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Assessing Risk Premia Estimators' Robustness to Spurious Factors

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Abstract

In this paper we do a research on the risk premia estimators' robustness to spurious factors. Including spurious factors in linear asset pricing models result in unreliable estimation techniques. This research is relevant to investors and analysts, as it allows them to focus their investment strategies solely on relevant sources of risk. We use the 25 portfolios formed on size and book-to-market (5x5) from Kenneth French for the period between January 1972 and January 2023. The six different estimators investigated are the Fama Macbeth(FM) estimator, the Penalized FM estimator with a penalization that is inversely related to the beta, the Penalized FM estimator with a penalization that is inversely related to the correlation of the factor with the excess returns, the Elastic Net estimator, the Adaptive Elastic Net estimator and the Model Selection procedure. These estimators are compared by means of simulated and real data. From the results it follows that both of the Penalized FM estimators are identifying the spurious factors the best. For the smaller models we see that the Adaptive Elastic-Net estimator in the short run and the Elastic Net estimator in the middle and long run identify the spurious factors and estimate the risk premia for the intercept and the useful factor the best.

Contents

1 Introduction

Asset pricing models play a vital role in understanding the relationship between asset returns and underlying risk factors. A large body of asset pricing literature tries to find the factors that explain the expected excess returns for assets. The first and most used linear factor model is the CAPM, introduced by Sharpe (1964) as well as Lintner (1965). This model is a single factor model, where one factor enters the SDF and is priced. In the case of the CAPM the factors that enters the SDF is the market factor. The recent asset pricing literature has identified an exploding number of candidate risk factors; see, e.g., Harvey et al. (2015) who catalogues 316 factors. However, most researchers believe that it is unrealistic to think that there may be hundreds of risk factors. Therefore, it is likely that most candidate factors in the literature are spurious, or "weak" factors, i.e., factors that are only weakly related to expected returns. Including these weak factors completely destroys the statistical properties of the main estimating methodologies(GMM and two-pass regression) in terms of identification, consistency and statistical inference(Kan & Zhang (1999a) and Kan & Zhang (1999b)). Because of this, modern asset pricing datasets with a large number of observed factors are not reliably estimated via standard methodologies.

In the literature some papers try to develop estimation methods that can handle weak factors. For example Gospodinov et al. (2014) proposed a sequential model selection procedure. The authors claim that this procedure restores the standard inference and can effectively remove the factors that have no explanatory power. Bryzgalova (2015) proposed a penalized two-step procedure. The author claims that this procedure eliminates the impact of weak factors, reestablishes consistency and the asymptotic Gaussian distribution of risk premia estimates, along with enhancing the precision of standard fit measurements. The method proposed by Bryzgalova (2015) is based on a penalization which is inversely related to the betas. The penalization is determined in this way, because the betas are the reflection of the low correlation of the useless(or weak) factors with the asset returns. We also investigate a penalization that is based on the correlation of the factors with the returns. When a factor is useless(or weak) it can be easily seen by the betas or directly by the correlations, because when a factor is useless(or weak) the correlation and the beta are close to zero.

In our research, we propose a new risk premia estimation procedure based on the adaptive elastic net penalty (Zou & Zhang (2009)). Moreover, we assess the robustness of our procedure to the presence of weak factors, and compare it to the procedures proposed by Gospodinov et al. (2014) and Bryzgalova (2015). The proposed Adaptive Elastic Net method by the authors is a combination of the strengths of the quadratic regularization as well as the lasso shrinkage with adaptive weights. Because we are verifying previous research, this paper is of scientific relevance.

Finding the factors that are weak and cannot explain the expected excess returns for assets is of importance for investors and analysts. When including weak factors, the usual estimation techniques are unreliable because the risk premia of weak factors may be deemed spuriously significant, while the risk premia of strong factors may be overly rejected. Separating the strong from the weak factors is relevant to investors and analysts as well, as it allows them to focus their investment strategies solely on relevant sources of risk.

The research on weak factors in linear asset pricing models is necessary, because weak factors can bring in noise and unnecessary complexity into the model. It can lead to estimation issues, such as biased parameter estimates, inflated significance levels, and reduced model efficiency. Research on spurious factors helps to identify and address these estimation problems, leading to more robust and reliable estimation results. Also, weak factors can hide the interpretation of the coefficients related with the relevant risk factors. By eliminating the effect of weak factors, the coefficients are more meaningful and easier to interpret. Additionally, removing weak factors result in a less computationally intensive and time-consuming estimation process. So an estimation process that is more efficient and less prone to estimation issues. Lastly, removing weak factors helps to mitigate overfitting and promotes more robust and reliable modeling results, given that overfitting can cause insufficient out-of-sample performance and reduced generalizability.

To address these limitations, further research is needed to expand the understanding of weak factors in asset pricing models. This includes developing robust methodologies for identifying, estimating, and validating weak factors, exploring new data sources and techniques, and examining the practical implications of incorporating weak factors in investment strategies. By filling these gaps, the existing knowledge can be significantly advanced, leading to more accurate and comprehensive asset pricing models.

In this paper we investigate six different estimators. Namely, the Fama Macbeth(FM) estimator, the Penalized FM estimator with a penalization that is inversely related to the beta, the Penalized FM estimator with a penalization that that is inversely related to the correlation of the factor with the excess returns, the Elastic Net estimator, the Adaptive Elastic Net estimator and the Model Selection procedure. These estimators are compared by means of simulated and real data. In our simulation analysis, we consider a simulation for four different models and compare the estimators based on the risk premia, the bias, the R^2 values and the squared misspecification-robust Hansen-Jaghannatan distance. Also we perform empirical analysis by investigating six different models, with different factors. We look at the risk premia, the R^2 values and the Hansen-Jaghannatan distance. Also we check if the models are correctly identified.

The main findings are that both of the Penalized FM estimators are identifying the spurious factors the best. For the smaller models we see that the Adaptive Elastic-Net estimator in the short run and the Elastic Net estimator in the middle and long run identify the spurious factors and estimate the risk premia for the intercept and the useful factor the best.

This paper is organised as follows. In Section 2 we delve in to previous literature and position this paper in the previous literature. In Section 3 we provide an overview of the methodologies and the simulation setting. In section 4 we present the data used and give insights to it. In section 5 we describe the Emprical Analysis that we perform. In Section 6 we show the key findings and in section 7 the conclusion and discussion of further research can be found.

2 Literature

Extensive research, e.g., Jagannathan & Wang (1998), Kan & Zhang (1999b), Kleibergen (2009), Gospodinov et al. (2014),Kleibergen & Zhan (2015) and Burnside (2016), highlight the problematics connected with the inclusion of weak factors in an asset pricing model. These problematics involve the mis-identification of the factor risk premia coefficient, and the consequent unreliability of estimation and inference methods. Because of these failures it is necessary to investigate the weak factors.

In the paper Bryzgalova (2015) the author investigated the problem of finding an estimation technique to estimate the cross-sectional asset pricing models that can handle a large number of 'weak' factors. In this paper they defined weak factors as factors with a zero population factor loading (beta coefficient). These weak factors can lead to misleading results and undermine the validity of the model. This paper developed the Penalized Fama-Macbeth approach, which makes use of an adaptive lasso type of penalty where the weights are inversely related to a measure of size factor beta, thereby eliminating the impact of spurious factors asymptotically.

Also, in the paper Gospodinov et al. (2014) the authors tried to find an approach that can handle weak factors. In this paper the authors define weak factors as useless, i.e., a factor that is independent to test asset returns. The approach suggested in the paper is the sequential model selection procedure. The procedure works as follows. At each step, the misspecification-robust SDF coefficient and the corresponding t-statistic are computed. If the smallest t-statistic is insignificant, then the factor is removed and we proceed to the next step. Otherwise, the procedure stops. According to the paper, this selection procedure is resilient against irrelevant factors and potential model inaccuracies, and it is successful in removing factors that cease to enhance the model's pricing capability.

The papers Gospodinov et al. (2014) and Bryzgalova (2015) have contributed to the literature by suggesting a method that can handle the potential presence of spurious factors. Our paper is contributing to the literature by trying to validate that these methods can handle spurious factors. Also, the examination is done with more recent data then used by the two papers. The difference between the two papers is that the paper Bryzgalova (2015) pay attention to models using β-representation, that shows the partial pricing effect regarding distinct risk factors, while the paper Gospodinov et al. (2014) is identifying the weak factors and subsequently eliminating them. They both use the simulation constructed by Gospodinov et al. (2014).

Furthermore, in the paper Zou & Zhang (2009) the adaptive elastic net method is introduced. According to the paper the adaptive elastic net method is combining the strengths of the quadratic regularization as well as the lasso shrinkage with adaptive weights. The authors claim that this method tackles better the collinearity issue compared to oracle-like methods. But it is still not investigated if the adaptive elastic net method can handle a lot of weak factors. So, we contribute to the literature by looking to the performance of the adaptive elastic net in this framework.

Their has also been research where they assumed that the parameters are already identified, in contrast with what we assume. The paper Lewellen et al. (2010) show that in the case that a group of assets displays any robust factor structure, every variable linked with that hidden risk factors could be recognized as a variable that has an effect on the cross-section of returns. This paper is contributing to the literature through suggesting solutions to the problem. Unfortunately, these solutions do not cause improved identification.

3 Methods

In this study we investigate the finite sample properties of various methodologies for estimating and evaluating linear asset pricing models by means of extensive simulation and empirical studies. Firstly, we explain the linear factor model and the Fama-Macbeth procedure. Subsequently, we delve into the approaches we use. Finally, we elaborate on the simulations conducted in our study.

3.1 Linear factor model and Fama-Macbeth procedure

Our linear factor framework is comprised of equations (1), (2) and (3).

$$E(R_t^e) = \beta_f \lambda_{0,f}, \tag{1}$$

$$cov(R_t^e, F_t) = \beta_f var(F_t), \tag{2}$$

$$E(F_t) = \mu_f, \tag{3}$$

where $t = 1, ..., T$, R_t^e is a $n \times 1$ vector that contains the excess portfolios returns on time t, i_n is a $n \times 1$ vector of ones, β_f is a $n \times k$ matrix of the betas of the portfolios with respect to the factors, $\lambda_{0,f}$ is a $k \times 1$ vector of risk premia on the factors and μ_f is a $k \times 1$ vector containing averages of the factors. The model formed by the equations (1), (2) and (3) can also be written as equations (4) and (5). In this case the disturbances u_t and v_t respectively $n \times 1$ and $k \times 1$ vectors are added.

$$R_t^e = \beta_f \lambda_{0,f} + \beta_f v_t + u_t, \tag{4}$$

$$F_t = \mu_f + v_t. \tag{5}$$

Furthermore we demean the factors and excess returns, see equation (6) and (7) and we define the mean of the disturbance μ_t as in equation (8).

$$\bar{F}_t = F_t - \frac{1}{T}\sum_{t=1}^{T} F_t, \tag{6}$$

$$\bar{R}_t^e = R_t^e - \frac{1}{T}\sum_{t=1}^{T} R_t^e, \tag{7}$$

$$\bar{v} = \frac{1}{T}\sum_{t=1}^{T} v_t. \tag{8}$$

Demeaning and getting rid of the μ_f makes it possible to update equation (4) to equation (9), where the disturbance ϵ_t is defined as in equation (10).

$$R_t^e = \beta_f(\bar{F}_t + \lambda_{0,f}) + \epsilon_t = \beta_f \lambda_{0,f} + \beta_f \bar{F}_t + \epsilon_t, \tag{9}$$

$$\epsilon_t = u_t + \beta_f \bar{v}, \tag{10}$$

where β_f is a $n \times (k)$ matrix of the asset betas, and $\lambda_{0,f}'$ is a $(k) \times 1$ vector of the risk premia. Equation (11) is showing the first step, which is computing the esimates of β_f that is resulting from regressing the excess returns on factors, of the Fama-Macbeth procedure introduced in

Fama & MacBeth (1973).

$$\hat{\beta}_f = \sum_{t=1}^{T} \bar{R}_t^e \bar{F}_t' \left(\sum_{j=1}^{T} \bar{F}_j \bar{F}_j' \right)^{-1}, \tag{11}$$

where $\hat{\beta}_f$ is a $n \times k$ matrix. The beta in this setting is showing how a factor associates with the excess returns, but it lacks to show if the indicated association is priced as well as if it might be exploited to clarify the distinction among essential rates of return on different securities. Step two, a single OLS cross-sectional regression, has the goal to control if asset holders ask some premium due to their exposure to factor's risk. The second step includes regressing the mean excess returns on factor loadings:

$$\hat{\lambda}_{OLS} = \left(\hat{\beta}' \hat{\beta} \right)^{-1} \hat{\beta}' \bar{R}^e, \tag{12}$$

where $\hat{\beta}_f$ is a $n \times k$ matrix and $\hat{\lambda}_f$ is a vector of $k \times 1$ containing the risk premia estimates.

