.86%	3929.17%	76.37%	-43.79%	129.05%	243.38%	57.78%
Turnover	2049.663	292.187	582.434	1910.982	187.184	513.724	2751.427	290.086	518.365	911.507	388.762	390.001
Wealth	-0.02	0.15	49.97	0.03	20.05	0.22	332.77	4.77	0.36	0.00	9.03	0.58
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	No	No	n/a	Yes	Yes
Return NS	111.82%	98.01%	82.58%	85.10%	64.42%	65.59%	122.63%	59.11%	55.95%	75.19%	49.57%	42.20%
Turnover NS	23.175	21.573	22.984	23.057	22.842	23.839	23.160	23.687	23.894	23.449	23.383	23.998
Wealth NS	167.65	159.29	144.78	146.67	130.57	130.18	176.23	125.27	122.53	138.27	118.13	111.60
Beat Sample NS	n/a	No	No	n/a	No	No	n/a	No	No	n/a	No	No
(choose every time)	N=94			N=188			N=282			N=376		
	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman	Sample	OGK	Spearman
Return	-9.03%	76.41%	166.21%	-107.52%	104.13%	109.44%	-103.49%	84.76%	690.30%	231.25%	118.59%	72.43%
Turnover	1218.070	69.752	129.383	441.931	147.226	236.291	542.101	149.841	1162.678	789.729	190.049	377.218
Wealth	0.00	85.96	67.71	-0.03	39.48	13.75	-0.19	33.06	318.077	1.19	27.16	1.16
Beat Sample	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes	n/a	Yes	Yes
Return NS	78.20%	65.72%	45.34%	78.24%	56.08%	44.15%	80.54%	43.69%	44.50%	80.54%	42.15%	39.96%
Turnover NS	24.328	24.013	24.649	23.990	23.568	24.236	23.761	23.271	23.902	23.761	23.389	23.858
Wealth NS	139.39	130.04	113.32	139.90	123.04	112.86	142.03	113.62	113.52	142.03	112.26	110.01
Beat Sample NS	n/a	No	No	n/a	No	No	n/a	No	No	n/a	No	No

Using the *choose every time* strategy we see that the turnover decreases for the unconstrained portfolios. However, the turnovers do not decrease when more assets are used in the portfolio like it does for the Global Minimum Variance portfolio. What is also interesting to see is the value of wealth for the robust methods is higher compared to the *choose and hold* strategy. This is also not the case for the GMV portfolio, with the extreme value of wealth of 318.08 for Spearman's correlation method when N = 282, compared to a value of wealth of -0.19 for the sample method. This is especially interesting because the portfolio of N = 282 was the best performing for the sample method when *choose and hold* strategy was used. If we take the portfolio where the number of assets equals 94 into consideration we see a very interesting fact that explains part of the reason why the robust methods do so much better than the sample method. The estimations for the means are very different from each other, especially when it comes to which assets have a negative mean. Of the 12 assets that the sample method estimates with a negative mean only 5 also have a negative mean for the OGK method. Furthermore, only 4 out of these 12 have a negative real return so the sample method loses money by going short on these portfolios. Most of these assets also had a negative mean due to a couple of negative outliers but the medians of these assets were mostly positive. This once again shows the benefit of using the robust methods. For the constrained portfolio we see the sample method, similar to the other characteristics, being once again the best method. The shrinkage the constraint brings to the estimations is enough to help the problems that the sample method had. Furthermore not being able to short sell helps also with the negative means that caused the big losses for the unconstrained portfolios.

470 asset portfolio Tangency

Lastly, we look at the portfolio where all the 470 assets are used. For the unconstrained portfolio, we immediately see that even we such a large number of assets the tangency portfolio is too aggressive to be a profitable portfolio. We do however see as expected the robust methods outperform the sample method where the OGK method performs the best. For the constrained portfolios we see the sample method being the best due to having high returns. These high returns are a consequence of the sample method putting a higher proportion of the wealth on some assets where the robust methods are more spread out. This works for the sample method because as mentioned before the shrinkage that the constraint creates gives enough to let the tangency portfolio not be so aggressive.

