ERASMUS UNIVERSITY ROTTERDAM
ERASMUS SCHOOL OF ECONOMICS
Bachelor Thesis Economics & Business
Specialization: Financial Economics

To shrink or not to shrink?

Covariance shrinkage and portfolio performance with high estimation risk.

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Finish date: 13-07-2024

The views stated in this thesis are those of the author and not necessarily those of the supervisor, second reader, Erasmus School of Economics or Erasmus University Rotterdam.

ABSTRACT

This study tests the performance of 16 Mean-Variance optimizations and 3 Black-Litterman (BL) models, with and without Ledoit-Wolf (LW) covariance matrices and two sets of allocation constraints, in four samples, for two short rolling windows (60 and 120 months). These portfolios are rebalanced monthly and have their expected returns estimated by rolling sample means. Similarly, DeMiguel et al. (2009a), no strategy consistently outperforms the 1/N portfolio across all samples, although LW shows reductions in estimation risk. 1/N also constantly achieves superior turnovers. Unlike Bessler et al. (2017), BL portfolios also do not outperform the benchmark. Therefore, the present article supports the economic literature on the failure of Mean-Variance portfolios to outperform the naïve portfolio. Through the analysis of risk factor exposures, it is possible to observe a trade-off when using models with less protection against estimations risk. This presents said risk as a possible idiosyncratic model.

Keywords: covariance matrix, estimation error, portfolio performance, mean-variance, Black-Litterman

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SECTION 1 Introduction

Real-world applications of portfolio theory, particularly of the classical model developed by Markowitz (1952), face a series of blunders. Firstly, it is hard to estimate its inputs. Secondly, the allocation process can lead to extreme allocations. As a result, mean-variance (MV) optimizations do not systematically outperform a benchmark, 1/N portfolio (DeMiguel, Garlappi and Uppal, 2009a; Bessler, Opfer and Wolff, 2017). To combat these problems, shrinkages are applied to this estimator to reduce the estimation risk and generate more moderate allocations (Jorion, 1986).

Estimation error is particularly high for expected returns when using sample means (Merton, 1980; Chopra and Ziemba, 2013), and so it has gathered a lot of attention leading to widespread research on the performance benefits of Bayes-Stein shrinkage estimators (Jorion, 1986; DeMiguel et al., 2009a; Kirby and Ostdiek, 2012; Bessler et al., 2017). Meanwhile, estimation risk for the covariance matrix, acting as a risk model, has been less researched (Ledoit and Wolf, 2017), even though its classical estimator, the sample, performs poorly (Ledoit and Wolf, 2003a).

To address the above-mentioned problem, Ledoit and Wolf (2003a, 2003b) developed a shrinkage model to reduce estimation risk in the covariance matrix, particularly when the number of assets is high relative to the number of observations. Additionally, many investment managers use allocation constraints to deal with the challenges of Markowitz's (1952) MV optimizations, which act as a sort of covariance shrinkage (Jagannathan and Ma, 2003).

Thus, the purpose of the present study is to assess the performance benefits, in terms of Sharpe Ratio (SR) and Turnover, as in DeMiguel et al. (2009a), of using said covariance shrinkage methods. The objective is to simulate as closely as possible private practice, where estimation periods are short, often 60 to 120 months (DeMiguel et al., 2009a), and, therefore, estimation risk is high. Moreover, to keep ceteris paribus and assess the ability of the improved matrix, no effort is made to improve return estimates, which are made using sample means. The portfolio rebalancing takes place monthly, as in DeMiguel.

The first hypothesis is that the SPs of the different optimization strategies applied, which are all based on or related to MV, differ from the benchmark. Notably, because previous literature (DeMiguel et al., 2009a; Bessler et al., 2017) found no statistically superior performance, it cannot be assumed, and so a two-tailed test is applied. Secondly, the second hypothesis is that the strategies have higher turnover than the benchmark. Thirdly, the last hypothesis is that the strategies have unexplained returns in common risk factor models.

In the present paper, it is found that Ledoit-Wolf (LW) matrices do not relevantly reduce estimation error in the short rolling window, but they positively impact risk. Said error is related to market risk. Further, no optimization systematically outperforms the benchmark strategies across all samples, although Minimum Variance portfolios and their LW extensions tend to do well. Moreover, 1/N has the lowest turnover out of all strategies. These conclusions support DeMiguel's results (2009a).

Notably, unlike in Bessler et al. (2017), the BL model does not perform systematically better in present samples. Hence, hypothesis one cannot be accepted, while support for hypothesis two is found but not statistically tested.

Furthermore, a trade-off between estimation risk and market risk is found and discussed. Upon using these models, one could be said to be diversifying away from the systematic risk but towards more idiosyncratic, estimation risk. Finally, hypothesis three cannot be accepted for the equity risk factor. It could be accepted for other risk models (Bond and Forex) due to strategies' exposure to equity, which is not modelled.

Through this study, 19 strategies are tested across 4 samples and two windows, leading to over 38 thousand estimations being performed.

This paper is organized into 5 sections. Following the introduction, Section 2 is the Theoretical Framework, where literature on portfolio theory, performance, estimation error, risk factors, and diversification is discussed. In Section 3, the different samples are presented along with descriptives and the reasoning behind their addition. Afterwards, Section 4 presents the performance measurements, their hypothesis testing, portfolio characteristics and methods, and the risk factor models. Section 5 contains all results, their discussion, and their relation to the literature, while Section 6 (Conclusion) summarises all findings and their implications given the literature.

SECTION 2 Theoretical Framework

2.1 Portfolio Theory

Markowitz (1952) forever marked portfolio theory by developing its famous Mean-Variance model (MV). Based on a set of asset return estimates and a risk model (covariance matrix), the model produces, in theory, the optimal mean returns given any level of risk, or vice versa. Investors can then, given their quadratic preference function, maximize their utility by choosing a certain efficient mean-variance portfolio (Jobson and Korkie, 1980). While the classical MV model uses sample means and covariance matrix as the expected returns and the risk model, multiple modern extensions were developed and are commented on below.

One of the fundamental problems that the MV portfolio faces is estimation risk (Frankfurter, Phillips and Seagle, 1971). As future returns, variances and covariances are not known, they first must be estimated to then serve as an input. These estimates then lead to a certain asset allocation that is different from the ex-post, theoretically preferable one due to errors in the estimation (Jorion, 1986). Notably, the estimation error is much larger for expected returns than covariance matrices (Merton, 1980). In fact, the error is roughly 10x greater for the former than for the latter (Chopra and Ziemba, 2013)

As such, a solution presented to the estimation problem was based on applying Stein's (1955) shrinkage estimator to expected returns. By "reducing" them towards a common, less extreme value, the subsequent model allocation has systematically lowered risk (Jorion, 1986). Despite this, DeMiguel (2009a) and Bessler (2017) still find that classical MV and its Bayes-Stein shrinkage do not systematically outperform a naïve portfolio that invests equal weights on all assets (1/N).

Other shortcomings that plague the model are corner solutions, extreme weights and high sensitivity to variations of expected returns (DeMiguel, 2009a; Bessler et al., 2017). Combined, these challenges lead to a poor MV out-of-sample performance (Jorion, 1985; Jorion, 1986; DeMiguel et al., 2009a; Fabozzi et al., 2013; Bessler et al., 2017).

2.1.1 Portfolio Performance

As mentioned, DeMiguel et al. (2009a) found that MV portfolios and its extensions fail to systematically outperform out-of-sample the 1/N portfolio, thus questioning the value of sample information. These results were found across multiple equity samples and performance measures.

The underlying reason, they argue, for 1/N's better performance is that its equal weights are less distant to the true optimal MV allocation (if the true values of returns and covariance were known) than the weights calculated with an estimation error. That is, the "estimation error" of 1/N is lower than the classical MV. Secondly, 1/N performs particularly well because the underlying assets are diversified portfolios of stocks and not individual stocks.

In response, Kirby and Ostdiek (2012) argue that DeMiguel et al. (2009a) put the MV model at an inherent disadvantage, with no constraints and extremely high conditional expected returns and weights. Defending the value of sample information, they propose that two "simple" methods statistically outperform 1/N: Volatility Timing and Reward-to-Risk.

Somewhat similar to DeMiguel et al. (2009a), Bessler et al. (2017), in a multi-asset universe, find that MV portfolios and their Bayes-Stein extensions do outperform 1/N, but not significantly. In their study, the model developed by Black and Litterman (1992) (BL) achieved significantly better results than 1/N and MV portfolios, with lower risk, more diversification, and moderate allocations. Additionally, BL's less variant return estimate, owing to the addition of the reliability of estimates as an input, leads to reduced turnover.

Bessler et al. (2017) also find that BL performs better than 1/N during recessions due to its ability to shift towards commodities and bonds during downturns. This may have significant implications for the strategy's exposure to equity risk factors relative to its peers.

Applied to currency markets, the MV performs much better. As part of expected returns is known ex-ante (the interest rate), the estimation error is reduced to the exchange rate variation (Ackerman, Pohl and Schmedders, 2017). This result is not extended to crypto-currency markets, as shown by Platanakis, Sutcliffe and Urquhart (2018). These findings are relevant to the present research as many of the assets in some of the samples are subject to exchange rate fluctuations.

Importantly, due to its extreme allocations, a measure of portfolio turnover or post-transaction cost performance is important. Kirby and Ostdiek (2012) find that MV outperforms the benchmark pre-transaction costs, but not post-costs.

Supported by the CAPM's low volatility anomaly in equity markets, low-volatility, or, in this case, Minimum Variance (MinVar) portfolios have shown superior mean returns versus the market and lower variance (Clarke, De Silva and Thorley, 2011)

2.2 Estimation Error

2.2.1 Covariance Matrix

As mentioned before the estimation error related to expected return is greater than the related to the covariance matrix. As such, the former received more attention in academia than the latter (Ledoit and Wolf, 2017). Despite this, Ledoit and Wolf (2003b) claim that "nobody should be using the sample covariance matrix for the purpose of portfolio optimization".

The sample covariance matrix performs particularly poorly when the number of observations is not significantly larger ("at least one order of magnitude") than the number of assets (Ledoit and Wolf, 2003a). This is the case in some of the samples in this study.

Instead of the sample estimate, Ledoit and Wolf provide two shrinkage covariance matrix estimators: the single factor and the constant correlation.

The first one is based on Sharpe's (1963) single index model estimator for stock returns. The shrinkage consists of reducing the variances and covariances towards the ones implied by the model. A crucial assumption is that the index/market portfolio significantly explains the variation in these assets' returns, mattering less if it is value-weighted or equal-weighted. This shrinkage results in a model that is biased and misspecified compared to the true covariance matrix, in exchange for reducing the estimation error. The sample covariance matrix has the opposite characteristics.

The second shrinkage method is the constant correlation method and consists of reducing all pairwise correlations to an average correlation of all assets. Again, this leads to a biased representation of the true matrix but entails less estimation error.

The crucial problem solved by Ledoit and Wolf (2003a) was the estimation of the optimal shrinkage intensity, which determines how much the resulting risk model is weighted towards the misspecified and sample covariances.

Recently, new developments with non-linear shrinkage estimators for the covariance matrix triumph over the above-mentioned, linear methods and the 1/N portfolio. Ledoit and Wolf (2017) found a 50% lower out-of-sample volatility for its shrunk non-linear minimum variance portfolio versus the benchmark.

Alternatively, Jagannathan and Ma (2003) propose that imposing short sale constraints to the MV optimization acts like a shrinkage to the covariance matrix. This is the case as assets strongly correlated with the asset set are underweighted or shorted, leading to an error in the cases where covariances are biased upward.