3.2 Sequential model selection procedure

The sequential model selection procedure is proposed by Gospodinov et al. (2014). It is designed to address the potential bias and inefficiency caused by including irrelevant factors in linear asset-pricing models. By employing misspecification-robust inference techniques, the methodology provides robust estimation and testing procedures that account for the presence of these irrelevant factors.

The sequential factor screening procedure is as follows. At each step, the misspecification-robust SDF coefficient and the corresponding t-statistic are computed. If the smallest t-statistic is insignificant, then the factor is removed and we proceed to the next step. Otherwise, the procedure stops.

Also for the sequential model selection approach they employ the Bonferroni method, explained in Romano et al. (2008), to control the total incorrect discoveries and maintain the asymptotic validity. The total number of SDF parameters being checked we define as K and we define $p_{T,i}$ as the p-value for checking the hypothesis $H_0 : \gamma_i = 0$. Call back that every single of the hypotheses $H_0 : \gamma_i = 0$ for $i = 1, ..., K$ are checked by relating the model-error-resistant t-statistic opposed to the critical value of the $N(0, 1)$ distribution for a significance level α. Afterwards, the Bonferroni method rejects the null hypothesis when $p_{T,i} \leq \alpha/K$ rather than $p_{T,i} \leq \alpha$ like the usual way.

For the implementation of this procedure in R Core Team (2023) we use the function GKR-

FactorScreening in the package intrinsicFRP of Quaini (2024).

3.3 Penalized FM estimator

In Bryzgalova (2015) the author introduced the Penalized FM (Factor Model) estimator as a new approach to estimate parameters in linear asset pricing models. The Penalized FM estimator expands the traditional FM estimator by incorporating penalization terms that encourage sparsity and enhance model selection in the presence of potentially irrelevant risk factors. In this setting the first stage of the Fama-Macbeth algorithm, equation (11), stays the same. For the second stage, equation (13), we define the correct values of risk premia being $\lambda_0 = (\lambda_{0,c}, \lambda_{0,F})$ and assume they are in the compact parameter space $\Theta \in \mathbb{R}^k$.

$$\hat{\lambda}_{pen\_beta} = \arg\min_{\lambda \in \Theta} \left(\bar{R}^e - \hat{\beta}\lambda\right)' W_T \left(\bar{R}^e - \hat{\beta}\lambda\right) + \eta_T \sum_{j=1}^{k} \frac{1}{||\hat{\beta}_j||_1^d} |\lambda_j|, \tag{13}$$

where d as well as η_T greather than 0 are the parameters we tune and $||\cdot||_1$ is the L_1 norm of a vector, so $||\hat{\beta}_j||_1 = \sum_{i=1}^{n} |\hat{\beta}_{i,j}|$.

As can be seen the Penalized FM estimator exists of two parts: the first part is the same as a standard OLS and the second part is the penalized term. The difference with the adaptive lasso is that the alteration implemented in this case guarantees that the motivating factor regarding the shrinkage term is neither the magnitude linked to the risk premium, but rather the inherent characteristics of the betas.

Our extension on this estimator is that we also look at a different penalization term. Instead of a penalization that is inversely related to the size factor beta we investigate a penalization that is inversely related to the correlation of the factor with the excess returns. For this estimator the first stage of the of the Fama-Macbeth algorithm, equation (11), stays the same. For the second stage, equation (14), we define the correct values of risk premia being $\lambda_0 = (\lambda_{0,c}, \lambda_{0,F})$ and assume they are in the compact parameter space $\Theta \in \mathbb{R}^k$.

$$\hat{\lambda}_{pen\_corr} = \arg\min_{\lambda \in \Theta} \left(\bar{R}^e - \hat{\beta}\lambda\right)' W_T \left(\bar{R}^e - \hat{\beta}\lambda\right) + \eta_T \sum_{j=1}^{k} \frac{1}{||\hat{z}_j||_1^d} |\lambda_j|, \tag{14}$$

where d as well as η_T greather than 0 are the parameters we tune and $||\cdot||_1$ is the L_1 norm of a vector, so $||\hat{z}_j||_1 = \sum_{i=1}^{n} |\hat{z}_{i,j}|$, where $\hat{z}_{i,j} = cov(\bar{F}_{i,j}, \bar{R}^e_i)$.

3.4 Elastic Net

Zou & Hastie (2005) presented the Elastic Net method, which brings together and balances the

L1(Lasso) as well as L2(Ridge) regularization penalties and results in coefficients who can be easily interpreted. This method is performing feature selection and it is inducing sparsity. The L1 term is stimulating sparsity by shrinking the coefficients of features, who are not correlated with the excess returns to zero and in this way it is performing feature selection. The L2 penalty helps to stabilize and control the magnitude of the coefficients that are not zero and improves robustness to multicollinearities in factors. In this setting the first stage of the Fama-Macbeth algorithm, equation (11), stays the same. For the secondary stage estimations of the Elastic Net for the risk premia see equation (15).

$$\hat{\lambda}_{enet} = (1 + \frac{\alpha_2}{n}) \left[\arg\min_{\lambda \in \Theta} \left(\bar{R}^e - \hat{\beta}\lambda \right)' W_T \left(\bar{R}^e - \hat{\beta}\lambda \right) + \alpha_2 ||\lambda||_2^2 + \alpha_1 ||\lambda||_1 \right], \tag{15}$$

where α_1 and α_2 are the tuning parameters.

From Zou & Hastie (2005) it follows that the Elastic Net method should handle collinearity and represents a deficiency in the oracle property, while from Zou (2006) it follows that the adaptive lasso attains the oracle property of the SCAD and is endowed with the instability associated with the lasso relating to high-dimensional data. That is why we consider the adaptive elastic net proposed by Zou & Zhang (2009), which combines the elastic net with the adaptive lasso. Firstly the elastic net estimator defined in equation (15) should be computed to obtain the estimates $\hat{\lambda}_{j,enet}$. The optimization problem of equation (16) can be solved to obtain the adaptive elastic net estimates.

$$\hat{\lambda}_{aenet} = (1 + \frac{\alpha_2}{n}) \left[\arg\min_{\lambda \in \Theta} \left(\bar{R}^e - \hat{\beta}\lambda \right)' W_T \left(\bar{R}^e - \hat{\beta}\lambda \right) + \alpha_2 ||\lambda||_2^2 + \alpha_1^* \sum_{j=1}^{k} \frac{1}{||\hat{\lambda}_{j,enet}||_1^d} |\lambda_j| \right], \tag{16}$$

where the L1 regularization parameter α_1^* could be different of α_1 and the L2 regularization parameter α_2 is the same for the elastic net and adaptive elastic net. The α_1 and α_2 are the tuning parameters. Tuning these parameters has the goal to control the balance between the L1 and L2 penalties, so the effectiveness of the method. We determine the optimal parameters by using 5 fold cross-validation.

3.5 Simulations

We look to simulations for the sample sizes as in previous literature(Bryzgalova (2015)): $T = 30, 50, 100, 250, 500, 1000$ and investigate the small-sample performance of our estimators explained before.

We do simulations for the following four models:

1. A model that consists of a constant term and a useful factor

2. A model that consists of a constant term, a useful factor and two useless factors

3. A model that consists of a constant term, a useful factor and two weak factors.

4. A model that consists of a constant term, three useful factors, and eleven useless factor.

The returns are simulated out of a normal distribution with the observational mean along with the variance of the excess returns.

The useful factor of model 1,2 and 3 is simulated out of a normal distribution with the observational mean along with the variance of the market excess returns. For model 4 the three useful factors have a covariance with the returns of 0.8, 0.5 and 0.3 with respectively a mean and variance equal to the observational mean and variance of the market excess returns, the profitability factor and the investment factor of the Fama and French five factor model.

The useless factors for model 2 are simulated out of a normal distribution with respectively the mean as well as the variance calibrated to the mean and variance of the SMB and HML factors of Fama and French. The useless factors have a covariance of zero with the excess returns, with each other and with the other factors. For model 4 the means and variances of the useless factors are all calibrated to the observational mean and variance of the HML factor. Also they have a covariance of zero with the excess returns, with each other and with the other factors.

The weak factors for model 3 are simulated out of a normal distribution with the mean as well as the variance calibrated to respectively the mean and variance of the SMB and HML factors of Fama and French. The difference between the weak factors and the useless factors is that the weak factors have a fastly vanishing covariance with the returns because the covariance of the factor and the returns is scaled with $1/(T^{3/4})$. The weak factors have a covariance of zero with each other and with the other factors.

I calculate regular measures of fit:

– The cross-sectional OLS-based R^2

$$R^2_{ols} = 1 - \frac{var\left(\bar{R}^e - \hat{\lambda}_{ols}\hat{\beta}\right)}{var(\bar{R}^e)} \tag{17}$$

– The squared misspecification-robust Hansen-Jagannathan distance of Kan & Robotti (2008), a modification of the prominent Hansen-Jagannathan distance of Hansen & Jagannathan (1997)

$$HJM^2 = min_d(E[R] - Cov[R, F]d)'V[R]^{-1}(E[R] - Cov[R, F]d) \tag{18}$$

Finally, I compare these models with the results of each seperately.

4 Data

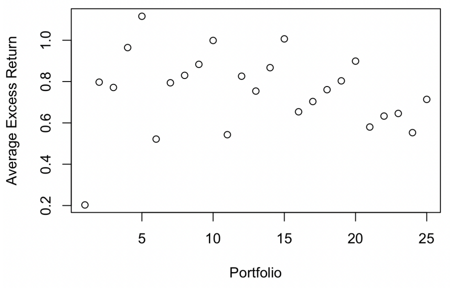
For our research we use the 25 portfolios formed on size and book-to-market (5x5) from Kenneth French for the period between January 1972 and January 2023. So, we have a total of 613 observations. See figure (1) for some insights in the data of respectively the average and standard deviation of the excess returns of the 25 portfolios formed on size and book-to-market (5x5). It is clear that for portfolios with high book-to-market firms the return is higher than for low book-to-market, but portfolios with high book-to-market firms have also a higher standard deviation. Also portfolios with smaller firms have a higher average return than portfolios with larger firms, this also follows from Roll (1981). Furthermore it is clear that portfolios with smaller firms have a higher standard deviation.