Table 26: 470 asset portfolio Tangency

	N=470		
	Sample	OGK	Spearman
Return	-31.04%	46.28%	-26.80%
Turnover	857.208	198.596	444.900
Wealth	0.00	15.63	0.08
Beat Sample	n/a	Yes	Yes
Return NS	80.55%	31.99%	43.45%
Turnover NS	23.760	22.945	23.949
Wealth NS	142.05	104.70	112.64
Beat Sample NS	n/a	No	No

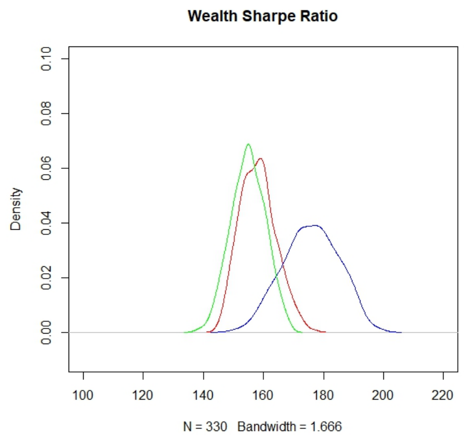
Concluding remarks high dimensional Tangency portfolios

Concluding this section on the tangency portfolio we go over the main findings for both the unconstrained and the constrained portfolios. For the unconstrained portfolio, the conclusion is the same as it was for the low-dimensional case. The unconstrained tangency portfolio is not a stable portfolio to make profits independent of which method is used. Furthermore, this conclusion is also independent of whether the *choose and hold* or the *choose every time* strategy is used. This is different from the low-dimensional case where all the characteristics could produce stable profitable portfolios with the *choose every time* strategy. So although the overall conclusion is that investors should not use the unconstrained tangency portfolios the robust methods did help with stabilizing the weights. Here the casewise robust methods were better than the cellwise robust methods. The constrained portfolios were again stable and more importantly profitable for the investor. For the strategies the *choose and hold* strategy overall performed the best. Furthermore due to the shrinkage in the estimates that the constraints brought were the sample method often the best method to use. This was mainly because the robust methods over-diversified the portfolio which led to low returns. This was for both strategies the case. This difference between the sample method and the robust methods also grew when the number of assets grew in the portfolio. The casewise robust method was for the *choose and hold* strategy almost always the better robust method to use having higher returns as well as lower turnover. However, for the *choose every time* strategy it depended on the number of assets in the portfolio. The OGK was often the better method for the portfolios where N = 94 or N = 188 but the Spearman's correlation method was better when the number of assets was above that.

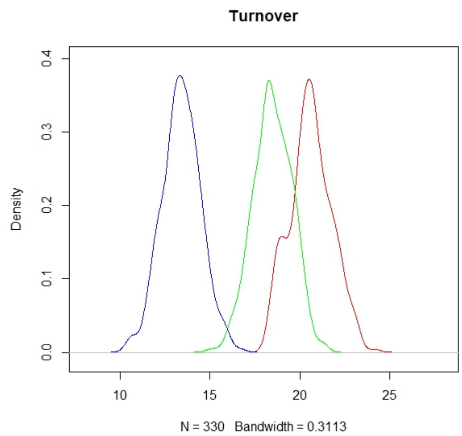
5.3 Takeaways + further analysis

So what can we take away from these results? What is the conclusion? First of all, for the low-dimensional case, we can conclude the robust methods used were not able to produce the desired results where the turnovers were lower than the sample method. This was especially the case for the Global Minimum Variance portfolio but also the results for the Tangency portfolio were not consistent enough to conclude that the robust methods were an improvement on the sample method. In the high-dimensional case, the robust methods did help reduce the turnover. This was especially the case for the GMV portfolio when short selling was allowed. With these portfolios, the investors benefited from using the robust methods to be more profitable. Is this because the high dimensional portfolios are more suited for the robust methods or is the reason that the OGK method and Spearman's correlation method are more equipped to use for stabilizing the weights? To see which it is, this section does the low-dimensional portfolio estimations with the OGK and Spearman's correlation method. It does so by using the Global Minimum Variance portfolio in the low-dimensional framework of 330 portfolios for each of the three characteristics for the *choose and hold stategy*. The reason for only using the *choose and hold stategy* is because for both the low and high dimensional case this was the best-performing portfolio. The results are shown below where for the figure the red line represents the sample method, the blue line the OGK method and the green line represents the Spearman's correlation method.