The reduction of the covariance matrix's estimation error is especially important in the context of MinVar portfolios, used in the present research, as they rely solely on them for its allocation. A MinVar strategy is, therefore, an MV portfolio that is optimal "if all expected returns are equal" (Jorion, 1985).

2.2.2 Expected Returns

Absolute asset pricing was labelled "the central problem in finance", especially regarding to the macroeconomic foundations of risk (Pitsillis, 2005). With the sample mean's particularly poor predictive power for returns (Merton, 1980; Jobson and Korkie, 1980; Chopra and Ziemba, 2013), many have sought to find better models. Green, Hand and Zhang (2013) identified over 330 return predictive signals. That is, firm-specific characteristics that have predictive power over returns.

Many superior models, in terms of out-of-sample performance, for expected returns have been developed. They include market valuation ratios, interest rate term structure, corporate patterns, and cross-section pricing (Campbell and Thompson, 2008).

Additionally, there are macroeconomic factors that investors may care about and influence returns, including equity risk factors (Fama and French, 1993; Fama and French, 2015), pure foreign exchange factors (Lustig, Roussanov and Verdalhan, 2011), bond risk factor and volatility risk factors (as described in Daniel, Hodrick and Lu, 2014). As diversification should be seen not only from an asset class perspective but also from an underlying risk factor consideration, the inclusion of common market risk factors is, therefore, important.

As mentioned before, the Bayes-Stein shrinkage estimator is a commonly used method for reducing estimation error in expected returns.

2.3 Black-Litterman Model

The BL model originally developed by Black and Litterman (1992) also attempts to tackle the estimation error. Particularly, it merges a prior estimate, inferred from the CAPM equilibrium (or via base portfolio weights, as in Bessler et al., 2017), with additional views derived from any pricing model (He and Litterman, 2002). The model has an informational advantage as the reliability of the view's expected returns is taken into account (Bessler et al., 2017). This way, the asset allocation moves away from the prior and towards the views for assets that are more reliable. For a visual representation, see Bessler's Figure 1.

The economic intuition is that "the optimal portfolio for an unconstrained investor is proportional to the market equilibrium portfolio plus a weighted sum of portfolios reflecting the investor's views" (He and Litterman, 2002).

2.4 Diversification Benefits

The asset set, hereby referred to as "Sample", considered for the MV process can influence the amount of risk reduction, given the level of return. The level of correlation is essential: less of it means that there are more achievable diversification benefits. As international assets are less correlated than national ones, better mean-variance portfolios can be achieved through international diversification (Levy and Sarnat, 1970).

Jorion (1985) argues that most of the benefits of international diversification come in terms of risk reduction, instead of return gains. With that in mind, MinVar should be more used in international portfolio selection.

Moreover, alternative investments, by having a lower correlation with traditional public markets, offer additional risk-reward diversification benefits (Schweizer, 2008).

2.5 Hypothesis and Objectives

The goal of the present research is to assess the performance improvements in terms of Sharpe Ratio attributed to the use of Ledoit-Wolf (2003a) covariance matrixes and the Black-Litterman model used in Bessler et al. (2017) in a realistic setting of high estimation error due to a short sample window. This is motivated by the under-research of the covariance matrix's effect on portfolio performance relative to expected returns.

The first and main hypothesis is that the optimization strategies perform statistically differently than the 1/N benchmark. Due to the contentious literature, it cannot be assumed that they outperform it (De Miguel, 2009a; Bessler et al., 2017), so a two-sided hypothesis is used. The null hypothesis is that the sharpe ratio of the strategies is equal to that of the benchmark.

A second hypothesis is that the 1/N has a lower turnover than the optimized strategies, although this is not statistically tested in the present study. This is based on the same finding by DeMiguel et al. (2009a)

A third hypothesis is that optimized strategies have unexplained returns (alpha) in risk models, due to their ability to diversify away from common risk factors. The null hypothesis is that alphas are equal to zero. To the best of the author's knowledge, no previous study for this set of optimization strategies.

SECTION 3 Data

3.1 Samples

To ensure the extrapolation of the results and robustness, multiple samples are used. These samples vary from equity-only industry, national and regional portfolios to an international multi-asset class setting. This is similar to literature as DeMiguel (2009a) and Kirby and Ostdiek (2012) use equity samples, and Bessler et al. (2012) a multi-asset sample. A summary of the data used can be found in Table 1.

Importantly, the Industry Sample cover a longer period than the other samples, and thus it is not directly comparable. All returns are in terms of USD, in totals (capital gains plus dividends) and gross (before tax and transaction costs).

In total, 50,126 monthly return observations on 124 assets across the four samples are analyzed and used as input for the optimizations.

Table 1 – Sample Descriptions

Sample	Source	Number of Asset Classes	Period	Market
Industry Sample	Kenneth French Data Library	49	July 1969 – March 2024	U.S. Equities
Country Sample	Bloomberg	47	May 2004 – April 2024	Global Equities
Regional Sample	Bloomberg	13	June 2003 – April 2024	Global Equities
International Sample	Bloomberg*	15	July 2005- April 2024	Global Multi-Assets Class

Notes: *data mostly from Bloomberg but includes indexes from other sources, which are described in Table 3. This table describes the 4 samples used in the present study, to increase robustness. The first 3 samples cover only equity, while the last covers multiple asset classes in multiple regions of the world. All data was retrieved from Bloomberg, except for the first sample, which comes from the Kenneth Rench Data Library.

3.1.1 Industry Sample

Similarly to DeMiguel et al. (2009a) who used Kenneth R. French's Data Library for industry portfolio, the same is used in the present study. In their case, they used 10 industry portfolios. However, since the goal of this paper is to assess the performance of portfolio strategies with emphasis on the covariance matrix improvement, the data set with 49 industry portfolios is used, hereby named "Industry Sample". That is because the sample covariance matrix performs particularly poorly for large numbers of assets (Ledoit and Wolf, 2003b). Moreover, the increased number of portfolios is more representative of the U.S. equity investment opportunity set.

Notably, this is a U.S. equity-only sample, which is still not representative of the entire available investment universe. All companies listed in the NYSE, AMEX and NASDAQ are included in the portfolios and are assigned to them based on their SIC code. Monthly returns data is available for all industry portfolios from July 1969 to March 2024, which corresponds to the analyzed period for this sample.

The Industry Sample has an average pair-wise correlation across all assets of 0.56. In Figure 1 (Appendix B), it is possible to observe the correlation table. Gold stands out for its low correlation with other assets.

The average monthly return and standard deviation across all assets are 1% and 6.69% respectively. The standard deviation between the asset's average monthly return is 0.16%. In Figure 2 it is possible to observe that there is no clear relationship between average return and volatility in the sample.

3.1.2 Country Sample

The addition of international equity assets to the investment universe is justified by empirical findings that they "materially improve" "risk-return positions" (Levy and Sarnat, 1970; Jorion, 1985). Driessen and Leaven (2007) also find great benefits to international diversification in equity portfolios. Hence, for this sample, Bloomberg's equity index for Large and Mid-sized companies in 47 countries, 24 of which are considered developing economies. Returns are monthly and cover the period from May 2004 to April 2024.

The average pair-wise correlation for this sample is 0.58 across all assets. In Figure 3 it is possible to see that the correlation between developed countries' returns (top left corner) is much higher than the emerging assets.

The average monthly return across all assets is 0.84%, with a between-asset standard deviation of 0.16%. The average standard deviation across all assets is 6.96%. In Figure 4 it is possible to observe a clear, positive graphical relationship between asset's returns and volatility.

3.1.3 Regional Sample

The Regional Sample is similar to the Country Sample in the sense that both cover global equities. The key difference between them is the level of portfolio aggregation and the number of assets. This sample has a higher level of aggregation (regional versus national) and, thus, a smaller amount of assets. This change is relevant as Ledoit and Wolf (2003a) find that the sample covariance matrix performs poorly when there are many assets. Hence, it is expected that the difference in performance between LW portfolios is smaller for this sample when compared to the Country Sample, despite the small difference in the underlying investment universe.

In Table 2 it is possible to observe the 13 assets that compose this sample, and their descriptive statistics for the sample period of June 2003 to April 2024. Returns are monthly. The total average monthly returns and standard deviation are 0.84% and 5.66% and, as expected, similar to the country sample. They are also positively related as seen in Figure 6. The standard deviation between assets is 0.13%. As expected from returns data, the asset’s returns are negatively skewed. Notably, the data for the Regional Sample is also from Bloomberg and all include Large and Mid-sized companies of the given region.

For this portfolio, however, there is some overlap between the portfolios. For instance, the same stocks may be included on Bloomberg’s World, Developed Europe, and Developed EMEA indexes or the Emerging Markets and Latin American Indexes. Therefore, for this sample, the 1/N is at a disadvantage, due to its lower underlying diversification, and is expected to relatively underperform its itself in other samples.

The overlap leads to a relatively higher average pair-wise correlation of 0.80. As such, MV optimizations are also expected to relatively underperform due to smaller diversification benefits. In Figure 5 it is possible to observe that the World Index is almost perfectly correlated with the Developed Equity Index. The former was kept in the analysis, however, as an investor may still have a preference to invest in the relevant market index despite the presence of highly correlated alternatives. For instance, a retail investor could be long in the S&P 500 and in some of the stocks that compose it.

3.1.4 International Sample

The International Sample attempts to accurately reproduce the available, broad investment universe. Hence, it goes beyond just equity and includes bonds (corporate and government), commodities, real estate, and other alternative investments like private equity. Schweizer (2008) finds that these alternative investments complement traditional assets, such as equity and bonds, improving the portfolio’s risk-return profile, thus providing the motivation for this sample.

Moreover, for this sample, the emphasis is on avoiding the overlap in underlying assets seen in the previous sample.

In Table 3 it is possible to observe the assets included in the sample and their descriptive statistics for the period from July 2005 and April 2024. The average monthly return across all assets in the whole sample is 0.52%, and the standard deviation is 4.63%. The standard deviation between the asset’s average returns is 0.30%. In Figure 8, it is possible to observe a graphical, positive relationship between average return and volatility. The average pair-wise correlation across all assets is 0.53, the lowest across all samples. The correlation table is presented in Figure 7.

3.2 Risk Factor Data

Multiple sources are used to build the various factor models employed in the present study. With consistency in mind, the Risk-Free Rate (RF) used for the calculation of excess returns is the 1-month treasury bill return from Ibbotson and Associates Inc., present in the Kenneth French Data Library.

For the equity risk factors, the Fama-French 5-factor model is chosen (Fama-French, 2015). Those factors represent the excess equity market (value-weighted of all U.S. stocks) return and 4 self-financing portfolios: SMB, shorts nine small-capitalization U.S. stock portfolios and longs nine large-capitalization stock portfolios; HML, longs two portfolios of high-book-to-market (“value”) U.S. stocks and shorts two low ratio portfolios (“growth”); RMW, longs two high operating profitability (“robust”) portfolios and shorts two low operating profitability (“weak”) ones; CMA, longs two low-investment (“conservative”) portfolios and shorts two high-investment (“aggressive”) ones. Monthly returns for these factors are available from 1963 to April 2024.

Following Daniel et al. (2014), the bond risk factor model contains three factors: the excess equity market return, the 10-year treasury bond excess returns, representing interest rate risk, and a self-financing factor for term-structure risk. The latter is long on the 10-year treasury and short on the 2-year note. Monthly return data for these factors was sourced from CRSP and were available from 1963 to December 2023.