Table 1: Mean and standard deviation of the Fama/French factors for the period between January 1972 and January 2023.

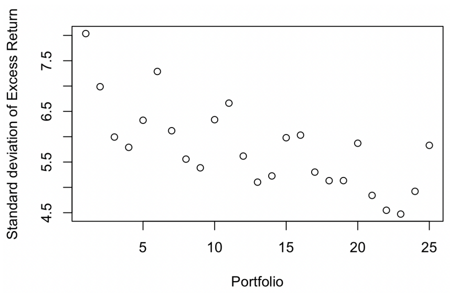
	Mean	Stdv
F1	0.591	4.604
F2	0.120	3.050
F3	0.313	3.097
F4	0.303	2.316
F5	0.318	2.039

Additionaly, we use the Fama/French factors formed using the 6 value-weight portfolios formed on size and book-to-market. These Fama/French factors are the excess market return, SMB, HML, RMW and CMA. The excess market return is calculated by the formule $Rm - Rf$, where Rm is the return on the market and Rf is the risk free rate. See Table (1) for some insights in the data of the excess market return and the Fama/French factors.

The return on the market is the value-weight return of all CRSP firms incorporated in the United States and listed on the New York Stock Exchange, American Express, or NASDAQ. Also, we use the real per capita non-durable goods, the real per capita durable consumption and L, per capita real disposable personal income, for the United States from the Bureau of Economic Analysis. We look at the period between January 1972 and October 2022. Furthermore, from Lettau & Ludvigson (2001) we use the consumption-aggregate wealth ratio (cay). In this case we look at the period between January 1972 and September 2019. It is quarterly data, but we have linearly interpolated it following Vissing-Jørgensen & Attanasio (2003).



(a) Averages excess returns of 25 portfolios(5x5)



(b) Standard deviations of 25 portfolios(5x5)

Figure 1: The averages and standard deviations of excess returns of 25 portfolios(5x5) formed on size and book to market ratio

5 Empirical Analysis

To show the relevance of our theoretical results we also perform empirical analysis.

5.1 Asset-pricing models

We investigate six asset-pricing models. The first model is the simple static CAPM with a SDF specification as in equation (19) below.

$$y_t^{CAPM}(\gamma) = \gamma_0 + \gamma_1 vw_t, \tag{19}$$

where vw is the excess return of the Fama/French factors.

The second model we investigate is the three factor model of Fama & French (1993), where the SDF specification can be seen in equation 20. This model is the CAPM extended with the two Fama/French factors smb and hml, where smb is the return of the portfolios of stocks with small capitalization minus the stocks with large capitalization, and hml is the return of the portfolios of stocks with high book to market ratio minus the stocks with low book to market ratio.

$$y_t^{FF3}(\gamma) = \gamma_0 + \gamma_1 vw_t + \gamma_2 sml_t + \gamma_3 hml_t, \tag{20}$$

The third model we analyze is the scaled CAPM specification with human capital (C-LAB) of Jagannathan & Wang (1998) and considered by Lettau & Ludvigson (2001). The SDF specification is

$$y_t^{C-LAB}(\gamma) = \gamma_0 + \gamma_1 cay_{t-1} + \gamma_2 vw_t + \gamma_3 lab_t + \gamma_4 vw_t \cdot cay_{t-1} + \gamma_5 lab_t \cdot cay_{t-1}, \tag{21}$$

where we follow Jagannathan & Wang (1996) and make use of a two-month moving average to define $lab_t = (L_{t-1} + L_{t-2}/(L_{t-2} + L_{t-3}) - 1$ as the growth rate in per capita labor income.

The fourth model is the unconditional consumption model(CCAPM) with the following SDF specification

$$y_t^{CCAPM}(\gamma) = \gamma_0 + \gamma_1 c_{nd,t}, \tag{22}$$

where we define $c_{nd,t}$ as the growth rate in real per capita nondurable consumption.

The fifth model is the conditional consumption model(CC-CAY) introduced by Lettau & Ludvigson (2001), with the following SDF specification

$$y_t^{CC-CAY}(\gamma) = \gamma_0 + \gamma_1 c_{nd,t} + \gamma_2 cay_{t-1} + \gamma_3 c_{nd,t} cay_{t-1}, \tag{23}$$

which is a result of scaling the constant term as well as the c_{nd} factor by a constant as well as cay.

The sixth and last model is highlighting the cyclical role of durable consumption(D-CCAPM) and it is introduced by Yogo (2006), with the following SDF specification

$$y_t^{D-CCAPM}(\gamma) = \gamma_0 + \gamma_1 vw_t + \gamma_2 c_{nd,t} + \gamma_3 c_{d,t}, \tag{24}$$

where we define $c_{d,t}$ as the growth rate in real per capita durable consumption.

5.2 Violation of identification condition

We examine if their is any correlation between the diverse risk factors and the asset returns as well as if the six asset-pricing models as stated in section 5.1 are correctly identified.

Theoretically adding a weak factor is leading to a violation of the important identification condition that the $N \times K$ matrix $B = E[x_t \bar{f}_t']$ is of full column rank. This is why it is interesting to test the hypothesis that B is of rank $K - 1$. Because $\hat{B}$ changes when the data is re-scaled, $\hat{B}$ has to be normalized. We normalize by $\hat{\Theta} = \hat{U}^{-\frac{1}{2}} \hat{B} \hat{S}_{\bar{f}}^{-\frac{1}{2}}$ and its equivalent population counterpart $\Theta = U^{-\frac{1}{2}} B S_{\bar{f}}^{-\frac{1}{2}}$, where $\hat{S}_{\bar{f}} = \frac{1}{T} \sum_{t=1}^{T} \bar{f}_t \bar{f}_t'$ and $S_{\bar{f}} = E[\bar{f}_t \bar{f}_t']$. It can be observed that $\hat{\Theta}$ is not changing when the data changes and $rank(\Theta) = rank(B)$. We define $\hat{\Pi}$ as a consistent estimator of the asymptotic covariance matrix of $\sqrt{T} vec(\hat{\Theta} - \Theta)$. We conduct a singular value decomposition on $\hat{\Theta}$, following Ratsimalahelo (2001) and Kleibergen & Paap (2006), such that $\hat{\Theta} = \tilde{U}\tilde{S}\tilde{V}'$, where $\tilde{U}'\tilde{U} = I_N, \tilde{V}'\tilde{V} = I_K$ and $\tilde{S}$ is a $N \times K$ matrix with singular values of $\hat{\Theta}$ in decreasing order on its diagonal. We define $\tilde{U}_2$ as the last $N - K + 1$ columns of $\tilde{U}$, $\tilde{V}_2$ as the last column of $\tilde{V}$ and

$$\tilde{\Pi} = (\tilde{V}_2' \otimes \tilde{U}_2')\hat{\Pi}(\tilde{V}_2 \otimes \tilde{U}_2). \tag{25}$$

The test with null hypothesis of $rank(\Theta) = rank(B) = K - 1$ is

$$W^* = T\tilde{s}_K^2 \tilde{\Pi}^{11} \xrightarrow{d} \chi^2_{N-K+1}, \tag{26}$$

where we define $\tilde{s}_K$ as the smallest singular value of $\hat{\Theta}$ and $\tilde{\Pi}^{11}$ as the (1,1) element of $\tilde{\Pi}^{-1}$.

6 Results

In this section we show the results. First we show the results of the simulations separately for each model, then we compare the results of the models with each other and lastly we present the results for the empirical analysis. We define the short run as T=30,50 and 100, the medium run as T=250 and the long run as T=500 and 1000.

6.1 Model 1

In table 2 you can see the estimates of the risk premia and the biases for model 1. Model 1 is only estimated using the Fama Macbeth estimator and it consists of an intercept and a useful factor. From table 2 it can be seen that for a higher T the risk premium for the useful factor is more correctly estimated. For example for a T equal to 30, the estimate of the risk premium of the useful factor has a bias of -0.294 and for a T equal to 1000, a bias of -0.126. The bias is closer to zero for a T equal to 1000, which signs a more correctly estimation of the risk premium.

Also we can see in this table that the useful factor have for all T a negative risk premium. So for example for T equal to 50, the risk premium for the useful factor is equal to -0.421. A negative risk premium means that the factor have a negative effect on the returns of the assets. Also the intercept is always positive. For example for a T equal to 100 the intercept is equal to 1.263. A positive intercept means that when all factors are set to zero the assets generate a positive risk premium. Probably because we look to excess returns this is the risk premium of the risk free rate.

Table 2: The estimates and biases of the risk premium for a model with one useful factor

	Estimates		Biases	
	Intercept	Useful Factor	Intercept	Useful Factor
30	1.111	-0.377	0.176	-0.294
50	1.178	-0.421	0.109	-0.250
100	1.263	-0.486	0.025	-0.185
250	1.326	-0.542	-0.039	-0.129
500	1.335	-0.547	-0.048	-0.124
1000	1.335	-0.545	-0.048	-0.126

Note. The table presents the estimates and the biases of the risk premia in a model with an intercept and a useful factor. We present findings for various values of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

6.2 Model 2

In table 3 the estimates and in table 4 the biases can be found for model 2. Model 2 is estimated using the five different estimators and it consists of an intercept, a useful factor and two useless factors. From table 4, several insights emerge..

Firstly we can observe that for all estimators and T's the intercept is always positive and the useful factor is always negative. For example the intercept of the Elastic Net estimator is 0.910 and the useful factor is -0.281 for T equal to 30. As concluded earlier the useful factor have then a negative effect on the excess returns because the risk premium is negative. Also, when all factors are set to zero the assets generate a positive risk premium because the intercept is positive .

Secondly we can see that in the short run the Adaptive Elastic Net estimator performs in general better than the Elastic Net estimator, which is in the middle and long run the other way around. Also the Penalized FM estimator that is inversely related to the correlation performs in general better in the short and middle run then the Penalized FM estimator inversely related to the betas, in the long run it is the other way around. In all runs, the FM estimator performs in general better than both of the Penalized FM estimators in estimating the risk premia of the intercept and the useful factor. However, both of the Penalized FM estimators exhibit a lower bias in estimating the useless factors across all runs.

Based on the bias the best estimator for the intercept and the risk premium of the useful factor in the middle run is the Elastic Net Estimator and in the short and long runs is the FM estimator. However, in these cases the useless factors are not set to zero. For the short run and long run the Penalized FM estimator with a penalization that is inversely related to the correlation is estimating the risk premia of the useless factors the best. For the middle run the

Penalized FM estimator with a penalization inversely related to the size factor beta is estimating the risk premia of the useless factors the best.

The estimator that is performing the best in general in estimating the intercept and all factors in the short run would be the Adaptive Elastic Net estimator and in the middle and long run the Elastic Net estimator.