When we do this for the Sharpe ratio characteristic, we see these robust methods get the desired result. The densities are shown in Figure 7a. First of all, in the right figure, we can see that the turnover of the robust methods is lower than the sample method. This is especially true for the density of the OGK method where it has no to little overlap with the density of the sample method. This is different from other robust methods used in the low-dimensional section. The lower turnover also results in high values of wealth for the OGK method which can be seen in the left figure. For Spearman's correlation method, the values of wealth are similar to the sample method indicating that the returns are lower. So at least for the Sharpe ratio as characteristic we can conclude that the OGK and Spearman's correlation method produce more stable weights. However is this only the case for the Sharpe Ratio or can we replicate the results for the other characteristics as well?



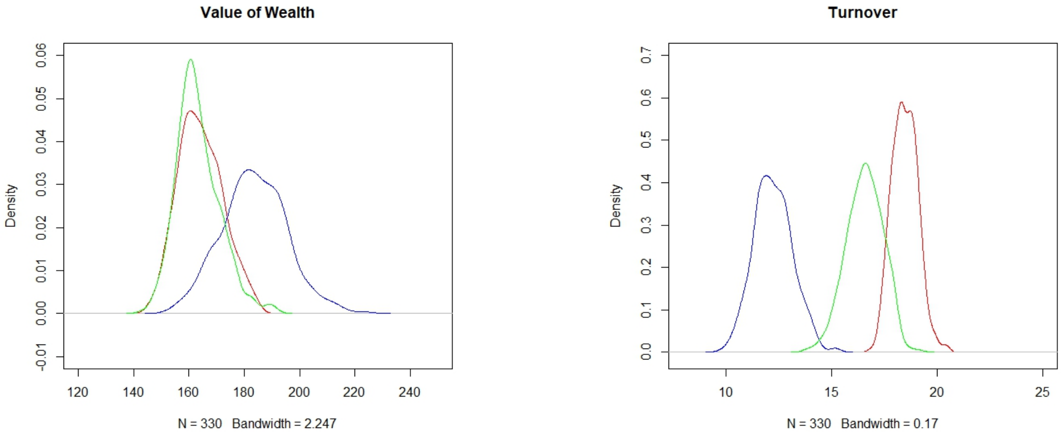
(a) Value of Wealth



(b) Turnover

Figure 7: Sharpe Ratio unconstrained portfolio choose and hold strategy

In Figure 8a and Figure 8b we see the value of wealth and the turnover when the Sortino Ratio is used and then we do see the results can be replicated with a different characteristic. This gives another indication that these robust methods are equipped to stabilize the weights. Furthermore, we see the density of the turnovers for the OGK and Spearman's correlation are almost identical to the Sharpe Ratio. For the sample method, the density is more centered between 18 and 19 with a density as high as 0.6 indicating lower turnovers compared to the Sharpe Ratio. For the value of wealth, the OGK performs the best while the sample method and Spearman's correlation have similar values of wealth indicating that the cellwise robust method struggles to get high returns.