Lustig, Roussanov, and Verdelhan’s (2011) two-factor model for pure foreign exchange risk is also used, as this risk is prevalent in the international investment universe used in the present study. To produce their factors, they assembled 6 portfolios from a total of 35 currencies by sorting based on interest rates. The two risk factors are then: RX, the average return on all six portfolios, and HML, the highest-rate portfolios minus the lowest-rate one. Gross monthly returns for these factors from 1983 to May 2021 were sourced from Lustig, Roussanov, and Verdalhan’s (2011) MIT database.

To measure exposure to equity market volatility, daily returns data for the (VIX) index is sourced from CBOE from 1990 to April 2024. The data is resampled to depict the monthly percentage change. The index represents the implied near-term expectations for price variation in the S&P500.

SECTION 4 Method

The methodology of the present study closely follows that of DeMiguel et al. (2009a) and Bessler et al. (2017), but with a focus on the performance improvements of a better covariance matrix. Hence, instead of using Bayes-Stein shrinkages for returns estimates, Ledoit-Wolf sample covariance matrix shrinkages are applied. They are not mutually exclusive, however. To not give any portfolio an informational edge by reducing estimation risk in returns, the sample mean returns are used as expected returns for all strategies.

As such, multiple portfolio optimization strategies are assembled:

1. The Naïve, 1/N portfolio, (“1/N” or “benchmark”)
2. Mean-Variance: Maximum Sharpe Ratio/Tangency Portfolio, (“Max Sharpe”)
3. Mean-Variance: Minimum Variance Portfolio (“MinVar”)
4. Max Sharpe Ratio: Ledoit-Wolf shrunk covariance extensions (“LW Max Sharpe”)
5. Minimum Variance: Ledoit-Wolf shrunk covariance extensions (“LW MinVar”)
6. Black-Litterman with three different priors (“BL”)

4.1 Performance Evaluation

The portfolios are evaluated based on the goal of the portfolio strategies – maximizing the Sharpe Ratio ***(SR)***. The measurement is to be defined as:

$$SR_i = \frac{Returns_i - RF}{Vol_i} \quad (1)$$

With i referring to the asset.

This evaluation method is chosen based on past literature (DeMiguel, 2009a; Kirby and Ostdiek, 2012; Bessler et al., 2017), but also based on Markovitz’s (1952) assumption that a higher SR is preferred by investors.

Another important metric is portfolio turnover (TO), which is to be defined as the amount of trades as a percentage of the portfolio, for each period. Moreover, TO is undesirable, as it leads to increased transaction costs, which are not accounted for in the present study. It is also relevant as MV portfolios often have weights that vary extremely (DeMiguel, 2009a; Bessler et al., 2017).

Mathematically, TO is:

$$TO_{p,t} = \sum_{i=1}^{N} \frac{Weight_{i,t} - Weight_{i,t-1}}{Weight_{i,t-1}} \quad (2)$$

Where t is the period (month), p is the portfolio and N is the total number of assets in the sample. For simplicity, TO will be presented as an average for each portfolio strategy for each sample. Moreover, even with no trades, the portfolio weights change throughout the period due to the different returns experienced by each asset. Hence, the weight of the last period, in this formula, represents the allocation of the beginning of the month t-1 after experiencing the growth seen in that month, all rebased so that the portfolio size is equal to 1.

Importantly, both SR and TO are measured out-of-sample. Here, the interest is to act as an investor would, with no data from the "future". Hence, both performance measurements are computed for the immediate period after parameters have been estimated and the portfolio assembled.

4.2 Portfolio Characteristics

Similar to DeMiguel et al. (2009a), the portfolio optimization's parameters (expected returns and covariance matrix) are re-calculated at the end of each month based on a rolling window of sample data. After each re-calculation, the portfolio is rebalanced.

Due to the limited data (approximately 20 years for most samples) and common practice in the asset management industry, a 60 and 120-month rolling window is used (DeMiguel et al., 2009a). That is, every month the parameters are re-calculated based on the past 5 or 10 years of returns. The two rolling windows were added for robustness.

Importantly, for all strategies, the point of view of an American investor is taken and, thus, returns are in USD. Sample mean returns are used in all portfolios as the focus is on the improvements of the covariance matrix.

The portfolio strategies are tested with two sets of weight constraints: (0, 1) and (-1, 1). The former does not allow for short selling and allocations in assets of more than 100%, while the latter does not restrict short selling up to -100% in a single asset. Moreover, the sum of weights is always equal to 1.

4.3 Portfolio Optimization Strategies

4.3.1 The Naïve portfolio

The Naïve, or 1/N, portfolio is the benchmark for all other optimization strategies. This portfolio simply invests an equal weight into all available assets in the sample, with no consideration for sample information and no parameters. The weights are rebalanced at the end of every month, following:

$$W_i = \frac{1}{N} \quad (3)$$

4.3.2 Mean-Variance: Max Sharpe Ratio

The Mean-Variance (MV) framework was developed by Markowitz (1952) to compile a portfolio theory that maximizes a mean-variance investor's preference. As the goal of this study is to assess the Sharpe ratio and turnover performance of MV versus the benchmark, the Tangency Portfolio which maximizes the SP, is used. DeMiguel et al. (2009a) used a different MV variant that maximizes investors' utility function, instead of the SP, which is widely common in literature.

Importantly, for the optimization problem, two parameters must specified: the expected return and the covariance matrix (risk model).

This strategy is addressed as Max Sharpe but also referred to as classical due to its reliance on sample estimates.

4.3.3 Mean-Variance: Minimum Variance

This portfolio attempts to minimize variance with no regard to the expected return. Thus, it requires no input for expected returns, only needing the covariance matrix. MinVar is the optimal strategy when return estimates are the same (Jorion, 1985). Despite not attempting to maximize the SP, it tends to perform well as it is not subject to estimation risk in the expected returns, only in the covariance matrix, which is less severe (Merton, 1980) but still significant (Ledoit and Wolf, 2003a).

4.3.4 Ledoit-Wolf Shrunk Covariance Matrices

As mentioned in the theoretical framework, Ledoit and Wolf (2003a) provide a form of shrinkage for the covariance matrix that causes a model misspecification but improves estimation risk. The misspecification is because the shrunk matrix does not attempt to reflect the true economic reality – the asset's variance and covariance. In doing so, however, it uses simpler estimators that are measured with less error, while also producing more moderate estimates due to moving estimates towards a central, constant value, thus reducing estimation risk. The degree to which the final matrix is moved toward the shrunk matrix depends on the estimation errors, misspecification, and the correlations between the two.

This improved covariance matrix is particularly important in the case of the MinVar portfolio since it only used the risk model to allocate weights.

The shrinkage consists of finding an optimal weight between the sample covariance matrix, S, and a more structured estimator, F, as in (4). Adding structure to the matrix is done by reducing the number of factors that determine it, a difficult task as there is no consensus on the number and nature of the factors, apart from the market index (Ledoit and Wolf, 2003a).

$$(4) \quad \hat{S} = \frac{k}{T}F + \left(1 - \frac{k}{T}\right)S$$

Where k, the optimal shrinkage constant, is estimated by:

$$(5) \quad k = \frac{(p - r)}{c}$$

In (5), p, r and c are estimators of "the covariance between the estimation errors of S and F", "the error of the covariance matrix", and "the misspecification of" F (Ledoit and Wolf, 2003a). Finding this optimal shrinkage constant is one of the author's most relevant contributions. T represents the number of observations, in this case, months.

In the present study, three types of Fs are used: single-factor, constant correlation, and constant variance. All of these reduce the number of factors in the matrix, improving structure. In the latter, all the variances in the matrix are replaced by the mean of all assets' variance, and correlations are set to zero. In the constant correlation matrix, all correlations are replaced by the average pair-wise correlation. These average correlation and variance are re-calculated according to the rolling basis.

4.3.4.1 Single-Factor Shrinkage

This shrinkage is based on Sharpe's (1963) return model, represented in Equation (6), and explained in detail in Ledoit and Wolf (2003a).

$$(6)\ r_{it} = \alpha_i + \beta_i Market_t + \varepsilon_{it}$$

Assuming the variance in residuals (within stocks) is constant, the model implies the covariance matrix following (7). β is a vector with assets i's betas and Δ a matrix with constant residual variances.

$$(7)\ \Phi = \sigma^2_{Mkt}\beta\beta' + \Delta$$

The nature of the market index used does not matter much, as long it significantly explains variation in returns (Ledoit-Wolf, 2003a). Here, an equal-weighted index is used.

4.3.5 Black-Litterman

The following section is based on Bessler et al. (2017), He and Litterman (2002) and Cheung (2010). As mentioned in Section 2.3, the Black-Litterman model combines priors, market equilibrium-derived return estimates, to views, leading to better estimators that may be used for MV optimization. In its essence, Black and Litterman (1992) created "weighted average" estimators for expected returns and covariance matrix, based on parameters τ, Q and Ω. τ dictates how distant the resulting portfolio can be from the benchmark, meaning that a high value would lead to estimators that closely resemble the views. Following Walters (2013), this estimator is set to 0.05.

For the present research, the prior expected returns are found by following equations (8), where S is the sample covariance matrix, w is the weights of the market index or reference portfolio and δ the market-implied risk premium or risk aversion coefficient.

$$(8)\ \Pi = \delta S w_{benchmark}$$

Following Bessler et al. (2017), the benchmark weights provided are not from the market index, but from other benchmark portfolios. Specifically, the naïve portfolio, the MinVar portfolio, and the MinVar with constant variance shrinkage. Hereby referred to "BL 1/N", "BL MinVar" and "BL MinVar constant variance", respectively. The idea is to use these well-performing portfolios that do not use the expected returns as input through the BL process. δ is set to 1 as its estimation would require a series of returns from the priors, which is not possible at the start of the sample.

Equation (9) and (10) shows BL's returns and covariance matrix estimates, where P is matrix on the existence of views, Q are the views and Ω a matrix measuring the reliability of views.

$$(9)\ \mu_{BL} = [(\tau S)^{-1} + P'\Omega^{-1} 1P]^{-1}[(\tau S)^{-1}\Pi + P'\Omega^{-1}Q]$$

$$(10)\ \Sigma_{BL} = S + [(\tau S)^{-1} + P'\Omega^{-1} 1P]^{-1}$$

For the present study, Q was set as the rolling sample mean returns and Ω as the sample variance. The reason behind this was to have a ceteris paribus comparison between the other strategies and BL, which would not be possible if the latter had a different returns estimation model. P is full of ones and constant and (sample mean) views are always provided.

After the estimation, the resulting model (11) is subjected to an MV optimization to maximize the Sharpe Ratio.

$$(11)\quad E(R) \sim N(\mu_{BL}, \Sigma_{BL})$$

4.4 Sharpe Ratio Hypothesis Testing

It is important not only to know what the values of the strategies' SRs are but also how they statistically differ. With that in mind, the SP hypothesis testing based on Jobson and Korkie (1981b) is used, with Memmel's (2003) corrections. Under their study, the SP's mean is straightforward – the (rolling) sample mean difference, where i and n are the different strategies:

$$(12)\quad H_0: SP_p - SP_n = 0$$

$$(13)\quad H_a: SP_p - SP_n \neq 0$$

The hypothesis is tested via the following Z-score.

$$(14)\quad z = \frac{SR_p - SR_n}{\sqrt{\frac{1}{T}}\,[2\left(1-\rho_{p,n}\right) + \frac{1}{2}(SR_p^2 - SR_n^2 - SR_p SR_n(1+\rho_{p,n}^2))]}$$

T represents the number of periods, which are in months. The SPs, calculated and tested based on monthly values, are shown annualized in tables 17-20 in Appendix A.