Table 3: The estimates of the risk premium for a model with one useful factor and two useless factors

	True Parameter values	Estimates				
		FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
				Panel A: T=30		
Intercept	1.287	0.937	0.878	0.910	0.940	0.923
Useful factor	-0.671	-0.332	-0.198	-0.281	-0.319	-0.241
Useless factor	-	0.175	0.096	0.163	0.162	0.080
Useless factor	-	0.024	-0.003	-0.002	0.003	-0.007
				Panel B: T=50		
Intercept	1.287	1.089	0.943	1.019	1.024	0.995
Useful factor	-0.671	-0.419	-0.208	-0.326	-0.340	-0.255
Useless factor	-	0.023	0.040	0.030	0.043	0.026
Useless factor	-	0.005	-0.002	0.011	0.013	0.001
				Panel C: T=100		
Intercept	1.287	1.171	0.965	1.063	1.070	1.012
Useful factor	-0.671	-0.508	-0.252	-0.383	-0.397	-0.289
Useless factor	-	-0.038	-0.016	-0.030	-0.036	-0.011
Useless factor	-	0.025	0.024	0.031	0.032	0.008
				Panel D: T=250		
Intercept	1.287	1.572	0.947	1.313	1.206	0.994
Useful factor	-0.671	-0.807	-0.178	-0.553	-0.449	-0.220
Useless factor	-	-0.807	0.000	-0.672	-0.413	-0.001
Useless factor	-	-0.430	0.000	0.044	-0.040	-0.001
				Panel E: T=500		
Intercept	1.287	1.266	0.909	1.093	1.018	0.883
Useful factor	-0.671	-0.569	-0.174	-0.383	-0.323	-0.136
Useless factor	-	0.135	0.023	0.115	0.123	-0.014
Useless factor	-	-0.003	0.023	0.009	0.020	0.019
				Panel F: T=1000		
Intercept	1.287	1.298	0.860	1.111	1.012	0.821
Useful factor	-0.671	-0.577	-0.103	-0.378	-0.294	-0.056
Useless factor	-	0.363	0.096	0.271	0.233	-0.013
Useless factor	-	-0.192	-0.056	-0.059	-0.083	0.006

Note. The table presents the estimates of the risk premia in a model with an intercept, a useful factor and two useless factors. We present findings for various values of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

Table 4: The biases of the risk premium for a model with one useful factor and two useless factors

	Biases				
	FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
			Panel A: T=30		
Intercept	0.350	0.409	0.377	0.347	0.364
Useful factor	-0.339	-0.473	-0.390	-0.352	-0.430
Useless factor	0.175	0.096	0.163	0.162	0.080
Useless factor	0.024	-0.003	-0.002	0.003	-0.007
			Panel B: T=50		
Intercept	0.198	0.344	0.268	0.263	0.292
Useful factor	-0.252	-0.463	-0.345	-0.331	-0.416
Useless factor	0.023	0.040	0.030	0.043	0.026
Useless factor	0.005	-0.002	0.011	0.013	0.001
			Panel C: T=100		
Intercept	0.116	0.322	0.224	0.216	0.275
Useful factor	-0.163	-0.419	-0.288	-0.274	-0.382
Useless factor	-0.038	-0.016	-0.030	-0.036	-0.011
Useless factor	0.025	0.024	0.031	0.032	0.008
			Panel D: T=250		
Intercept	-0.285	0.340	-0.026	0.081	0.293
Useful factor	-0.136	-0.493	-0.118	-0.222	-0.451
Useless factor	-0.807	0.000	-0.672	-0.413	-0.001
Useless factor	-0.430	0.000	0.044	-0.040	-0.001
			Panel E: T=500		
Intercept	0.021	0.378	0.194	0.269	0.404
Useful factor	-0.102	-0.497	-0.288	-0.348	-0.535
Useless factor	0.135	0.023	0.115	0.123	-0.014
Useless factor	-0.003	0.023	0.009	0.020	0.019
			Panel F: T=1000		
Intercept	-0.011	0.428	0.176	0.275	0.466
Useful factor	-0.094	-0.568	-0.293	-0.377	-0.615
Useless factor	0.363	0.096	0.271	0.233	-0.013
Useless factor	-0.192	-0.056	-0.059	-0.083	0.006

Note. The table presents the biases of the risk premia in a model with an intercept, a useful factor and two useless factors. We present findings for various values of the number of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

6.3 Model 3

In table 5 the estimates and in table 6 the biases can be found for model 3. Model 3 is estimated using five different estimators and it consists of an intercept, a useful factor and two weak factors. From table 6, several insights emerge.

Firstly we can observe that for all estimators and T's the intercept is always positive and the useful factor is always negative. For example the intercept of the Adaptive Elastic Net estimator is 1.117 and the risk premium of the useful factor is -0.410 for T equal to 50. This is in line with models 1 and 2.

Secondly it is clear that in the middle and long run the Elastic Net estimator performs in general better than the Adaptive Elastic Net estimator, for T=30,50 this is the other way around only in estimating the intercept and the risk premium of the useful factor.

The Penalized FM estimators are doing it both approximately in the same way, there is no clear better estimator.

Based on the bias the best estimator in the short and long run for the intercept and the risk premium of the useful factor is the FM estimator and in the middle run it is the Elastic Net estimator. The best estimator for the weak factors in the short run is the Penalized FM estimator with a penalization inversely related to the size factor beta, in the middle run it is the Penalized FM estimator with a penalization that is inversely related to the correlation, in the long run it is both of the Penalized FM estimators.

Table 5: The estimates of the risk premium for a model with one useful factor and two weak factors

	True Parameter value	Estimates				
		FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
				Panel A: T=30		
Intercept	1.287	0.995	0.980	0.980	1.029	0.965
Useful factor	-0.671	-0.310	-0.215	-0.267	-0.321	-0.196
Weak factor	-	-0.059	-0.065	-0.072	-0.083	-0.073
Weak factor	-	0.129	0.037	0.107	0.130	0.047
				Panel B: T=50		
Intercept	1.287	1.161	0.995	1.070	1.117	1.002
Useful factor	-0.671	-0.473	-0.239	-0.359	-0.410	-0.254
Weak factor	-	0.056	-0.032	0.006	0.009	-0.010
Weak factor	-	0.027	0.001	0.012	0.010	-0.021
				Panel C: T=100		
Intercept	1.287	1.172	0.975	1.064	1.059	0.924
Useful factor	-0.671	-0.511	-0.262	-0.385	-0.388	0.215
Weak factor	-	0.106	0.029	0.065	0.074	-0.001
Weak factor	-	0.110	0.026	0.077	0.081	0.036
				Panel D: T=250		
Intercept	1.287	1.567	0.963	1.254	1.125	0.972
Useful factor	-0.671	-0.794	-0.195	-0.480	-0.363	-0.195
Weak factor	-	-0.701	0.004	0.691	-0.706	0.004
Weak factor	-	-0.428	-0.154	0.005	-0.090	0.141
				Panel E: T=500		
Intercept	1.287	1.265	0.903	1.096	1.027	0.865
Useful factor	-0.671	-0.569	-0.167	-0.387	-0.331	-0.134
Weak factor	-	0.337	0.081	0.243	0.243	0.077
Weak factor	-	0.229	0.082	0.212	0.252	0.082
				Panel F: T=1000		
Intercept	1.287	1.285	0.871	1.094	0.999	0.834
Useful factor	-0.671	-0.583	-0.134	-0.381	-0.302	-0.100
Weak factor	-	0.426	0.051	0.244	0.252	0.071
Weak factor	-	0.265	0.041	0.192	0.224	0.036

Note. The table presents the estimates of the risk premia in a model with an intercept, a useful factor and two weak factors. We present findings for various values of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

Table 6: The biases of the risk premium for a model with one useful factor and two weak factors

	Biases				
	FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
			Panel A: T=30		
Intercept	0.292	0.307	0.307	0.258	0.322
Useful factor	-0.361	-0.456	-0.404	-0.250	-0.475
Weak factor	-0.059	-0.065	-0.072	-0.083	-0.073
Weak factor	0.129	0.037	0.107	0.130	0.047
			Panel B: T=50		
Intercept	0.126	0.292	0.217	0.170	0.285
Useful factor	-0.198	-0.432	-0.312	-0.261	-0.417
Weak factor	0.056	-0.032	0.006	0.009	-0.010
Weak factor	0.027	0.001	0.012	0.010	-0.021
			Panel C: T=100		
Intercept	0.115	0.312	0.223	0.228	0.363
Useful factor	-0.159	-0.409	-0.286	-0.283	-0.456
Weak factor	0.106	0.029	0.065	0.074	-0.001
Weak factor	0.110	0.026	0.077	0.081	0.036
			Panel D: T=250		
Intercept	-0.280	0.324	0.033	0.162	0.315
Useful factor	-0.123	-0.476	-0.191	-0.308	-0.476
Weak factor	-0.701	0.004	0.691	-0.706	0.004
Weak factor	-0.428	-0.154	0.005	-0.090	0.141
			Panel E: T=500		
Intercept	0.022	0.384	0.191	0.260	0.422
Useful factor	-0.102	-0.504	-0.284	-0.340	-0.537
Weak factor	0.337	0.081	0.243	0.243	0.077
Weak factor	0.229	0.082	0.212	0.252	0.082
			Panel F: T=1000		
Intercept	0.002	0.416	0.193	0.288	0.453
Useful factor	-0.088	-0.537	-0.290	-0.369	-0.571
Weak factor	0.426	0.051	0.244	0.252	0.071
Weak factor	0.265	0.041	0.192	0.224	0.036

Note. The table presents the biases of the risk premia in a model with an intercept, a useful factor and two weak factors. We present findings for various values of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

6.4 Model 4

In tables 7 and 8 the estimates and in tables 9 and 10 the biases can be found for model 4. Model 4 is estimated using five different estimators and it consists of an intercept, three useful factors and eleven useless factors.

From the tables 7 and 8 it is clear that the estimated intercept is in all runs positive. The investment factor which is the third useful factor has in general a positive risk premium in the middle and long run and a negative risk premium in the short run. This means that when an investor invest in more conservative portfolios and not in more aggressive portfolios the returns are higher in the middle and long run, but lower in the short run. The first and second useful factor, the market excess return and the robust minus weak(RMW) factor, are both sometimes positive and negative. Sometimes the market and RMW factors have a positive effect and sometimes a negative effect on the excess returns.

From the tables 9 and 10 it is clear that both of the Penalized FM estimators in general are estimating the risk premium of all factors more correctly than the other estimators. Explicitly for the useless factors both of the Penalized FM estimators are estimating the risk premium a lot better than the others.