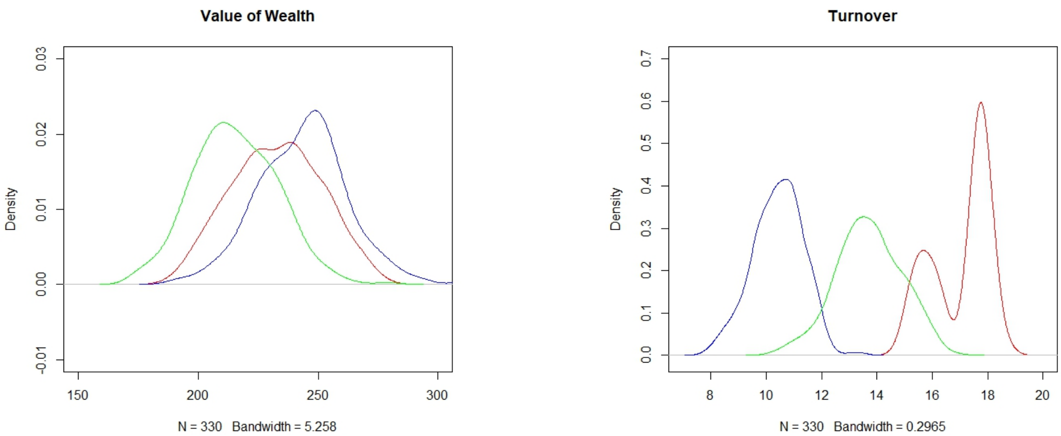


(a) Value of Wealth

(b) Turnover

Figure 8: Sortino Ratio unconstrained portfolio choose and hold strategy

When the high mean is used as a characteristic, denoted in Figure 9a, we see a different picture for the values of wealth. The densities of the sample method and the OGK method are overlapping so we can conclude the difference in values of wealth is less significant. However, it is interesting that the turnovers do differ significantly, and similar to the other characteristics the turnover of the OGK method does not overlap and thus is for all 330 portfolios lower compared to the sample method. This results in the conclusion that for the high mean characteristic, the sample method returns are notably different from the OGK method. Lastly, Spearman's correlation method is performing considerably worse than the other two methods, once again confirming the conclusion that the casewise robust method is better at handling the weight stability than the cellwise robust method is.



(a) Value of Wealth

(b) Turnover

Figure 9: High Mean unconstrained portfolio choose and hold strategy

5.3.1 Mahalanobis distance methods vs pairwise covariance methods

This paper thus shows that robust methods based on Mahalanobis distances do not work for this dataset and for estimating weights over time. The weights are more extreme and volatile than for the sample method and this also causes higher turnover. This is not the case for the robust methods based on pairwise covariances. As we have seen in the figures above the weights stabilize more which leads to lower turnover. This section focuses on the differences between these methods and compares the Minimum Covariance determinant method with the Orthogonalized Gnanadesikan-Kettenring method and the cellwise Minimum Covariance determinant method with Spearman's correlation method to see where the methods based on the Mahalanobis distances go wrong. This is done for the unconstrained GMV portfolio based on the 15 highest Sortino ratio with the choose and hold strategy as here the differences are the biggest.

MCD method vs OGK method

First of all, this paper analyses the difference between two casewise robust methods. Comparing the return, turnover and wealth of both methods we already see big differences with the MCD having a return of 101.34%, a turnover of 23.547 and the value of wealth of 158.74 where the return of the OGK method is 137.07%, the turnover 10.526 and the value of wealth is 213.28. This shows the usefulness of the OGK method because these numbers also outperform the sample method. But now the question begs, where does the MCD method go wrong?

Looking at the covariance matrices of both methods we see that in terms of determinant, the covariance matrix of the OGK method has a higher value than the covariance matrix of the MCD method. This leads to the determinant of the inverse of the MCD method being more than 10 times the value of the inverse of the OGK method. The higher determinant for the inverse also is an indication of higher weights. This is exactly what we see for the first time period where the weights for the MCD are more extreme than the OGK method. Analyzing the contamination process gives us in the first time period that out of the 90 rows, the MCD found 19 to be contaminated whereas the OGK method found 17 rows to be contaminated. Furthermore, both methods agreed on 82 out of the 90 rows whether they were contaminated or not. So in this sense, the difference is not that large. What we however do notice is the distance measurements of the OGK method are much larger than the distance measurements of the caseMCD method. This could also help the OGK better identify the outliers which leads to more stability in the weights.