Notably, a two-tailed test is used as prior literature is contentious on the ability of these models to outperform the benchmark. All SPs are tested against the benchmark, and not for their nominal significance.

4.5 Risk Factor Regressions

As described in Data Section 3.2, exposures to four common market risks are considered: equity market risk, bond market risk, currency market risk, and volatility. Only the latter is not a proper risk factor, but only a monthly percentage change of the index.

The objective of this part of the methodology is not to find a causal relationship between the risk factors and the portfolio returns, hence no consideration with regards to multiple biases, such as Omitted Variable Bias (OVB) is made. The goal of the risk models is to assess if the strategies' returns are statistically exposed to common risk factors and if said factors can explain the variance of returns. Thus, a similar approach to Daniel et al. (2014) was employed. The focus is on the significance of the alpha (when relevant) and the r-squared.

For the following models, the choice of using the excess returns (ER) instead of nominal returns is based on CAPM's belief that returns are composed of two parts, the risk-free rate and some risk premium. Here, the interest is in the later component of returns. Most of the factors refer to American risk factors, which is in line with the study's American investor point of view.

4.5.1 Fama-French 5-factor model

The model developed by Fama and French (2015) is applied:

$$(15)\ ER_p = \alpha + \beta_1 ExMkt + \beta_2 SMB + \beta_3 HML + \beta_4 RMW + \beta_5 CMA + \varepsilon_p$$

Where ER and ExMkt are the excess portfolio returns and excess return on the market portfolios. The rest of the factors are described in depth in the Data Section 3.2.

In this regression, the alpha (α) is interpreted as returns that are not unexplained by the factors. It is expected that strategies in the Industry Sample have a higher exposure and explainability by these factors, as they are both confined to the U.S. equity market.

Strategies in the Country and Regional samples are still expected to be exposed to these factors as global equity markets are correlated to the American one, exemplified by the average pair-wise correlation of 0.58 and 0.80 respectively. Meanwhile, the International Sample's strategies are expected to have the lowest explainability by these U.S. equity market factors.

4.5.2 Bond Risk

For the bond risk, the portfolios' excess returns are regressed on the excess returns of the S&P500, 10-year treasury bond, and the difference between the 10-year and 2-year treasury in the following model, as in Daniel et al. (2014):

$$(16)\ ER_p = \alpha + \beta_1 ExS\&P500 + \beta_2 Ex10year + \beta_3 (10year - 2year) + \varepsilon_p$$

Here, again, the alpha can be interpreted as the portfolio's excess returns that are not explained by the model. Industry Sample and International Sample are expected to produce strategy returns with more exposure to these factors, as they have more exposure to the U.S. equity and U.S. credit, respectively. The latter sample includes both American government and corporate bonds as asset classes, for instance.

4.5.3 Currency Risk

Strategies' excess returns were regressed on the two risk factors described in Section 3.2:

$$(17)\ ER_p = \alpha + \beta_1 RX + \beta_2 HMLc + \varepsilon_p$$

All samples but the Industry Sample are expected to be exposed to these factors, as they are mostly international and exposed to currency risk versus the USD. This model is based on Lustig (2011), as applied in Denial et al. (2014).

4.5.4 Volatility

The exposure to volatility is the simplest measurement, as it is not a tradable factor. As a result, the alpha cannot be interpreted as unexplained excess returns. However, the results of the regression are still useful to determine the association between the returns and the U.S. equity market volatility.

$$(18)\quad ER_p = \alpha + \beta_1 VIX + \varepsilon_p$$

SECTION 5 Results & Discussion

In this Section, firstly, the mean returns, variances, etc of the portfolios are assessed. Subsequently, the performance measurements (Sharpe Ratio and Turnover) are presented and discussed, including method and sample considerations.

The focus of the paper is to evaluate the performance benefits of improved covariance matrices and not the improvements in terms of estimation error, although they are relevant.

Little attention is paid to the absolute values of the relevant statistics, as they vary substantially per sample. The interest is on the relative performance within the sample and rolling window, which can be compared across samples. The difference in rolling windows leads to different out-of-sample series, as the 120-month window takes 5 extra years to train versus the 60-month one.

5.1 Portfolio Descriptives

Descriptive statistics are available for all strategies in all samples in Tables 4 to 12 (Appendix A).

In Tables 4 and 8, the results for the 60 and 120-month rolling windows, respectively, in the Industry Sample can be observed. As expected, MinVar strategies do, indeed, deliver the lowest SDs for this Sample. Between them, the classical MinVar (-1, 1) has a higher variance, showing the extent of estimation error and how shrinkage by LW or constraint may lead to improved allocation given the objective. In fact, for Max Sharpe portfolios, the use of (0, 1) versus (-1, 1) constraints also reduced variance in all cases.

In Tables 5 and 9, the 60-month and 120-month rolling window results for the Country Sample are available. In this sample, once again MinVars have the lowest SDs. Importantly, once more, the greatest reduction in risk is seen for portfolios that use LW covariance matrices and that have constraints (-1, 1). As expected, the BL models with MinVar priors also have low SDs.

The results for the Regional Sample are in Tables 6 and 9. This sample is the most challenging of all for the MV framework, as the asset's correlation is 0.80, possibly making diversification gains lower. Similarly to the other samples, LW extensions of MaxSharpe (-1, 1) increased SD versus the classical MV. In this case, shrinking the covariance matrices does not impact risk, nor do the tighter constraints in the case of MinVar.

The results for the International Sample can be found in Tables 7 and 11. For this sample, LW shrinkages also do not systematically reduce SDs. The exception was the short-selling MinVar portfolios.

Similarly to other samples, the BL model produces minor or no risk improvements for the MinVar prior, but relevant ones for the 1/N. For instance, for the 60-month window, the BL with the Naïve prior has an SD of 1.19% versus the prior's 3.32%.

The main takeaways of this subsection are fourfold. First, short-selling constraints (0, 0) lead to lower standard deviations, perhaps acting as a type of shrinkage, in agreement with Jagannathan and Ma

(2003). Secondly, LW portfolios decrease risk more when short selling is allowed. This is intuitive since, with no shrinkage from constraints, (-1, 1) are more exposed to estimation risk and, thus, the marginal benefits, in terms of lower variance, of LW may be higher. Thirdly, the BL model only produces risk improvements versus the prior in the case of 1/N. Hence, the model adds more to the allocation when no sample information is considered.

Lastly, LW extensions to Max Sharpe (-1, 1) increase SD. This may be simply because Max Sharpe does not target a specific return or variance, but the max trade-off between the two. Therefore, given a new LW covariance matrix, the optimization targets a higher return and higher SD tangency point. This is further evidenced by these portfolio's relatively high mean returns in most of the samples.

Notably, out of all strategies, 1/N has one of the highest percentages of negative returns over all months (Table 12). On an equal-weighted average across all samples, it experiences negative returns in 42% of periods. This provides some evidence that the optimal allocations can deter losses.

5.2 Estimation Error

In Figures 9 and 10, it is possible to observe two series of absolute mean errors of the estimated covariance matrixes versus the realized matrix in the next 12 months. The first compares the error for the estimated sample covariances for the 60-month and 120-month periods. It is possible to note that there is a small reduction in estimation risk when the rolling window is increased. Additionally, the errors are related to market downturns, as they increase, for instance, in 2008 and 2020. This is also seen for expected return errors in Figure 11.

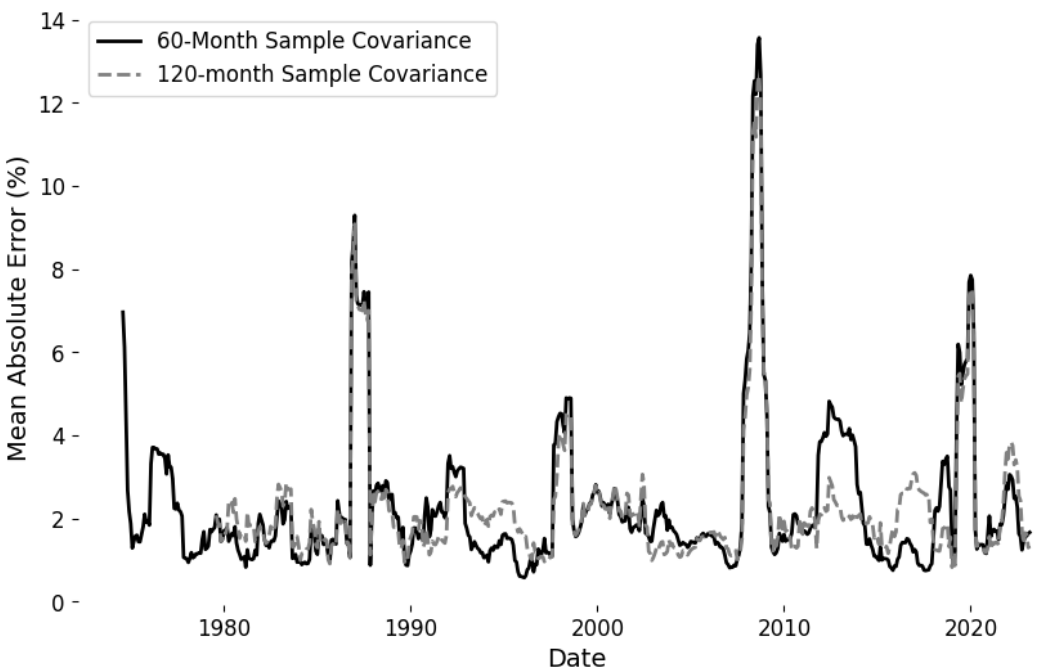

Figure 9 – Estimation Error for the sample covariance matrix, different rolling windows

Note: The graph shows a series of Mean Absolute Errors (%) for the estimates of the covariance matrix made with the 60-month and 120-month rolling windows, from July 1974 to February 2023. The data is from the Industries Sample, 49 industry portfolios from the Kenneth French Data Library. For the calculation of the error, the realized covariance matrix of the next 12 months after the estimation date is used.

In Figure 10, it is possible to see that the reduction in estimation error by using the LW constant correlation is negligible. In fact, it reduced the absolute mean errors by just 0.30% to a monthly average of 2.29%. Meanwhile, the constant variance leads to a relative 4.86% drop in the average monthly error, from 2.30% to 2.19%. Single factor reduces the average error by 1.14%. These findings are consistent across all samples. Therefore, it is expected LW extensions with constant variance will perform better.

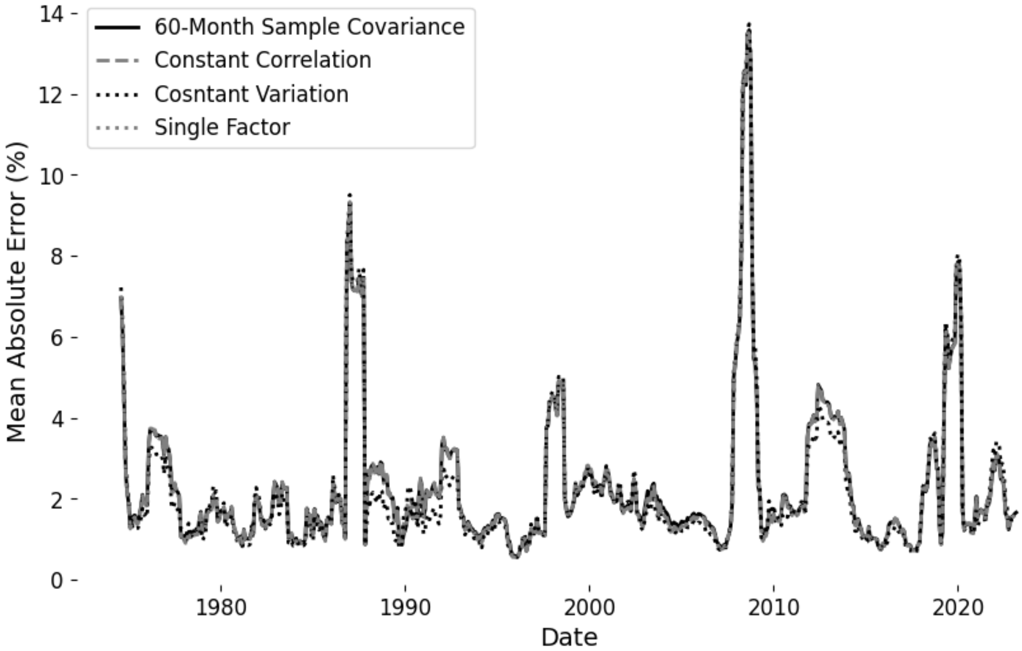

Figure 10 – Estimation Error for the Sample Covariance Matrix and the Ledoit-Wolf extensions.