Table 7: The estimates of the risk premium for a model with three useful factors and eleven useless factors for T=30,50,100

	True Parameter values	Estimates FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
		Panel A: T=30				
Intercept	0.644	0.325	0.545	0.482	0.425	0.544
Useful Factor	0.001	0.059	0.025	0.016	0.046	0.045
Useful Factor	-0.093	0.079	0.021	0.023	0.024	0.016
Useful Factor	0.328	-0.077	-0.021	-0.027	-0.045	-0.003
Useless Factor	-	-0.070	0.006	-0.004	-0.010	0.018
Useless Factor	-	0.088	-0.008	0.011	0.019	0.005
Useless Factor	-	-0.086	-0.099	-0.096	-0.120	-0.082
Useless Factor	-	0.044	0.021	0.011	0.014	-0.019
Useless Factor	-	-0.032	-0.016	-0.042	-0.055	-0.033
Useless Factor	-	-0.040	-0.018	-0.021	-0.025	-0.011
Useless Factor	-	0.043	0.009	0.020	0.050	0.003
Useless Factor	-	0.088	0.025	0.041	0.051	0.008
Useless Factor	-	0.105	0.021	0.040	0.048	0.038
Useless Factor	-	0.013	0.010	0.041	0.038	0.046
Useless Factor	-	-0.027	-0.013	-0.019	-0.035	-0.026
		Panel B: T=50				
Intercept	0.644	0.553	0.680	0.624	0.589	0.677
Useful Factor	0.001	-0.005	-0.034	-0.027	-0.024	-0.033
Useful Factor	-0.093	0.006	-0.011	0.000	0.001	-0.004
Useful Factor	0.328	-0.012	-0.001	-0.004	-0.006	0.005
Useless Factor	-	-0.027	0.007	-0.006	-0.003	-0.017
Useless Factor	-	0.075	-0.001	0.023	0.035	0.017
Useless Factor	-	0.005	0.013	-0.002	-0.012	0.001
Useless Factor	-	-0.042	0.006	-0.002	-0.004	0.007
Useless Factor	-	-0.008	0.014	0.015	0.004	0.008
Useless Factor	-	-0.022	0.002	0.009	0.011	0.013
Useless Factor	-	-0.006	-0.006	-0.012	-0.016	-0.013
Useless Factor	-	-0.044	-0.024	-0.009	-0.021	-0.007
Useless Factor	-	-0.002	0.002	-0.004	-0.004	0.006
Useless Factor	-	-0.017	-0.008	-0.014	-0.018	-0.020
Useless Factor	-	0.075	0.004	0.018	0.035	-0.003
		Panel C: T=100				
Intercept	0.644	0.645	0.702	0.675	0.661	0.710
Useful Factor	0.001	-0.147	-0.065	-0.055	-0.078	-0.026
Useful Factor	-0.093	-0.018	0.002	-0.016	-0.019	-0.001
Useful Factor	0.328	-0.009	0.005	-0.010	-0.006	-0.011
Useless Factor	-	0.030	0.021	0.002	0.007	0.011
Useless Factor	-	0.021	0.008	0.023	0.031	0.018
Useless Factor	-	0.020	0.006	0.012	0.022	-0.004
Useless Factor	-	0.086	0.037	0.041	0.062	0.026
Useless Factor	-	-0.018	0.003	-0.004	-0.001	0.000
Useless Factor	-	-0.080	-0.008	-0.012	-0.032	0.012
Useless Factor	-	0.053	0.010	0.049	0.065	0.017
Useless Factor	-	0.011	-0.001	-0.019	-0.014	-0.018
Useless Factor	-	0.008	0.014	0.011	0.008	-0.010
Useless Factor	-	-0.001	0.014	0.011	0.011	0.011
Useless Factor	-	0.044	-0.007	-0.017	-0.008	-0.005

Note. The table presents the estimates of the risk premia in a model with an intercept, three useful factors and eleven useless factors. We present findings for the values 30, 50 and 100 of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

Table 8: The estimates of the risk premium for a model with three useful factors and eleven useless factors for T=250,500,1000

	True Parameter values	Estimates				
		FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
				Panel D: T=250		
Intercept	0.644	0.692	0.734	0.707	0.695	0.736
Useful Factor	0.001	0.004	0.000	0.017	0.033	-0.042
Useful Factor	-0.093	-0.037	-0.002	-0.025	-0.027	-0.014
Useful Factor	0.328	0.031	0.037	0.037	0.045	0.023
Useless Factor	-	0.015	-0.003	0.017	0.024	-0.006
Useless Factor	-	0.051	0.032	0.003	0.007	0.005
Useless Factor	-	-0.092	-0.010	-0.030	-0.053	0.001
Useless Factor	-	-0.025	0.013	0.027	0.050	0.012
Useless Factor	-	0.021	0.012	0.038	0.041	0.021
Useless Factor	-	0.042	-0.006	-0.004	0.008	-0.010
Useless Factor	-	0.023	0.001	0.003	-0.004	0.008
Useless Factor	-	-0.033	-0.009	0.002	-0.003	-0.010
Useless Factor	-	-0.013	0.003	0.017	0.022	0.002
Useless Factor	-	-0.040	0.011	-0.005	-0.007	0.005
Useless Factor	-	-0.022	0.000	-0.022	-0.020	-0.019
				Panel E: T=500		
Intercept	0.644	0.753	0.731	0.724	0.718	0.731
Useful Factor	0.001	-0.098	-0.001	0.113	0.167	0.047
Useful Factor	-0.093	-0.031	0.013	-0.004	-0.017	0.002
Useful Factor	0.328	0.027	0.003	0.006	-0.012	-0.005
Useless Factor	-	-0.039	-0.018	-0.012	-0.061	-0.019
Useless Factor	-	-0.008	0.035	0.023	0.016	-0.005
Useless Factor	-	0.028	0.027	0.019	0.037	-0.001
Useless Factor	-	0.046	-0.014	0.013	0.020	-0.004
Useless Factor	-	-0.057	0.033	-0.028	-0.013	0.010
Useless Factor	-	0.228	-0.001	0.000	0.004	-0.019
Useless Factor	-	-0.125	-0.071	-0.154	-0.181	-0.034
Useless Factor	-	-0.052	0.019	-0.018	-0.007	0.025
Useless Factor	-	0.203	0.062	0.067	0.108	0.002
Useless Factor	-	-0.065	0.039	-0.005	0.018	-0.018
Useless Factor	-	0.001	-0.007	-0.009	0.000	-0.020
				Panel F: T=1000		
Intercept	0.644	0.772	0.748	0.743	0.747	0.755
Useful Factor	0.001	-0.088	-0.025	-0.076	-0.143	-0.010
Useful Factor	-0.093	-0.217	-0.083	-0.098	-0.126	-0.100
Useful Factor	0.328	0.052	0.026	-0.004	0.018	0.028
Useless Factor	-	0.137	0.014	0.040	0.065	-0.018
Useless Factor	-	0.105	0.010	0.026	0.058	-0.046
Useless Factor	-	-0.077	0.007	-0.058	-0.078	0.047
Useless Factor	-	0.123	0.023	0.049	0.097	0.027
Useless Factor	-	0.066	-0.043	-0.060	-0.065	-0.056
Useless Factor	-	-0.071	0.012	-0.061	-0.037	-0.013
Useless Factor	-	-0.041	-0.064	-0.019	0.012	0.033
Useless Factor	-	0.066	0.004	0.084	0.126	0.000
Useless Factor	-	0.066	0.039	-0.007	-0.003	-0.020
Useless Factor	-	0.014	-0.010	0.018	0.033	0.027
Useless Factor	-	0.058	-0.005	-0.051	-0.072	-0.038

Note. The table presents the estimates of the risk premia in a model with an intercept, three useful factors and eleven useless factors. We present findings for the values 250, 500 and 1000 of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

Table 9: The biases of the risk premium for a model with three useful factors and eleven useless factors for T=30,50,100

	Biases				
	FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
			Panel A: T=30		
Intercept	0.319	0.099	0.162	0.219	0.1
Useful Factor	-0.058	-0.024	-0.015	-0.045	-0.044
Useful Factor	-0.172	-0.114	-0.116	-0.117	-0.109
Useful Factor	0.405	0.349	0.355	0.373	0.331
Useless Factor	-0.070	0.006	-0.004	-0.010	0.018
Useless Factor	0.088	-0.008	0.011	0.019	0.005
Useless Factor	-0.086	-0.099	-0.096	-0.120	-0.082
Useless Factor	0.044	0.021	0.011	0.014	-0.019
Useless Factor	-0.032	-0.016	-0.042	-0.055	-0.033
Useless Factor	-0.040	-0.018	-0.021	-0.025	-0.011
Useless Factor	0.043	0.009	0.020	0.050	0.003
Useless Factor	0.088	0.025	0.041	0.051	0.008
Useless Factor	0.105	0.021	0.040	0.048	0.038
Useless Factor	0.013	0.010	0.041	0.038	0.046
Useless Factor	-0.027	-0.013	-0.019	-0.035	-0.026
			Panel B: T=50		
Intercept	0.091	-0.036	0.020	0.055	-0.033
Useful Factor	0.006	0.035	0.028	0.025	0.034
Useful Factor	-0.099	-0.082	-0.093	-0.094	-0.089
Useful Factor	0.340	0.329	0.332	0.334	0.323
Useless Factor	-0.027	0.007	-0.006	-0.003	-0.017
Useless Factor	0.075	-0.001	0.023	0.035	0.017
Useless Factor	0.005	0.013	-0.002	-0.012	0.001
Useless Factor	-0.042	0.006	-0.002	-0.004	0.007
Useless Factor	-0.008	0.014	0.015	0.004	0.008
Useless Factor	-0.022	0.002	0.009	0.011	0.013
Useless Factor	-0.006	-0.006	-0.012	-0.016	-0.013
Useless Factor	-0.044	-0.024	-0.009	-0.021	-0.007
Useless Factor	-0.002	0.002	-0.004	-0.004	0.006
Useless Factor	-0.017	-0.008	-0.014	-0.018	-0.020
Useless Factor	0.075	0.004	0.018	0.035	-0.003
			Panel C: T=100		
Intercept	-0.001	-0.058	-0.031	-0.017	-0.066
Useful Factor	0.148	0.066	0.056	0.079	0.027
Useful Factor	-0.075	-0.095	-0.077	-0.074	-0.092
Useful Factor	0.337	0.323	0.338	0.334	0.339
Useless Factor	0.030	0.021	0.002	0.007	0.011
Useless Factor	0.021	0.008	0.023	0.031	0.018
Useless Factor	0.020	0.006	0.012	0.022	-0.004
Useless Factor	0.086	0.037	0.041	0.062	0.026
Useless Factor	-0.018	0.003	-0.004	-0.001	0.000
Useless Factor	-0.080	-0.008	-0.012	-0.032	0.012
Useless Factor	0.053	0.010	0.049	0.065	0.017
Useless Factor	0.011	-0.001	-0.019	-0.014	-0.018
Useless Factor	0.008	0.014	0.011	0.008	-0.010
Useless Factor	-0.001	0.014	0.011	0.011	0.011
Useless Factor	0.044	-0.007	-0.017	-0.008	-0.005

Note. The table presents the biases of the risk premia in a model with an intercept, three useful factors and eleven useless factors. We present findings for the values 30, 50 and 100 of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

Table 10: The biases of the risk premium for a model with three useful factors and eleven useless factors for T=250,500,1000