cellMCD vs Spearman's Correlation

The second comparison this paper makes is the difference between the cellwise minimum covariance determinant method and Spearman's correlation method. These are both cellwise robust methods and similar to the other comparison this paper analyses the returns and turnover of these methods. Additionally, as the outliers are identified individually we can better see where the differences between the two methods are. First of all, looking at the value of wealth the difference is clear. The cellMCD method, as seen in Section **??** has a value of wealth of 116.86 whereas Spearman's correlation method has a value of wealth of 154.98. Furthermore, the turnover of the cellMCD method is also twice as large as the turnover of Spearman's correlation method with 36.671 against 16.184. Also, the returns of Spearman's correlation method are higher with 82.41% versus 69.57%. Looking at the time period individually we see why Spearman's correlation method does so much better than the cellMCD method. First of all, the determinant of the precision matrix of the cellMCD method is significantly higher than that of Spearman's correlation method. This is mainly because Spearman's correlation method creates much more zeros in the precision matrix. This also translates to a more even distribution of the weights in the portfolio. This helps certainly when the portfolio chooses the wrong assets to short or go long in.

6 Discussion

This paper has discussed the use of robust methods to estimate the covariance matrix as well as the means of portfolios to stabilize the weights of the portfolio. The results indicate for the low-dimensional case that the robust methods which rely on Mahalanobis distances do not perform better than the sample method. Here we saw the casewise robust methods slightly outperform the cellwise outlier methods because of lower turnover and so being better at stabilizing the weights. The robust methods, that are based on pairwise covariance or correlation, such as the OGK and Spearman's correlation method, did bring more stability to the weights which led to better values of wealth for the investors. For the high-dimensional case, the OGK and Spearman's correlation method also made a positive impact in stabilizing the weights. This biggest difference compared to the sample method was when no short-selling was allowed.

One of the limitations of this paper is the fact that for the unconstrained tangency portfolio, the results are slightly skewed because the wealth equation does not perform well when the wealth drops below 0. Another limitation is the fact that only big companies were used in the data set so the conclusions taken from this paper can only be used for big companies and further research needs to be done on whether these conclusions also hold for smaller companies or for example cryptocurrencies where volatility and outliers are bigger factors.

This paper did mainly focus on the performance of the robust methods and less on the side of active trading. For the portfolios, an initial value of 100 was used and there was no additional money put into the portfolio. For further research, this could also be interesting to look into. Strategies could be formed where certain indicators could lead to putting more money into the portfolio or retrieving money from the portfolio. For example, money can be retrieved from the portfolio after a certain time period where the returns were very high. This is to ensure that not a big part of the winnings is lost when the portfolio eventually drops in value. The opposite can be done when an asset has a relatively low market value to have an advantage once it goes back up again. Furthermore, another strategy could be used that every time period the investor uses 100 euros to invest and retrieves the full return after each time period. This strategy would not use cumulative returns to gain an advantage but would also allow for a value of wealth calculator that can deal with returns that are lower than -100%. On the other hand, strategies could be formed utilizing the cumulative returns by putting each time period more money into the portfolio. This is in the real world often done for trading indices where the volatility and where the risk is relatively low.

The last point of interest for further research is to look at varying time windows. This could be interesting because a longer time window means fewer trades are happening which could lead to lower turnovers. On the other hand, it could harm potential gains when an asset is held too long. The role of the robust methods and an analysis of the performance compared to different time windows could be an interesting topic to investigate. For this, the robust methods based on pairwise covariances are the best to use.

7 Conclusion

This paper analyzed the use of casewise and cellwise robust methods in portfolio allocations. It did this with daily stock data from the S&P500. The robust methods were used to estimate the means and covariance matrices that shaped the weights of the portfolio assets. This was done for the Global Minimum Variance portfolio and the Tangency Portfolio in a low-dimensional case and also a high-dimensional case. For the low-dimensional case, we found both the casewise and the cellwise robust methods based on the Mahalanobis distance were not able to stabilize the weights more than the sample method did for the Global Minimum Variance portfolios. This conclusion holds regardless of whether short-selling was allowed or not and also is independent of the strategy used. This conclusion is in line with the work in the paper (Pacreau and Lounici, 2024). For the robust methods themselves, the casewise robust methods often did perform better than the cellwise robust methods in terms of stabilizing the weights. This was often due to high turnovers from the cellwise robust methods due to higher portfolio allocations. The performance for the returns and the values of wealth was between the robust methods more random. For the Tangency portfolio, we found that the portfolio often did not make any money when short-selling was allowed. This was especially true for the *choose and hold* strategy where a lot of methods even came into the negatives. The value estimations had problems when this happened and as mentioned in the discussion other methods of using the money as an investor could help this. For the *choose every time* strategy this was not a problem as estimations were based on more recent information and so led to weights on the portfolio to be less extreme. Furthermore, the casewise robust methods did stabilize the weights more than the sample method when no short selling was allowed. However, this was not the case for the cellwise robust methods. For the portfolios where short selling was allowed the sample method again performed the best. So the Mahalanobis distance robust methods did certainly not improve the stabilizing of the portfolio weights. Constrained portfolios for both strategies made the portfolios more profitable and the turnover dropped dramatically.