Note: The graph shows a series of Mean Absolute Errors (%) for the estimates of the covariance matrix made with the 60-month and 120-month rolling windows, from July 1974 to February 2023. The data is from the Industries Sample, 49 industry portfolios from the Kenneth French Data Library. For the calculation of the error, the realized covariance matrix of the next 12 months after the estimation date is used.

In Figure 11 (Appendix B), it is possible to observe the expected returns estimation error of the rolling sample mean. As expected, the error is more severe for the smallest sample window. Over the whole sample, the 60 and 120 rolling windows Mean Squared Error (MSE) were, respectively, 2.7% (60 months of extra data) and 2.2%. Graphically, it is also possible to see that the sample mean is unable to predict downturns, as the MSE is negatively related to market volatility. The correlation between 60-month rolling estimates' MSE and the market returns is -0.56.

Importantly, the goal of the shrinkage is not to purely improve estimation, but to reduce estimation error and its risks in the context of an asset allocation (Jorion, 1986). As such, this method should theoretically also lead to less variant and extreme allocation, as it brings sample means towards a mean value, which should be reflected in the final weights. This is evidenced in the Section 5.3.2 (Turnover).

5.3 Portfolio Performance

As mentioned, the performance measurements are the Sharpe Ratio and the Turnover, as justified in the Methodology Section 4.1. The results for the different measurements are split between tables 13-20 and 21-22, respectively.

The nominal average annual Sharpe Ratios for each sample can be found in Tables 13 to 16, while their hypothesis testing versus the benchmark can be found in Tables 17-20.

5.3.1 Sharpe Ratio

5.3.1.1 Industry Sample

For the Industry Sample (Tables 13 and 17, Figure 12), the best performers are the MinVar (0, 1) portfolios with the LW covariances. Their average annual SP are 0.61 and 0.65 versus 1/N's 0.54 and 56 for the 60 and 120 windows, respectively. None of these differences are statistically significant, however. Additionally, the classical MinVar (0, 1) performs similarly to its extensions, but slightly less.

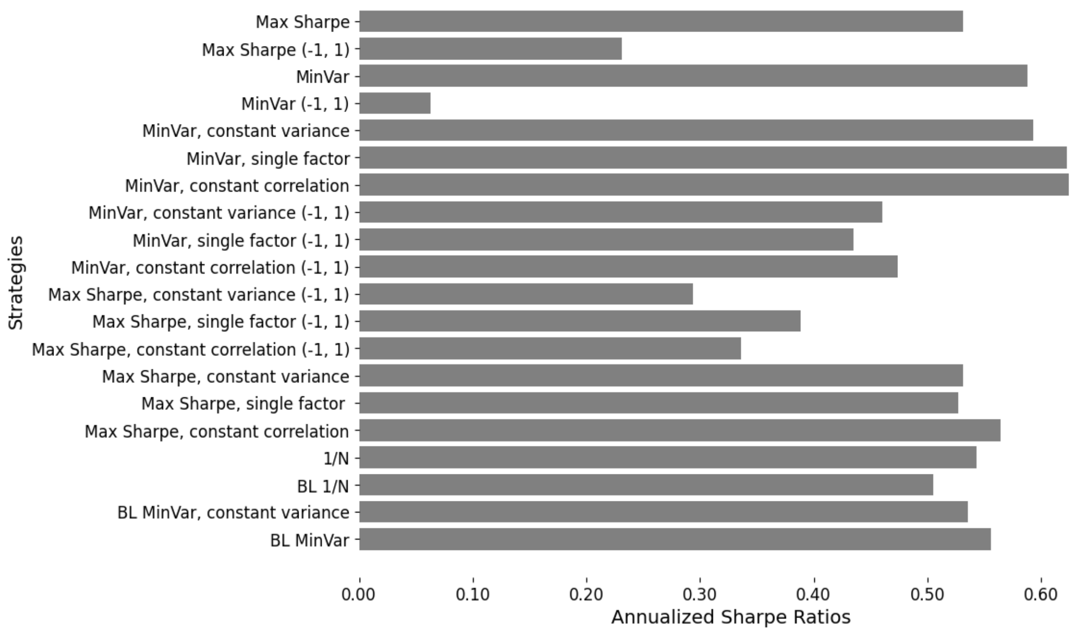

Figure 12 – Industry Sample: Sharpe Ratios.

Notes: The rows' values correspond to the annualized Sharpe Ratio realized over the whole period of 1974 to 2024. The strategies' inputs are calculated based on a 60-month sample rolling window. Data is from the Industry Sample.

Moreover, (-1, 1) portfolios underperform their (0, 1) counterparts in all strategies. This agrees with the private practice of using constraints and the theory that constraint acts as a form of shrinkage, improving portfolio performance (Jagannathan and Ma, 2003).

The worst performers are the classical (-1, 1) Max Sharpe and MinVar, with 0.23 and 0.06 annual SPs over the sample. They statistically underperform the benchmark at 10% and 1% significance, respectively. This is expected as these portfolios are exposed without any remedy to the estimation risk of the sample mean and (co)variances, which perform poorly (Merton, 1980; Jorion, 1986; Ledoit and Wolf, 2003a).

The superior performance, albeit insignificant, of the LW MinVar portfolios in the Industry Samples points towards a few conclusions. Firstly, LW shrinkage improves performance, even if the reduction in estimation error is small, as seen. Secondly, constraints act as shrinkage and are beneficial to performance. Thirdly, classical MVs perform poorly.

The first conclusion is particularly interesting for this sample as it has a high number of assets relative to the Rollings samples (49 vs 60 and 120 observations), leading the sample covariance matrix to contain a lot of estimation error (Ledoit and Wolf, 2003a).

5.3.1.2 Country Sample

In this Sample (Tables 14 and 18), the best performer for the 60-month rolling window is the MinVar (-1, 1) and its LW extensions (apart from the constant correlation). They have an average annual SP of 0.81 versus the benchmark of 0.52. Their difference is not significant. MinVar (-1, 1)'s performance is, however, not robust to the different rolling windows.

Figure 13 gives more insight into the reason why MinVar (-1, 1) performs differently given the two estimation windows. The dotted line is a measure of the total difference in weight allocation between the two allocations, while the solid black line is their return difference. Unexpectedly, the changes to the weight difference are not correlated with changes in the VIX index, market returns, currency factors (relevant for this sample), or between asset SDs. Graphically, a relation between the allocation difference and the latter can be seen, however. Therefore, this lack of explainability supports DeMiguel et al. (2009a) in their claim that 60 or 120 months are too short periods for MV optimizations, even when using shrinkages and constraints, due to high estimation error.

Further support for this is the fact that the best performer in the previous sample (LW MinVar (0, 1)) inverses positions for the Country Sample with a 120-month rolling window, with an average annualized SP of 0.22 versus the benchmark's 0.31. This finding is particularly troublesome for the MV framework as both samples are similar in terms of average returns, SDs, between SD of average returns, average pairwise correlation, and number of assets. All, except for the former, should have an impact on MV optimization processes (estimation of returns and covariance matrixes).

Similarly to the previous sample, the (-1, 0) Max Sharpe and its LW extensions were among the worst performers. This may be related to their reliance on poor, sample mean return estimates and lack of shrinkage from tighter constraints, among other problems.

Moreover, this sample exemplifies how challenging it is to find statistical differences in SPs, as no strategies had a significant difference from the benchmark.

Table 18 – Country Sample (60 and 120 months): Monthly Sharpe Ratio Hypothesis Testing

Strategies	Difference to benchmark (60 months)	Difference to benchmark (120 months)
Max Sharpe	0,02	0,04
Max Sharpe (-1, 1)	-0,04	0,06
MinVar	-0,02	0,01
MinVar (-1, 1)	0,09	0,02
MinVar, constant variance	-0,01	-0,03
MinVar, single factor	-0,02	-0,03
MinVar, constant correlation	-0,01	-0,02
MinVar, constant variance (-1, 1)	0,08	0,02
MinVar, single factor (-1, 1)	0,09	0,02
MinVar, constant correlation (-1, 1)	0,02	-0,03
Max Sharpe, constant variance (-1, 1)	-0,09	0,01
Max Sharpe, single factor (-1, 1)	-0,04	0,03
Max Sharpe, constant correlation (-1, 1)	-0,07	0,00
Max Sharpe, constant variance	0,02	0,04
Max Sharpe, single factor	0,01	0,04
Max Sharpe, constant correlation	0,00	0,05
1/N	0,00	0,00
BL - 1/N	0,03	0,00
BL MinVar, constant variance	-0,03	0,03
BL - MinVar	-0,05	0,04

Notes: This table contains the average difference in the monthly Sharpe ratio of each portfolio strategy versus the 1/N, and its statistical significance. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (180 and 120 observations respectively). Data and assets from the Country Sample are used. The statistical significance related to the difference between the strategies' Sharpe ratio and the benchmark, 1/N, using Memmel's (2003) method. ***p<.01, **p<.05, ***p<0.10

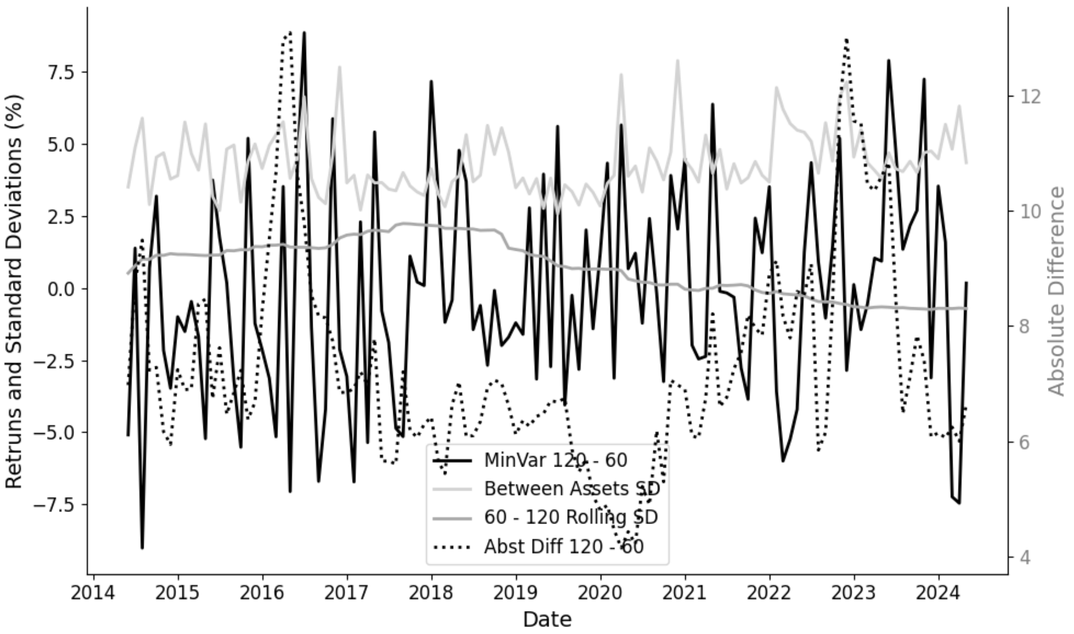

Figure 13 – Absolute Weight and Returns difference, 60 vs 120 rolling windows

Notes: For the MinVar (-1, 1) strategy in the Countries sample, 2014-2024. Abst difference is equal to the sum of the absolute weight difference for each asset at each period between allocations estimated with 60 and 120-month rolling windows. 60 – 120 rolling SD is the difference between the average 60-month rolling SD of all assets minus the same number but with a 120-month rolling window. Between asset SD is the SD in returns for all assets in the sample for a given month. MinVar 120 – 60 is the difference in return between the MinVar with the two estimation windows. Abs diff (left), Return difference (Right), Between Assets Standard Deviation (Right), and 120 – 60 rolling SD (right).