	Biases				
	FM	Pen-FM(betas)	Elnet	A-Elnet	Pen-FM(corr)
			Panel D: T=250		
Intercept	-0.048	-0.09	-0.063	-0.051	-0.092
Useful Factor	-0.003	0.001	-0.016	-0.032	0.043
Useful Factor	-0.056	-0.091	-0.068	-0.066	-0.079
Useful Factor	0.297	0.291	0.291	0.283	0.305
Useless Factor	0.015	-0.003	0.017	0.024	-0.006
Useless Factor	0.051	0.032	0.003	0.007	0.005
Useless Factor	-0.092	-0.010	-0.030	-0.053	0.001
Useless Factor	-0.025	0.013	0.027	0.050	0.012
Useless Factor	0.021	0.012	0.038	0.041	0.021
Useless Factor	0.042	-0.006	-0.004	0.008	-0.010
Useless Factor	0.023	0.001	0.003	-0.004	0.008
Useless Factor	-0.033	-0.009	0.002	-0.003	-0.010
Useless Factor	-0.013	0.003	0.017	0.022	0.002
Useless Factor	-0.040	0.011	-0.005	-0.007	0.005
Useless Factor	-0.022	0.000	-0.022	-0.020	-0.019
			Panel E: T=500		
Intercept	-0.109	-0.087	-0.08	-0.074	-0.087
Useful Factor	0.099	0.002	-0.112	-0.166	-0.046
Useful Factor	-0.062	-0.106	-0.089	-0.076	-0.095
Useful Factor	0.301	0.325	0.322	0.34	0.333
Useless Factor	-0.039	-0.018	-0.012	-0.061	-0.019
Useless Factor	-0.008	0.035	0.023	0.016	-0.005
Useless Factor	0.028	0.027	0.019	0.037	-0.001
Useless Factor	0.046	-0.014	0.013	0.020	-0.004
Useless Factor	-0.057	0.033	-0.028	-0.013	0.010
Useless Factor	0.228	-0.001	0.000	0.004	-0.019
Useless Factor	-0.125	-0.071	-0.154	-0.181	-0.034
Useless Factor	-0.052	0.019	-0.018	-0.007	0.025
Useless Factor	0.203	0.062	0.067	0.108	0.002
Useless Factor	-0.065	0.039	-0.005	0.018	-0.018
Useless Factor	0.001	-0.007	-0.009	0.000	-0.020
			Panel F: T=1000		
Intercept	-0.128	-0.104	-0.099	-0.103	-0.111
Useful Factor	0.089	0.026	0.077	0.144	0.011
Useful Factor	0.124	-0.01	0.005	0.033	0.007
Useful Factor	0.276	0.302	0.332	0.31	0.3
Useless Factor	0.137	0.014	0.040	0.065	-0.018
Useless Factor	0.105	0.010	0.026	0.058	-0.046
Useless Factor	-0.077	0.007	-0.058	-0.078	0.047
Useless Factor	0.123	0.023	0.049	0.097	0.027
Useless Factor	0.066	-0.043	-0.060	-0.065	-0.056
Useless Factor	-0.071	0.012	-0.061	-0.037	-0.013
Useless Factor	-0.041	-0.064	-0.019	0.012	0.033
Useless Factor	0.066	0.004	0.084	0.126	0.000
Useless Factor	0.066	0.039	-0.007	-0.003	-0.020
Useless Factor	0.014	-0.010	0.018	0.033	0.027
Useless Factor	0.058	-0.005	-0.051	-0.072	-0.038

Note. The table presents the biases of the risk premia in a model with an intercept, three useful factors and eleven useless factors. We present findings for the values 250, 500 and 1000 of the number of time-series observations (T) using 100,000 simulations. These simulations are conducted under the assumption that returns are derived from a multivariate normal distribution. The means and covariance matrix are adjusted to match the characteristics of the 25 size and book-to-market Fama/French portfolio returns over the period 1972:1-2023:1.

6.5 Empirical Analysis

In table 13 the results of the empirical analysis are shown. Six different models are estimated with six different estimators. The models CAPM and CCAPM are only estimated with the FM estimator. For all models we have listed for the FM estimator the risk premia, the standard errors, the p-values and the R^2 values. For the other estimators we have listed the risk premium and the R^2 values. Also we have performed the Wald test for all models and factors and we have done the model selection procedure and stated which factors of which models do survive the procedure.

Firstly the risk premia of the CAPM model is estimated. The Market factor is surviving the model selection procedure. Also the Wald test is equal to 0.044, which means that the factors risk premia is significantly different from zero for a significance level of 0.05. The R^2 value is equal to 12. For the FF3 model where the SMB and HML factors are added, the R^2 value increases a lot from 12 to 61 for the FM Estimator. The added factors do not survive the model selection procedure, but the p value of the Wald test for HML is equal to zero, so the risk premium for HML is significantly different from zero. This can be seen for the FM estimator. Probably the Elastic Net and the Adaptive Elastic Net is assuming that the factors SMB and HML are useless, because they estimate a low risk premia for these factors. That is why the R^2 value is lower. We also have calibrated the useless and weak factors in our simulations to the mean and variances of SMB and HML, so our assumption is in line with the empirical application.

Afterwards instead of the market factor, the non durable consumption factor is estimated, the CCAPM model. Following the model selection procedure this factor do not survive it. Also it has a Wald test with a p-value of 0.84, which means that the risk premium is not significant. Also the R^2 value is equal to zero, so this factor is not explaining the excess returns. Adding the cay factor and the interaction of both result in the CC-CAY model. Where the non durable consumption factor still not survive the model selection procedure and not have a Wald test with a p-value lower than 0.05 so the risk premium is not significant. The R^2 value increases a lot from 0 to 69 for the FM estimator, this is probably because cay and the interaction have a risk premium that is significant and have a p-value which is lower than 0.05. So adding these two factors result in more explaining power of the excess returns. It can also be seen that the Penalized FM estimator inversely related to the correlation is estimating the risk premia for the non durable consumption factor and the interaction with the cay factor the lowest, which is logical.

Also the C-Lab model, where the market factor, cay, lab and the interactions are added, is estimated. This model improves the R^2 value a lot to 95 for the FM estimator, so together the factors explain the excess returns the best. It follows that the interaction of lab and cay have a low risk premium for the FM, Elastic Net and Adaptive Elastic Net estimators, also it is not chosen by both of the the Penalized FM estimators. Only the market factor survives the model selection procedure. The Cay and Lab factors have a p value lower than 0.05 so they are significant also they have a risk premium higher than 0 for FM, Penalized FM inversely related to the size factor, Elastic Net and Adaptive Elastic Net estimators.

Furthermore the D-CCAPM model, where we investigate the market factor and the non-durable and durable consumption factors, is estimated. It follows that only the market factor survives the model selection procedure and only the market factor have a p-value for the wald test of 0.05 and lower. Also only the market factor is chosen by the elastic net and adaptive elastic net as factor. So probably the durable and nondurable consumption factors have no effect on the excess returns.

From table 11 it is clear that CCAPM, CC-CAY and C-Lab have a p-value that is higher than 0.05. Which means that the null hypothesis of the beta having no full rank is not rejected and the model maybe not correctly identified. This is in line with what we see in table 13.

Table 11: Beta rank test of Kleibergen & Paap (2006)

	Statistic	P-value
CAPM	607.97	0
FF3	585.04	0
CCAPM	51.14	0.302
CC-CAY	11.88	0.772
C-Lab	16.26	0.094
D-CCAPM	30.93	0

We compare the different estimators with each other by checking if the useless factors are correctly set to zero and the useful factors have a risk premium which is not equal to zero. We see that for D-CCAPM model where as stated above the durable and non durable consumption are probably useless and the excess market return is useful, all estimators set correctly the useless factors to zero. However both of the Penalized-FM estimators set also the useful factors to zero. This was also a problem of the Penalized FM estimators in the simulations.

For the C-Lab model where the market excess return only survived the model selection procedure, we can see that both of the Penalized FM estimators estimate the risk premium of the useless factors close to zero or do not select the factors. The Adaptive Elastic Net estimator estimate the risk premium of the market excess return as zero and the risk premia for the useless factors not equal to zero. This shows that probably the Adaptive Elastic Net estimator cannot handle spurious factors.

For the CC-CAY model where the non durable consumption, cay and the interaction do not survive the model selection procedure, the Elastic Net and Adaptive Elastic Net estimators estimate a non zero risk premium for the non durable consumption factor while the Penalized FM estimators estimate a risk premium closer to zero. The Penalized FM estimators estimate the risk premium of the other useless factors also closer to zero.

From table 12 it is clear that between the models the C-Lab model has the best fit, because the HJM^2 is 0.108 and is the lowest of the models. This also followed from the R^2 value. But if we do not look at not correctly identified models we see that FF3 has the lowest HJM^2 of 0.151.

Table 12: The squared misspecification-robust Hansen-Jagannathan distance

	HJM^2
CAPM	0.169
FF3	0.151
CCAPM	0.186
CC-CAY	0.118
C-Lab	0.108
D-CCAPM	0.160

Table 13: Empirical Analysis

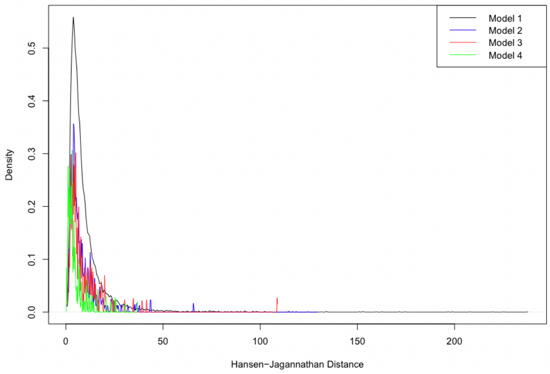
				Fama-MacBeth				Pen-FM(betas)		Pen-FM(correlation)		Elnet		AElnet	
		GKR	p-value		st.error	p-value	R^2		R^2		R^2		R^2		R^2
Model	Factors	(2014)	(Wald)	λ_j	(OLS)	(OLS)	(%)	λ_j	(%)	λ_j	(%)	λ_j	(%)	λ_j	(%)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
CAPM	Intercept	-	-	1.347***	0.3124	0.0003	12								
	MKT	yes	0.044	-0.554	0.2887	0.0846									
FF3	Intercept	-	-	1.287**	0.3864	0.0032		1.313		0.753		1.266		1.429	
	MKT	yes	1.0	-0.671	0.3706	0.0844	61	-0.569	61	0	61	-0.641	58	-0.749	50
	SMB	no	0.4	0.104	0.0536	0.0653		0		0		0.096		0.076	
	HML	no	0	0.336***	0.0645	0		0.076		0		0.319		0.174	
CCAPM	Intercept	-	-	0.791***	0.1908	0.0004	0								
	$c_n d$	no	0.84	-0.013	0.0624	0.8385									
CC-CAY	Intercept	-	-	0.468**	0.1569	0.0071		0.713		0.488		0.509		0.535	
	$c_n d$	no	0.58	0.022	0.0405	0.5843	69	0.013	63	0.009	65	0.019	62	0.027	66
	cay	no	0	0.029***	0.0063	0.0001		0		0.074		0.045		0.017	
	$c_n d\ cay$	no	0	0.009***	0.0015	0		0.005		0.004		0.006		0.008	
C-Lab	Intercept	-	-	0.210	0.1349	0.1355		0.253		1.258		0.413		0.057	
	cay	no	0.0097	0.008*	0.0033	0.0181		0.004		0		0.004		0.006	
	vw	yes	0.15	-0.148	0.1030	0.1661	95	-0.064	94	-0.524	91	-0.153	90	0	94
	lab	no	0	0.015***	0.0020	0		0.012		0		0.010		0.016	
	$vw\ cay$	no	0.14	0.014	0.0098	0.1561		0		-0.020		-0.010		0.012	
	$lab\ cay$	no	0.11	0	0.0001	0.1314		0		0		0		0	
D-CCAPM	Intercept	no	-	1.649***	0.3294	0		0.753		0.753		0.832		1.327	
	vw	yes	0.0054	-0.892*	0.3205	0.0111	30	0	28	0	30	-0.073	3	-0.535	13
	$c_n d$	no	0.27	0.112	0.1009	0.2789		0		0		0		0	
	c_d	no	0.84	-0.084	0.4260	0.8454		0		0		0		0	

Note. The table provides the fit and risk premia estimates for various models of the cross-section of stocks. The first column describes each estimated model. The second column lists the risk factors. The third column displays the p-value of the Wald test assessing whether a factor is useless. Column 4 indicates if a risk factor survives the sequential elimination procedure based on the misspecification-robust t_m-statistic of Gospodinov et al. (2014). Columns 5-8 show model estimation results using the Fama-MacBeth procedure with an identity weight matrix (W = In), including point estimates of the risk premia, OLS standard error and corresponding p-value, along with the cross-sectional R2. Columns 9-10 summarize Penalized FM estimation inversely related to the size factor, while Columns 11-12 summarize Penalized FM estimation inversely related to the correlation factor, each providing point estimates of risk premia and cross-sectional R^2. Columns 13-14 describe Elastic Net estimation, and Columns 15-16 describe Adaptive Elastic Net estimation, with both providing point estimates of risk premia and cross-sectional R^2.