For the high-dimensional case, this paper looked at portfolios where the number of assets was 94, 188, 282, 376 and 470. Both the casewise, the OGK method and the cellwise robust method, Spearman's correlation method were considered but these methods are based on pairwise covariances instead of being based on Mahalanobis distances. For the GMV portfolio, both the casewise and the cellwise robust methods were able to help stabilize the portfolio weights which led to a higher value of wealth. This was for all the portfolios but the biggest effect was when short selling was allowed. This is because the constraint introduced shrinkage to the portfolio which already leads to stabilizing the weights. Furthermore, for the unconstrained GMV portfolio, the weights of the sample method stabilized more when the number of assets increased. This was true for both strategies used in this paper. This was however not true for the robust methods.

For the Tangency portfolios, we saw the unconstrained portfolios were not profitable, but both robust methods made sure that the loss was minimized compared to the sample method. For the constrained portfolios the sample method was far better due to the shrinkage already given to the estimates by the constraints. The robust methods did not do so well due to overdiversification which led to lower returns. Furthermore, the turnover of the casewise robust methods was always lower but this was not the case for the cellwise robust method. Lastly, the robust methods based on pairwise covariances did so well for the high-dimensional case that this paper also analyzed how they would do for the low-dimensional case. This was only done for the unconstrained Global Minimum Variance portfolio where the *choose and hold* strategy was used. This paper found that these robust methods based on pairwise covariances also stabilized the weights more for the low dimensional case compared to the sample method. The OGK did this better than Spearman's correlation method giving more reason to think that the casewise robust methods are better to use than the cellwise robust methods. The biggest gain in the value of wealth was for both the Sharpe Ratio and the Sortino Ratio. The portfolios where the high mean was the characteristic were on the other hand less successful due to the returns of the sample method being much higher.

In conclusion, in this paper, we saw both the casewise and the cellwise robust methods based on the Mahalanobis distance for estimating the means and covariances are not capable of stabilizing the weights of a portfolio based on characteristics and the sample method does a better job in this. The instability of these methods was also found in the paper (Pacreau and Lounici, 2024). Although these methods did not work the casewise methods and the cellwise methods based on pairwise covariances did succeed in stabilizing the weights for both low-dimensional and high-dimensional portfolios, so investors would be better off estimating their portfolio weights based on means and covariances of these methods instead of the sample method.

8 Appendix A

These are all the companies where the assets were used in the database. This includes the 35 stocks that eventually were discarded.