5.3.1.3 Regional Sample

The SPs and their hypothesis testing for the Regional Sample can be found in Tables 15 and 19 respectively. As mentioned in the Data Section, this sample differs from the others because its assets overlap, putting the 1/N at an under-diversified position and increasing the correlation. The sample's average pair-wise correlation for the whole period is 0.80 versus the 0.56 average between the other samples. Moreover, this dataset is the one with the least assets, possibly leading to smaller sample covariance estimation errors (Ledoit and Wolf, 2003a).

The high correlation and low N lead to the "perfect" scenario for MV strategies to outperform the benchmark, 1/N. Indeed, all portfolios do so, except for Max Sharpe, constant variance (-1, 1) (Figure 14). Only Max Sharpe (0, 1), Max Sharpe LW single factor (0, 1), and Max Sharpe LW constant correlation have SPs that statistically differ from 1/N at 10%, while the rest is insignificant.

In the 120-month window, the MinVar (-1, 1) and BL 1/N have an annualized SP of 0.73 and 0.63, significantly beating the benchmark at 10% and 5% significance levels, respectively. The former's performance reinforcers Kirby and Ostdiek's (2012) and Bessler's et al. (2017) findings that the sample information has indeed value if used in the correct setting.

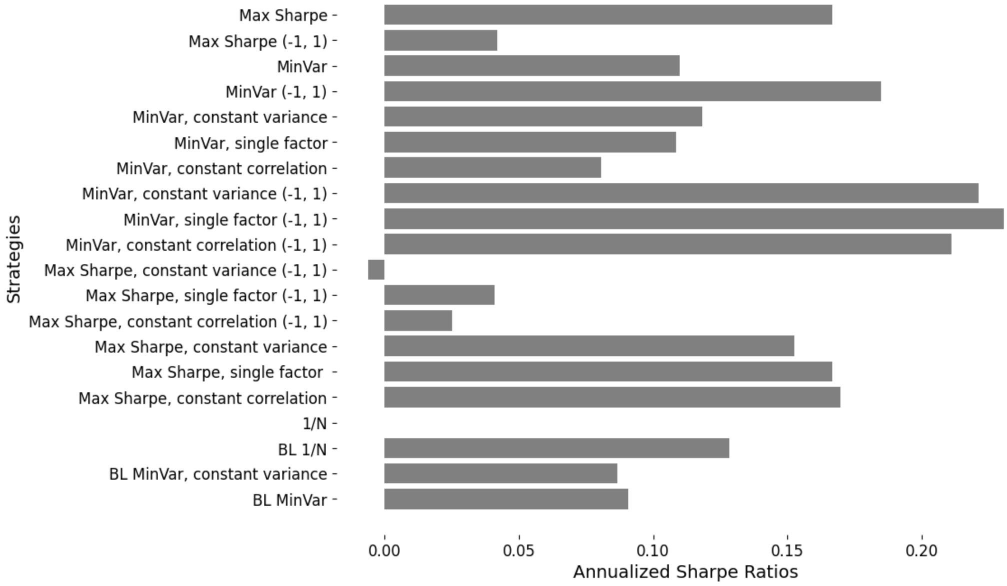

Figure 14 – Regional Sample: Difference of annualized Sharpe Ratio vs. 1/N, 60 months

Notes: The rows' values correspond to the annualized difference versus the 1/N in the Sharpe Ratio realized over the whole period of 2008 to 2024. The strategies' inputs are calculated based on a 60-month sample rolling window. Data is from the Regional Sample.

In this sample, MinVar (-1, 1) and its LW extensions have been the best performers. In fact, it has been a trend among all the samples that MinVar portfolios perform well. This, however, varies drastically, with (0, 1) doing better in some samples and having the lowest SP in the Countries Sample.

MinVar (-1, 1)'s superior performance versus (0, 1) may point to the fact that in an investment universe with high correlations, it is less risky to take a large short position. If the price of one asset falls, likely all other prices are falling, offsetting losses from longs with gains from shorts. In light of high estimation error due to the short window, the allocation may be more likely to "guess" the correct balancing short when most assets are highly correlated with themselves (this sample) than not (other samples). MinVar represents the purest form of this mechanism, as no input from returns is taken. Additionally, the fact that MinVar constant correlation (-1, 1) performed equally well as other similar strategies reinforces this point, as this strategy shrinks all correlations towards an average.

This rationale is backed up by Figure 15, showing MinVar (-1, 1)'s (120-month) allocation to overlapping, highly correlated assets. Not only do these allocations vary significantly, going from high longs of 100% weights to -100% in short periods, but also these highly correlated assets are often longed and shorted at the same.

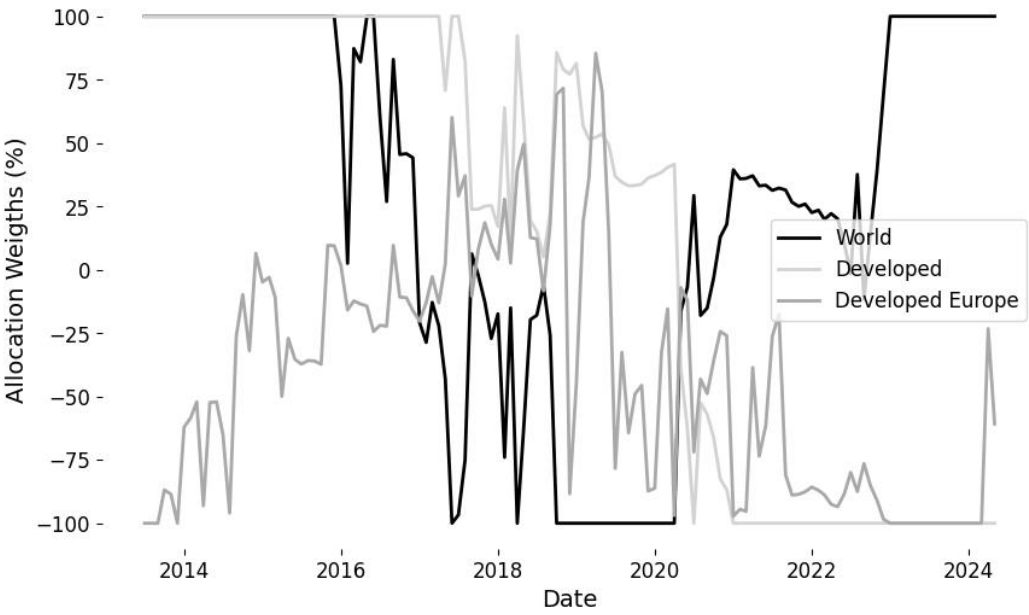

Figure 15 – Minvar (-1, 1) allocation through time, selected assets.

Notes: The figure shows the allocated weights for the MinVar (-1, 1) strategies for the World Index, Developed Index and Developed Europe Index (described in Table 2). The estimation window is 120 months and the sample used in the Regional one. The period covered is from 2013 to 2024.

When expected returns considerations are added to the optimization mechanism, performance decreases, evidenced by Max Sharpe (-1, 1) and its extensions' poor performance. This is also shown by BL MinVar, which deviates from the prior based on expected returns and its reliability and does not perform well in this sample.

Figure 15 shows how allocations can be extreme even with some constraints, a larger estimation window (120 months), and no exposure to returns estimation risk, such as in the case of MinVar (-1, 1). Figure 16 shows the conditional expected return implied by the allocation of 3 strategies through time. As can be observed, they move together but have different orders of magnitude. As expected, Max Sharpe (-1, 1) is much more aggressive. This is in agreement with Kirby and Ostdiek's (2012) criticism of DeMiguel's (2009a) study, where MV strategies had no constraints and expected returns of over 100%. Interestingly, the BL 1/N, which is not allowed to short, only had non-zero weights for 5 (out of 13) assets, leading to high concentrations. The assets and their respective average weight were North America (76%), Latin America (1%), Asia Developed (23%), APAC Developing (0.2%) and Asia Emerging (0.2%).

Notably, in a setting where there is no estimation error, high correlations are less interesting from an asset allocation perspective, as discussed in the Theoretical Framework.

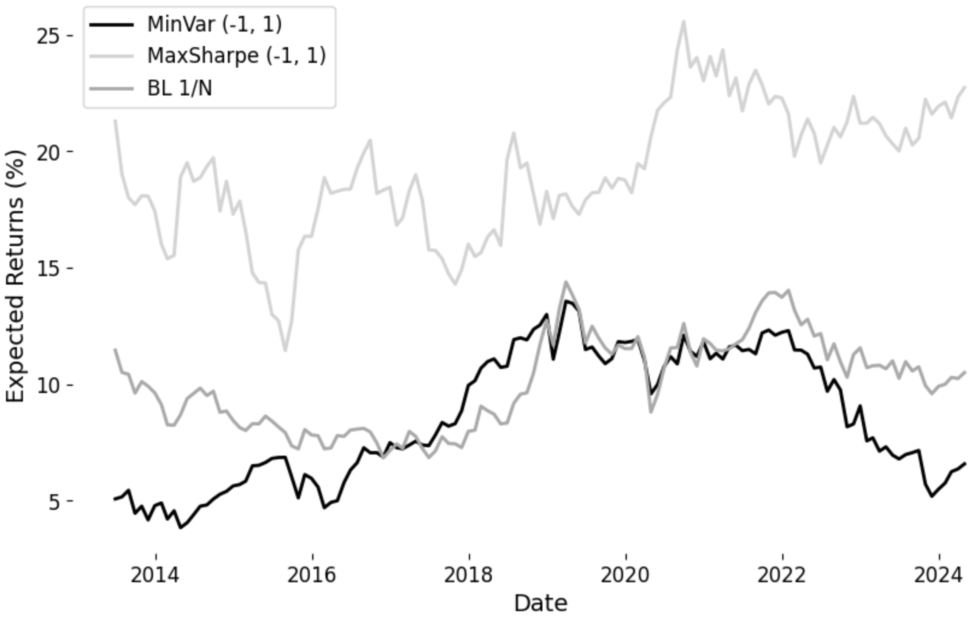

Figure 16 – Regional Sample: Expected Returns for each strategy given allocation

Notes: The figure shows the expected returns of the MinVar (-1, 1), Max Sharpe (-1, 1) and BL 1/N strategies. These returns are calculated by multiplying the vector with the model's input of returns estimates with their final allocation vector. The estimation window is 120 months and the sample used in the Regional one. The period covered is from 2013 to 2024.