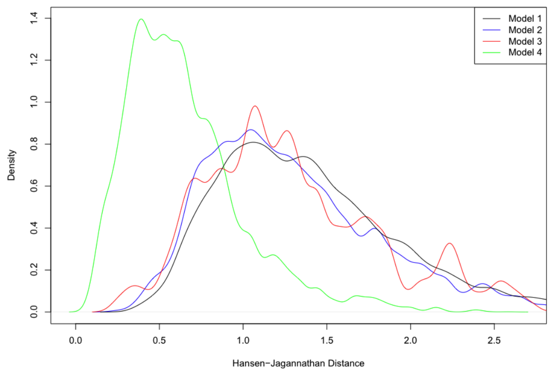
6.6 Comparing Models

In figure 2 the probability density function of the squared misspecification-robust Hansen-Jagannathan distance can be found for model 1, 2, 3 and 4. When the 11 useless factors and the 2 useful factors are added, so the comparison of model 1 and 4, the probability density function shift to the left in comparison with the probability density function of model 1, which means that the probability's for a less HJM^2 distance is higher, so the model fits the data better. It follows that model 4 fits the data the best, this is logical because it includes two more useful factors than the other models. Also it follows that when the two useless factors and the two weak factors are added, so the comparison of model 1 and 2, model 1 and 3 and model 1 and 4, the probability density functions are affected, but not too much.

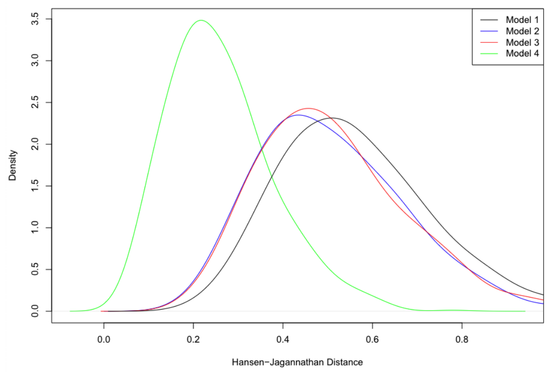
In the Appendix in figures 3, 4, 5 and 6 the probability density function for R^2 values for respectively model 1, 2, 3 and 4 can be found. What is clear for all models is that the probability for a high R^2 value is the highest for the FM estimator. This is logical because we have seen in the simulation and empirical application that the FM estimator is setting the useless factors not equal to zero, so it seems like the useless factors are contributing to the R-squared but that isn't true. We can also see the opposite. So both of the Penalized FM estimators, who are setting the useless factors to zero, have a higher probability on a lower R^2 value. We can also see that adding the weak and useless factors so comparing model 1 with models 2 and 3 show that it results in a higher 'falsely' R^2. Between the models 2 and 3 their is no clear difference of adding the useless and adding the weak factors. Model 4 with in comparison with model 1 added two useful factors and eleven useless factors results in a much higher probability on a higher R^2 value than model 1, 2 and 3. This also logical because adding useful factors should improve the R^2 because you fit the data better.



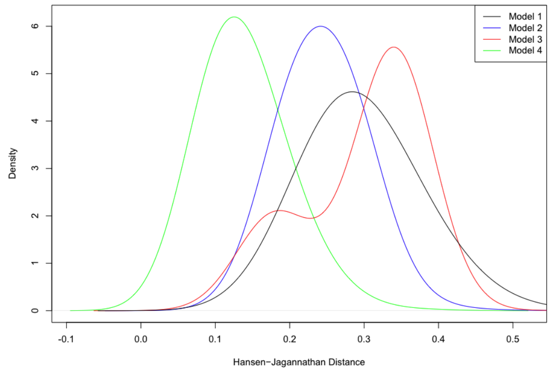
(a) T=30



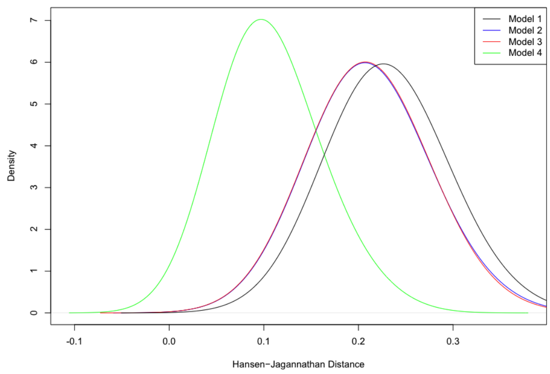
(b) T=50



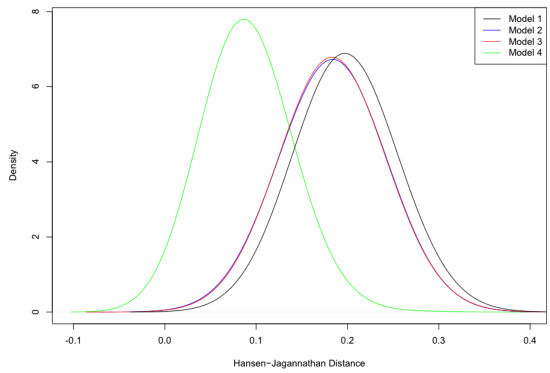
(c) T=100



(d) T=250



(e) T=500



(f) T=1000

Figure 2: Distribution of the squared misspecification-robust Hansen-Jagannathan distance

7 Conclusion

In this paper we tried to do a research on the risk premia estimators' robustness to spurious factors. From the results of the simulations and empirical application it is clear that both of the Penalized FM estimators are the best in handling the useless factors. Furthermore the risk premia for the intercept and the useful factor are estimated correctly for the short and long run by the FM estimator and in the middle run by the Elastic-Net estimator. But the FM estimator is also estimating the risk premia for the useless and weak factors higher than zero, so it can not handle spurious factors. The model that handles the useless factors the best is in its turn not giving the intercept and useful factor the correct risk premia. In general we can conclude that the Adaptive Elastic Net in the short run and the Elastic Net Estimator in the middle and long run estimate the risk premia more correctly than the FM estimator and handles the spurious factors better than both of the Penalized FM estimators. This all is in the case of the models with an intercept, a useful factor and two useless or weak factors. From the results of the big model with 11 useless factors it is clear that both of the Penalized FM estimators are estimating the risk premia more correctly than the other estimators. In this case these estimators are the less effected by the useless factors.

Upon reviewing our research, we found a few limitations. Firstly, we have used data from Fama/French, while it is widely used, it can still contain errors. Secondly, we assumed that the market factor is a useful factor for comparing estimators. This is also clear from the empirical application, but still it could be a wrong assumption, which would results in false conclusions. Lastly, assuming that useless factors have zero correlation with asset returns could lead to false conclusions if this assumption is incorrect.

For further research, we have a number of recommendations. Firstly, it could be useful to investigate if using the 17 industry portfolios would result in the same conclusions as using the 25 portfolios formed on size and book to market ratio. Secondly, exploring whether another method, besides k-fold cross-validation, which we have employed, could provide more accurate parameter estimates for the Elastic Net estimator. Thirdly, it would be beneficial to verify if the conclusions remain consistent when considering another useful factor aside from the one we have used.

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A Figures

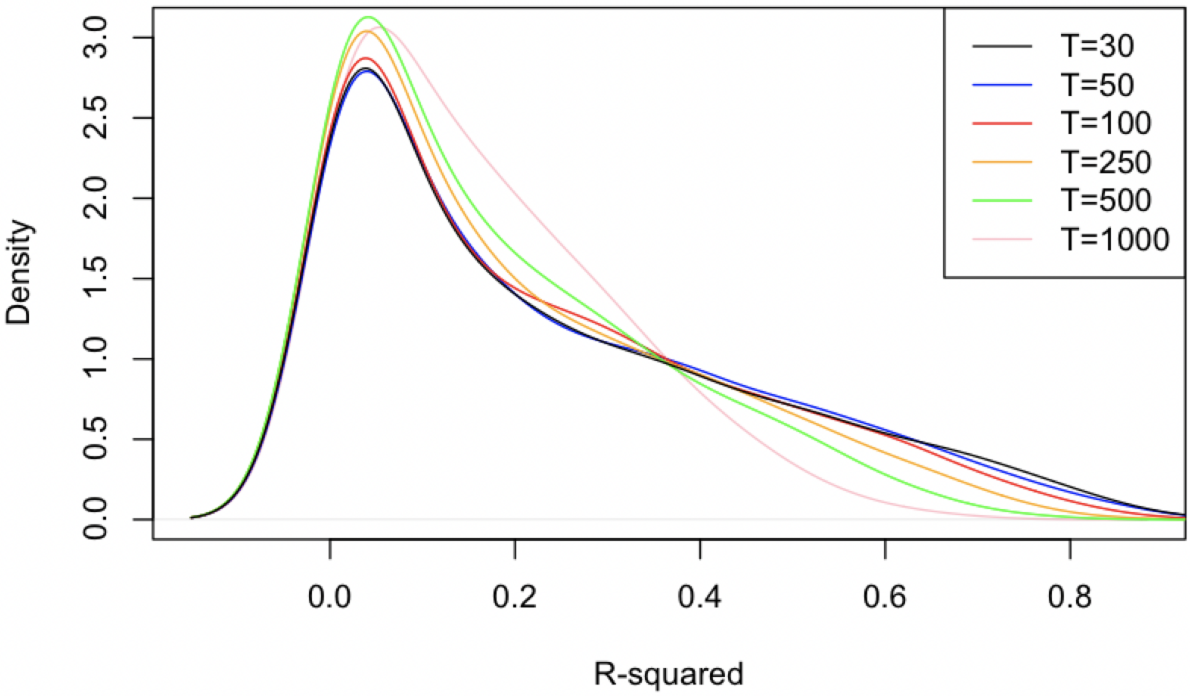
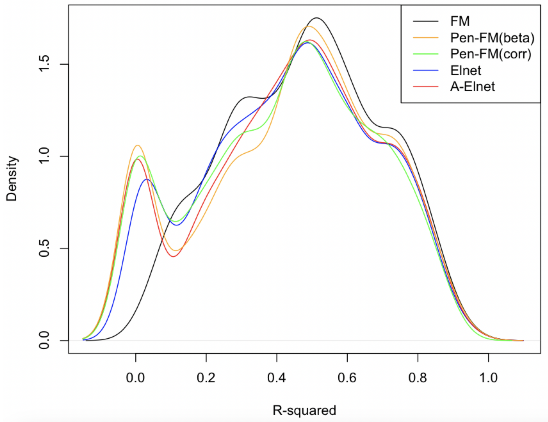