Assets: AAL, AAPL, AAP, ABBV, ABC, ABT, ACN, ADBE, ADI, ADM, ADP, ADSK, ADS, AEE, AEP, AES, AET, AFL, AGN, AIG, AIV, AIZ, AJG, AKAM, ALB, ALGN, ALK, ALLE, ALL, ALXN, AMAT, AMD, AME, AMGN, AMG, AMP, AMT, AMZN, ANDV, ANSS, ANTM, AON, AOS, APA, APC, APD, APH, APTV, ARE, ARNC, ATVI, AVB, AVGO, AVY, AWK, AXP, AYI, AZO, A, BAC, BAX, BA, BBT, BBY, BDX, BEN, BF.B, BHF, BHGE, BIIB, BK, BLK, BLL, BMY, BRK.B, BSX, BWA, BXP, CAG, CAH, CAT, CA, CBG, CBOE, CBS, CB, CCI, CCL, CDNS, CELG, CERN, CFG, CF, CHD, CHK, CHRW, CHTR, CINF, CI, CLX, CL, CMA, CMCSA, CME, CMG, CMI, CMS, CNC, CNP, COF, COG, COL, COO, COP, COST, COTY, CPB, CRM, CSCO, CSRA, CSX, CTAS, CTL, CTSH, CTXS, CVS, CVX, CXO, C, DAL, DE, DFS, DGX, DG, DHI, DHR, DISCA, DISCK, DISH, DIS, DLR, DLTR, DOV, DPS, DRE, DRI, DTE, DUK, DVA, DVN, DWDP, DXC, D, EA, EBAY, ECL, ED, EFX, EIX, EL, EMN, EMR, EOG, EQIX, EQR, EQT, ESRX, ESS, ES, ETFC, ETN, ETR, EVHC, EW, EXC, EXPD, EXPE, EXR, FAST, FBHS, FB, FCX, FDX, FE, FFIV, FISV, FIS, FITB, FLIR, FLR, FLS, FL, FMC, FOXA, FOX, FRT, FTI, FTV, F, GD, GE, GGP, GILD, GIS, GLW, GM, GOOGL, GOOG, GPC, GPN, GPS, GRMN, GS, GT, GWW, HAL, HAS, HBAN, HBI, HCA, HCN, HCP, HD, HES, HIG, HII, HLT, HOG, HOLX, HON, HPE, HPQ, HP, HRB, HRL, HRS, HSIC, HST, HSY, HUM, IBM, ICE, IDXX, IFF, ILMN, INCY, INFO, INTC, INTU, IPG, IP, IQV, IRM, IR, ISRG, ITW, IT, IVZ, JBHT, JCI, JEC, JNJ, JNPR, JPM, JWN, KEY, KHC, KIM, KLAC, KMB, KMI, KMX, KORS, KO, KR, KSS, KSU, K, LB, LEG, LEN, LH, LKQ, LLL, LLY, LMT, LNC, LNT, LOW, LRCX, LUK, LUV, LYB, L, MAA, MAC, MAR, MAS, MAT, MA, MCD, MCHP, MCK, MCO, MDLZ, MDT, MET, MGM, MHK, MKC, MLM, MMC, MMM, MNST, MON, MOS, MO, MPC, MRK, MRO, MSFT, MSI, MS, MTB, MTD, MU, MYL, M, NAVI, NBL, NCLH, NDAQ, NEE, NEM, NFLX, NFX, NI, NKE, NLSN, NOC, NOV, NRG, NSC, NTAP, NTRS, NUE, NVDA, NWL, NWSA, NWS, OKE, OMC, ORCL, ORLY, OXY, O, PAYX, PBCT, PCAR, PCG, PCLN, PDCO, PEG, PEP, PFE, PFG, PGR, PG, PHM, PH, PKG, PKI, PLD, PM, PNC, PNR, PNW, PPG, PPL, PRGO, PRU, PSA, PSX, PVH, PWR, PXD, PX, PYPL, QCOM, QRVO, RCL, REGN, REG, RE, RF, RHI, RHT, RJF, RL, RMD, ROK, ROP, ROST, RRC, RSG, RTN, SBAC, SBUX, SCG, SCHW, SEE, SHW, SIG, SJM, SLB, SLG, SNA, SNI, SNPS, SO, SPGI, SPG, SRCL, SRE, STI, STT, STX, STZ, SWKS, SWK, SYF, SYK, SYMC, SYY, TAP, TDG, TEL, TGT, TIF, TJX, TMK, TMO, TPR, TRIP, TROW, TRV, TSCO, TSN, TSS, TWX, TXN, TXT, T, UAA, UAL, UA, UDR, UHS, ULTA, UNH, UNM, UNP, UPS, URI, USB, UTX, VAR, VFC, VIAB, VLO, VMC, VNO, VRSK, VRSN, VRTX, VTR, VZ, V, WAT, WBA, WDC, WEC, WFC, WHR, WLTW, WMB, WMT, WM, WRK, WU, WYNN, WYN, WY, XEC, XEL, XLNX, XL, XOM, XRAY, XRX, XYL, YUM, ZBH, ZION, ZTS

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