5.3.1.3 International Sample

The SPs and hypothesis testing for the International Sample can be found in Tables 16 and 20, respectively. This sample differentiates itself from the previous ones since it allows for significant diversification away from equities, including (government and corporate) bonds, commodities, real estate, hedge funds, and private equity, for many regions. This is beneficial from an asset allocation perspective as it expands the investment frontier and leads to superior risk-return trade-offs (Schweizer, 2008).

In Figure 17, it is possible to see that, due to the benefits mentioned above, all optimization strategies outperform the benchmark for the 60-month rolling window in the International Sample. Note here that unlike in the Regional Sample, 1/N is not at an inherent disadvantage here, as there is no overlap in the indexes' coverage. Additionally, the average pair-wise correlation is 0.53, the lowest out of all the samples, meaning that the "guessing" mechanism that benefited MinVar (-1, 1)s in the previous sample applies to a lesser extent here. Hence, this result is to the strategies' "own merit", in defence of the value of sample information, like Bessler et al. (2017).

For the 60-month rolling window, Max Sharpe (0, 1), its single factor LW extension, MinVar single factor (-1, 1) and all BL models have statistically superior SPs versus the 1/N at 10% significance level. MinVar constant variance (-1, 1) and Max Sharpe constant variance (0, 0) have it at 5%. Much like the Regional and Country (60 months), the best performers are LW MinVar (-1, 1), especially constant variance and single factor. Constant correlation's underperformance relative to its close peers further supports the statement that "guessing" is not taking place here, since when correlation is shrunk, performance decreases. That is, when there are more potential benefits from diversification, correlation matters more.

These results are, however, not robust to different estimation windows (120-months), as, in this setting, MinVar (0, 1)'s LW extensions underperform the benchmark. They also fail to deliver the lowest SDs, which are achieved by MinVar (-1, 1) and its LW extensions.

The best performers for the 120-month window are MaxSharpe (-1, 1), its LW extensions and MaxSharpe (0, -1), perhaps showing the extra 5 years of data in the context of a sample with low-correlation assets, allowing for MV to benefit significantly. The latter two statistically differ from the benchmark at a 10% significance level.

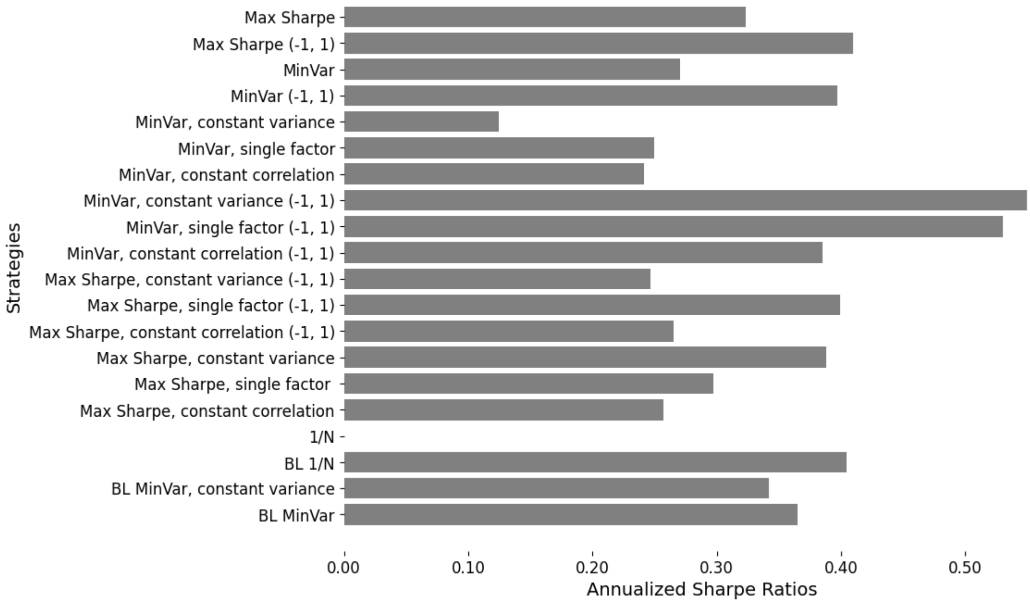

Figure 17 – International Sample: Difference of average annualized Sharpe Ratio vs. 1/N

Notes: The rows' values correspond to the annualized difference versus the 1/N in the Sharpe Ratio realized over the whole period of 2010 to 2024. The strategies' inputs are calculated based on a 60-month sample rolling window. Data is from the International Sample.

In Appendix B, Figure 18 shows the relationship between MinVar(-1, 1) and its LW constant variance extension. As it is possible to observe, the latter has a higher SD (1.18% versus 1.00%), but also higher returns (0.34% versus 0.24%). Despite its higher performance, it could be said that MinVar (-1, 1) constant variance is less effective than its classical counterpart (in this sample), as its objective was not to reach a higher SP, but to minimize variance.

Table 20 – International Sample (60 and 120 months): Monthly Sharpe Ratio Hypothesis Testing

Strategies	Difference to benchmark (60 months)	Difference to benchmark (120 months)
Max Sharpe	0,09*	0,10*
Max Sharpe (-1, 1)	0,12	0,13
MinVar	0,08	0,04
MinVar (-1, 1)	0,11	0,06
MinVar, constant variance	0,04	-0,03
MinVar, single factor	0,07	0,00
MinVar, constant correlation	0,07	0,00
MinVar, constant variance (-1, 1)	0,16**	0,07
MinVar, single factor (-1, 1)	0,15*	0,05
MinVar, constant correlation (-1, 1)	0,11	0,01
Max Sharpe, constant variance (-1, 1)	0,07	0,10
Max Sharpe, single factor (-1, 1)	0,12	0,11
Max Sharpe, constant correlation (-1, 1)	0,08	0,09
Max Sharpe, constant variance	0,11**	0,10*
Max Sharpe, single factor	0,09*	0,09*
Max Sharpe, constant correlation	0,07	0,10*
1/N	0,00	0,00
BL - 1/N	0,12*	0,08
BL MinVar, constant variance	0,10*	0,09
BL - MinVar	0,11*	0,09

Notes: This table contains the average difference in the monthly Sharpe ratio of each portfolio strategy versus the 1/N, and its statistical significance. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (166 and 106 observations respectively). Data and assets from the International Sample are used. The statistical significance related to the difference between the strategies' Sharpe ratio and the benchmark, 1/N, using Memmel's (2003) method. ***p<.01, **p<.05, ***p<0.10

5.3.2 Turnover

The second performance measurement, Turnover, is relevant due to its implications for transaction costs and taxes. The more allocations are shifted, more trading costs are incurred and more capital gain taxes that could be otherwise deferred are paid in the present. As noted beforehand, all the returns and SP presented thus far are gross of any transaction costs. Transaction costs can amount to 0.5% of the whole notional value traded (Balduzzi and Lynch, 1999).

In Tables 21 and 22, it is possible to observe the average Turnover for all the strategies in all samples. Three important conclusions can be taken from it. Firstly, as expected (Miguel et al., 2009; Bessler et al., 2017), Max Sharpes have the highest average turnover out of all strategies. Due to their aggressive profile (see discussion of Figure 17) and exposure to high estimation risk (returns and covariance matrix), it is unsurprising that turnover is high. MaxSharpe (-1, 1) had a turnover of 347% versus the benchmark's 2.8%, meaning that on average (across all samples), the former portfolio traded volumes over 3 times larger than its equity value each month. This would lead to transaction costs of 1.7% of the equity value monthly, which is unsustainable. A lesser, but still extreme value, of 198% is found for MinVar (-1, 1) as expected by Figure 16.

Secondly, short-selling constraints (0, 1) lead to substantial decreases in turnover. The average monthly turnover across all samples for MinVar (0, 1) is 9.1%, which is 95.4% smaller than its (-1, 1) counterpart.

Thirdly, LW covariance extensions lead to more reductions in turnover with (-1, 1). This is seen in the across-sample average of both rolling windows. It is also consistent with theory. Shrinkage leads to more moderate, less variant covariance matrices that tend towards a mean (Ledoit and Wolf, 2003a). Based on those models, intensive allocation shifting takes place (evidenced by Figure 12, Table 21, and 22) to reduce risk, especially in the case of short selling. If the matrix itself varies less, the allocations will also.

5.4 Risk Factor Exposure

In this subsection, a total of 640 regressions are run between the 20 strategies, 2 rolling windows, 4 samples, and 4 risk models. To be able to perform any sort of analysis, only 32 are displayed in Tables 23 and 24. Only 8 representative strategies are shown, in two samples. The equity, bond, and volatility models are shown for the Industries Sample, due to the larger series of data. The currency model was run on the Countries Sample due to its exposure to foreign exchange fluctuations.

It is important to note again that these models do not attempt to find any causal relationship but search for statistical exposure (or lack thereof) that these strategies' returns may have to the risk factors.

Table 21 – Turnover figures, 60 months rolling window.

Strategies	Industry Sample	Country Sample	Regional Sample	International Sample	Average
Max Sharpe	3,2%	3,4%	2,0%	2,5%	2,8%
Max Sharpe (-1, 1)	13,6%	14,7%	5,3%	2,8%	9,1%
MinVar	391,4%	255,4%	120,7%	26,1%	198,4%
MinVar (-1, 1)	12,5%	11,6%	5,8%	4,7%	8,7%
MinVar, constant variance	11,6%	12,8%	6,2%	2,6%	8,3%
MinVar, single factor	12,5%	11,1%	8,8%	2,8%	8,8%
MinVar, constant correlation	52,8%	45,4%	16,2%	9,6%	31,0%
MinVar, constant variance (-1, 1)	50,1%	39,5%	31,4%	11,2%	33,1%
MinVar, single factor (-1, 1)	47,4%	30,6%	21,1%	8,1%	26,8%
MinVar, constant correlation (-1, 1)	30,6%	27,4%	5,0%	14,4%	19,3%
Max Sharpe, constant variance (-1, 1)	631,6%	493,6%	188,5%	74,2%	347,0%
Max Sharpe, single factor (-1, 1)	27,7%	25,3%	6,2%	13,9%	18,3%
Max Sharpe, constant correlation (-1, 1)	28,4%	26,5%	5,0%	16,3%	19,0%
Max Sharpe, constant variance	30,3%	27,1%	5,2%	20,9%	20,9%
Max Sharpe, single factor	379,9%	300,0%	120,1%	60,3%	215,0%
Max Sharpe, constant correlation	344,4%	272,9%	155,5%	55,8%	207,2%
1/N	391,8%	259,1%	131,1%	56,1%	209,5%
BL - 1/N	25,2%	27,7%	16,1%	3,3%	18,1%
BL MinVar, constant variance	12,6%	20,6%	7,7%	1,6%	10,6%
BL - MinVar	12,0%	19,3%	7,7%	1,5%	10,1%

Notes: This table contains the average monthly turnover of 20 portfolio strategies over the entirety of each sample (Industry, Country, Regional, and International). The average is an equal-weighted average of all samples. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the

Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the International Sample are used. The samples, from left to right have 597, 180, 191 and 166 observations respectively.

In the first part of Table 23, it is possible to observe the Fama-French 5 model. All the resulting alphas are statistically insignificant, meaning that there are no statistically significant returns that are not explained by the model. This is expected as this is an equity sample.

Another important insight relates to the comparison of (0, 1) strategies and their (-1, 1) counterparts. The latter have lower market betas. For instance, Max Sharpe (-1, 1) has a significant market beta of 0.58, versus Max Sharpe (0, 1)'s 1. The former is also less exposed to other factors.