Figure 3: Distribution of R^2 for the model with one useful factor



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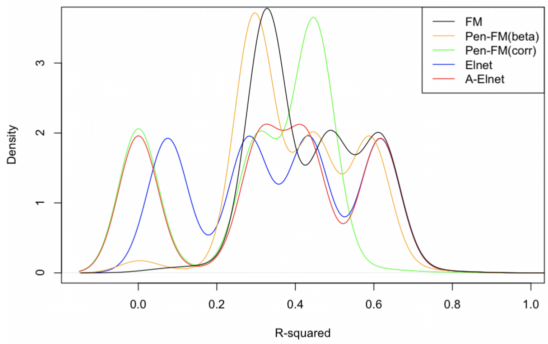

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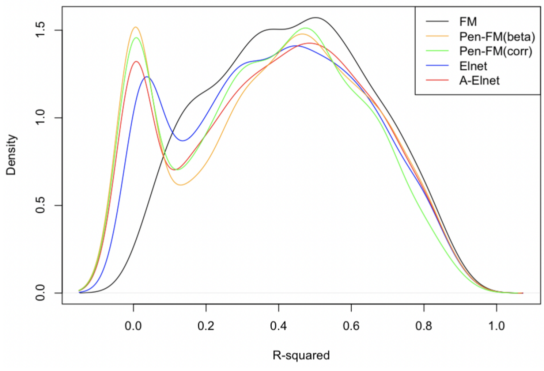

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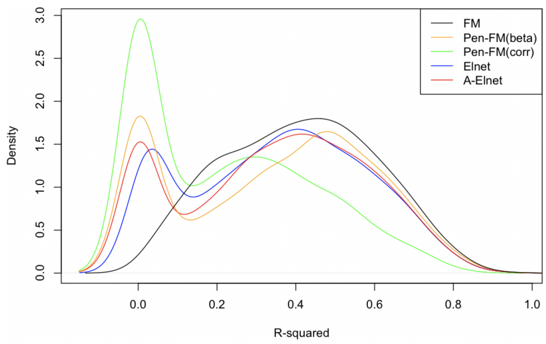

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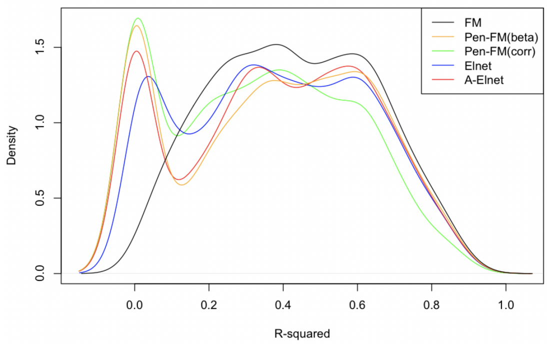

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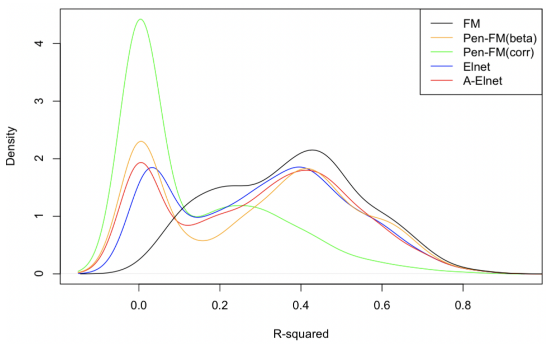

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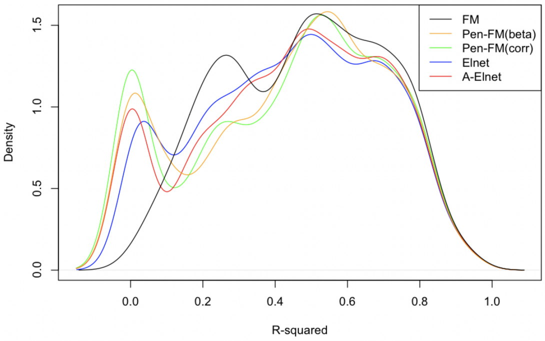
Figure 4: Distribution of R^2 for the model with one useful factor and two useless factors



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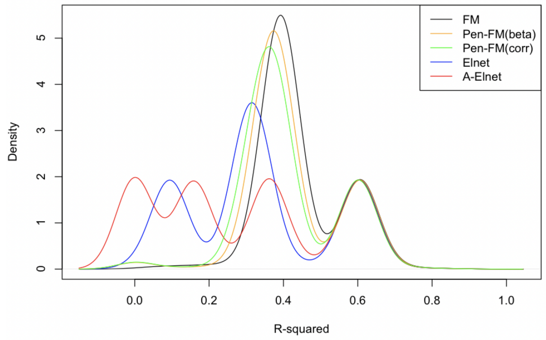

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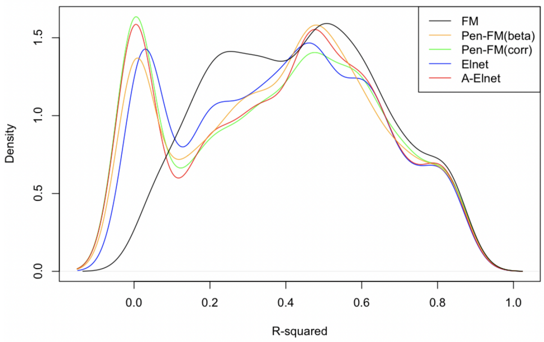

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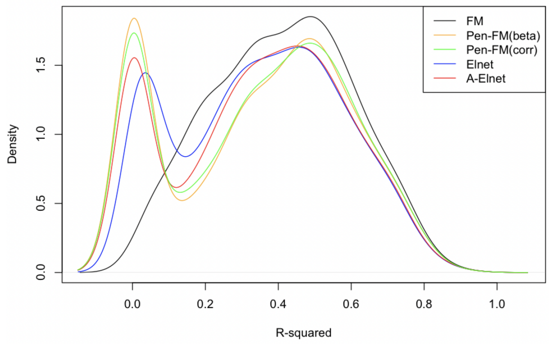

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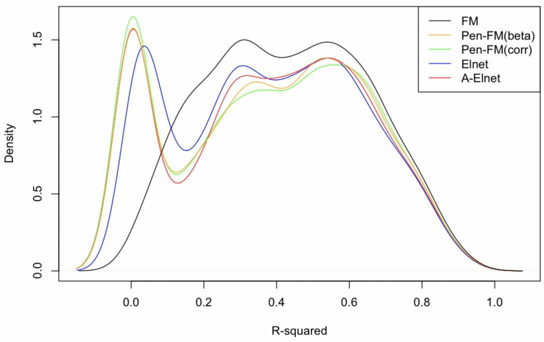

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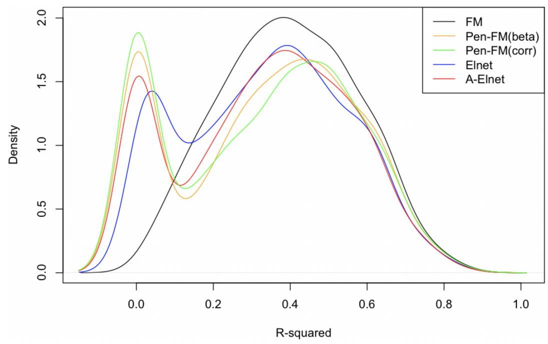

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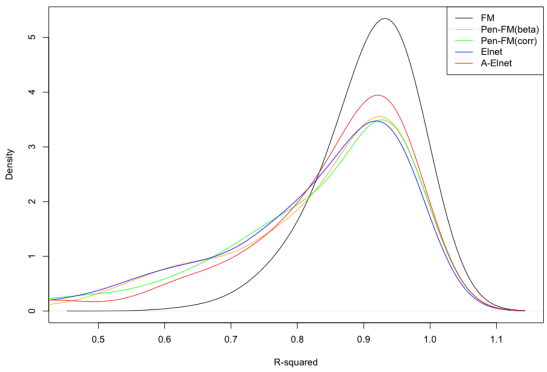
Figure 5: Distribution of R^2 for the model with one useful factor and two weak factors



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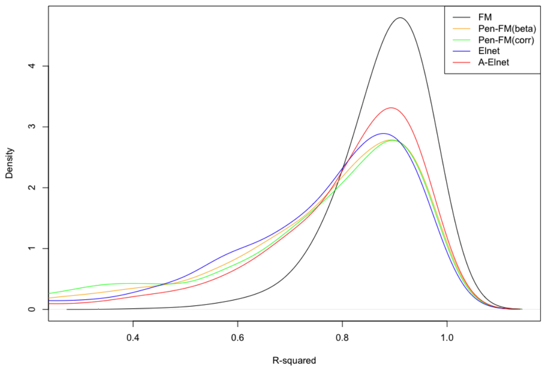

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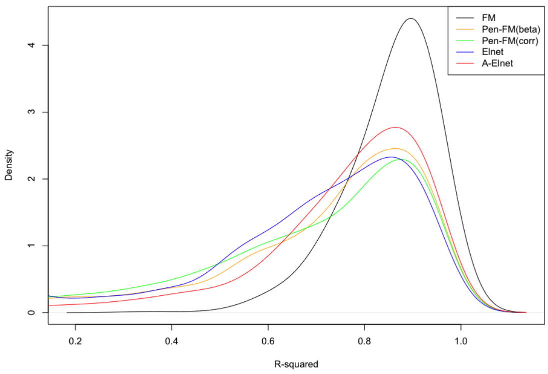

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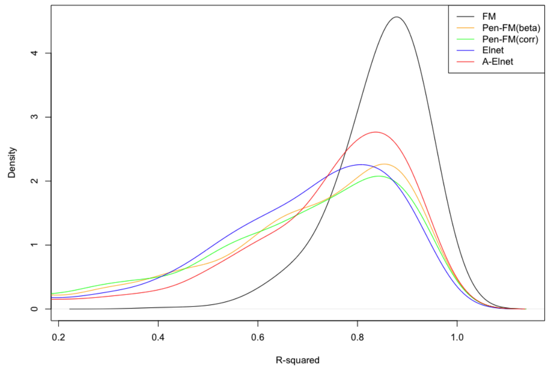

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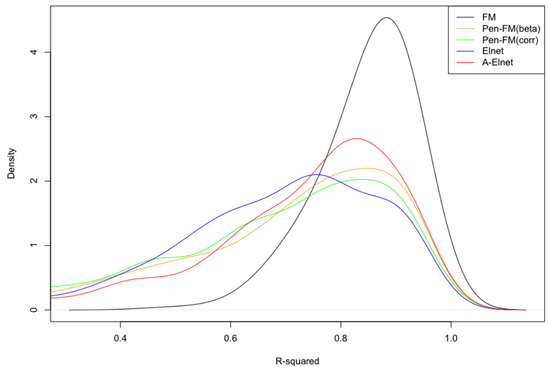

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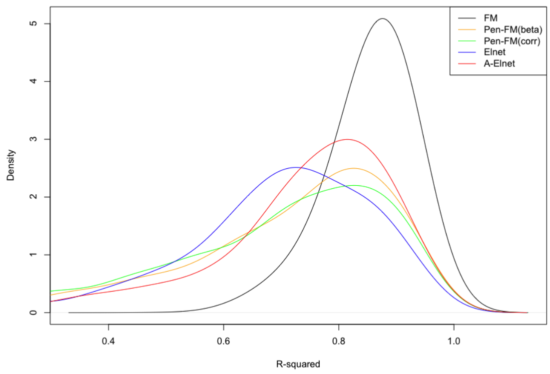

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Figure 6: Distribution of R^2 for the model with three useful factors and eleven useless factors