This may point towards the following economic explanation. Firstly, given its ability to short, (-1, 1)s are better suited to diversify away from common risk factors, leading to more uncorrelated portfolio returns. Secondly, said reduction in market risk is exchanged for an increase in estimation risk, as there are looser constraints. The estimation risk is associated with market volatility, as seen in Figure 12. However, in Figure 19 (Appendix B), this trade-off has proven to not be worth it for this sample, and (-1, 1) have smaller SPs for this sample.

Further support for the above rationale comes from the fact that (-1, 1)s, in Tables 21 and 22, have lower explained variance (r-squared) in all models, since estimation risk is not a factor that is modelled. For all models displayed, BL 1/N presents smaller (market) factor exposure and r squared. This is consistent with Bessler's (2017) that the portfolio can away shift from riskier assets at times of recession.

Moreover, some strategies in the Industry Sample have statistically significant alphas in the Bond model. This is expected since the sample is equity-only and, therefore, less explainable from an interest rate risk perspective.

Additionally, (-1, 1) models have a significant negative exposure to the VIX index, which, again, relates to Figure 12. When market volatility increases, estimation risk also rises and returns fall.

For the currency risk factor model for the Country Sample in Table 24, the 1/N (essentially an equal-weighted index) has statistically significant unexplained returns (alpha). The r-squared is high, however, at 0.77. Clearly, equity risk factors are missing from this model given the equity nature of the sample, possibly leading to the alpha. The optimization strategies perform relatively well in terms of reducing the r squared. Again, this points towards the rationale presented in this section of a factor-estimation risk trade-off.

Table 24 – Exposure to Currency Risk Factors – Countries Sample, 60-months window.

Factors	Max Sharpe	Max Sharpe (-1, 1)	MinVar	MinVar (-1, 1)	MinVar, constant variance (-1, 1)	Max Sharpe, constant variance (-1, 1)	1/N	BL 1/N
Currency								
Alpha	0.7%***	1.2%	0.5%**	1.5%***	0.8%***	0.6%	0.6%***	0.8%***
RX mean	1.65***	1.08**	1.29***	0.64***	0.79***	0.44	2.10***	1.78***
HML	0.49***	0.19	0.45***	0.16	0.14	0.03	0.59***	0.38***
R-squared	0.52	0.05	0.53	0.07	0.23	0.01	0.77	0.52

Notes: this model is based on Lustig et al. (2011), with data from 2009 to 2021. RX mean is the average return on 6 currency portfolios, while HML is the return on a portfolio with high nominal rates minus one with low. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60-rolling window is used to estimate the inputs used in the optimization of the portfolios (180 observations, from 2009-2024). Data and assets from the Country Sample are used. ***p<.01, **p<.05, ***p<0.10

CHAPTER 6 Conclusion

The performance of the 20 portfolios was assessed in light of the high estimation error caused by the small rolling estimation windows and the high number of assets to observations (particularly for the Industry and Country samples).

In this scenario, Ledoit and Wolf ‘s (2003a) shrunk covariance matrix reduced SDs and were among the best performers, despite the low reduction in estimation error. These results are not statistically significant, however. The same can be said about higher constraints, acting as shrinkage (Jagannathan and Ma, 2003). In many samples, the (0, 1) outperformed their (-1, 1) counterparts. LW and (0, 1) also reduced turnover.

Moreover, MinVar portfolios also performed well across all samples, constantly ranking among the top performers. This is crucial from the lenses of estimation risk, as these portfolios are not subject large part of said risk in the form of estimation error in the expected returns (Merton, 1980; Jorion, 1985). MinVar portfolios with (0, 1) constraints and LW extensions also performed well, and often above their (-1, 1) and sample covariance counterparts, attesting to the ability of these methods to reduce estimation risk.

The results of the present study are similar to Bessler et al. (2017), in which the authors find that MV portfolios do outperform 1/N but could not find any statistical significance. It also backs up DeMiguel (2009a), where MVs also did not outperform the benchmark. It disagrees with Korby and Ostdiek (2012), where MV only underperforms after transaction costs. In fact, the last author criticized DeMiguel because they did not impose any constraints and ran portfolios that had expected returns of over 100%. In the present study, even with constraints and lower return expectations, DeMiguel’s results appear robust. Hence, there is no evidence to accept hypothesis one, as it is not possible to reject the null hypothesis of equal performance between the benchmark and the optimization strategies.

The results in the Regional Sample point towards an interesting conclusion. In a high estimation risk environment, where it is harder to capture the benefits of diversification, high correlations and allowing shorts may help with risk reduction. That is because hedging market risk becomes less subject to estimation risk. Take the example of a situation where an investor wants exposure to an asset that is highly correlated with the market, but not the market risk itself. If there is a high degree of uncertainty in the estimated correlations, there is more risk that the investor will choose a hedge that is not actually correlated with the market when the average pair-wise correlation across the sample is low (given no other information is available).

Unlike in Bessler (2017), the Black-Litterman models does not outperform the benchmark consistently and with significance across all samples but did in a few cases. A key difference between the BLs employed here and Bessler’s may be the Ω used in this study (assets’ sample variance) and theirs

(MSE of the sample mean returns). In further disagreement, BLs have relatively concentrated weights and less diversification (for the Regional Sample).

Further in agreement with DeMiguel et al. (2009a), MVs have higher turnover than the 1/N and extreme allocations that shifted quickly. This gives further support to the author's claims that 60- and 120-month estimation windows are not enough. This is also supported by the lack of estimation window robustness. Thus, the findings are in support of the second hypothesis but are not statistically tested.

A trade-off between common risk factors and estimation risk is found. Portfolios with higher exposure to estimation risk, such as (-1, 1) and no LW covariance matrix extensions, have lower R-squareds. This means that less of their variation could be explained by those models, perhaps pointing towards a key factor, estimation risk, not being present.

Finally, the third hypothesis cannot be accepted for an equity model (Fama-French 5), as the null hypothesis of zero alpha could be rejected. In other settings, significance is found but mainly due to the lack of equity factors.

In line with the theory documenting the benefits of the addition of low correlation assets (Levy and Sarnat, 1970; Schweizer, 2008), the International Sample has the best performance of optimization strategies, with a lot of significance. This points towards a situation where sample considerations are important when choosing which model to apply. For instance, MinVar (-1, 1) may perform better in high correlation samples (as seen in the Regional Sample) and Max Sharpe in highly diversified ones. The performance in this sample also may be related to its exposure to forex fluctuations, consistent with Ackerman et al. (2017).

The main limitation of the present study is its ignorance of expected returns estimation error, which is relevant and was not remedied. That is intentional in the sense that the focus is on the ceteris paribus performance benefits of the LW covariance matrices. Another improvement could be the extension of estimation windows, which are not enough to provide systematic performance benefits (DeMiguel, 2009a). Finally, the portfolios in this study are rebalanced monthly, which generates more turnover. Other, longer, rebalancing periods could remedy the situation. Moreover, transaction costs can be accounted for.

For further research, many questions are left pending. Firstly, the relationship between Sharpe ratio performance and the error in the covariance matrix estimator, both of which are time-varying and related to the market. Secondly, modeling estimation risk into a risk factor. This would require some type of "optimal" MV estimation model that can be longed, financed by a short on the sample means MV, and could perhaps explain the trade-off mentioned above.

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APPENDIX A - Tables

Table 2 – Descriptive Statistics: Regional Sample, 2004-2024.

Asset	Mean	Std. Deviation	Excess Kurtosis	Skewness
World Equity Index	0.84%	4.53%	2.21	-0.75
Developed Markets Index	0.84%	4.47%	1.94	-0.71
Emerging Markets Index	0.92%	5.92%	2.48	-0.54
North America Index	0.92%	4.42%	1.71	-0.63
Latin America Index	1.12%	7.68%	2.53	-0.62
Developed Europe	0.76%	5.29%	1.58	-0.60
Emerging Europe	0.72%	8.83%	5.03	-1.12
Developed EMEA	0.76%	5.28%	1.61	-0.53
Emerging EMEA	0.77%	6.65%	2.73	-0.65
APAC Developed	0.72%	4.47%	1.33	-0.51
APAC Developing	0.98%	5.91%	1.64	-0.41
Asia Developed	0.65%	4.28%	1.06	-0.29
Asia Emerging	0.98%	5.91%	1.64	-0.41
Total	0.84%	5.66%		

Notes: these indices are made by and sourced from Bloomberg. They contain equity in large, medium and small companies from each relevant region. Returns were classified as the monthly change in the total returns of each index. Mean and Std. Dev. correspond to the average monthly return and standard deviation of monthly returns over the entire sample period of 2004-2024. Some of the assets in this sample overlap, meaning that the stocks of some companies may be included in more than one index.

Table 3 – Descriptive Statistics: International Sample, 2005-2024

Asset	**Source**	**Mean**	**Std. Deviation**	**Excess Kurtosis**	**Skewness**
Commodity Index	Bloomberg	0.06%	4.72%	1.83	-0.60
U.S. REITs	FTSE NAREIT	0.72%	6.38%	5.48	-0.61
Private Equity Buyout Index	Thomson Reuters	1.22%	6.56%	1.36	-0.27
US Long Corporate Bond Index	Bloomberg	0.41%	3.28%	2.52	-0.01
US Long Treasury	Bloomberg	0.31%	3.56%	0.66	0.40
S&P500	Bloomberg	0.90%	4.41%	1.13	-0.59
STOXX Europe 600	Bloomberg	0.64%	5.41%	1.28	-0.47
Developed Europe REIT	FTSE EPRA NAREIT	0.40%	6.61%	2.10	-0.55
10-Year German Government Bond	UBS	0.24%	2.94%	2.57	-0.13
Eurozone Investment Grade Corporate Bonds	Standard and Poor’s	0.19%	2.96%	1.48	-0.23
Emerging Market Equities*	Bloomberg	0.73%	6.04%	2.46	-0.49
Emerging Market Government Bonds	FTSE	0.45%	2.67%	7.99	-1.60
Developed	Bloomberg	0.59%	4.97%	1.87	-0.61

Equity Markets (Ex-U.S.) *					
Hedge Fund Index	Credit Suisse	0.43%	1.56%	4.78	-1.36
Emerging Markets REIT	Morningstar	0.47%	7.43%	1.72	-0.37
Total		0.52%	4.63%		

Notes: *these indices contain equity in large, medium, and small companies from each relevant region. These indexes are all total return (or equivalent) and gross. Mean and Std. Dev. correspond to the average monthly return and standard deviation of monthly returns over the entire sample period of 2005-2024. Data was sourced from Bloomberg.

Table 4 – Monthly Excess Returns for portfolio strategies: Industry Sample, 60-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.77%	4.94%	-0.52	2.83
Mean-Variance				
Max Sharpe	0.84%	5.47%	0.00	2.54
Max Sharpe (-1, 1)	0.75%	11.27%	-0.12	1.59
MinVariance	0.62%	3.64%	-0.41	2.97
MinVariance (-1, 1)	0.12%	6.50%	-0.15	1.77
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.84%	5.45%	0.00	2.69
Constant Variance (-1, 1)	1.11%	13.06%	-0.43	2.48
Single Factor	0.83%	5.48%	0.01	2.44
Single Factor (-1, 1)	1.40%	12.46%	-0.39	2.80
Constant Correlation	0.93%	5.72%	0.01	2.44
Constant Correlation (-1, 1)	1.42%	14.62%	-0.46	3.92
MinVariance				
Constant Variance	0.63%	3.67%	-0.43	3.20
Constant Variance (-1, 1)	0.48%	3.65%	-0.30	3.01
Single Factor	0.65%	3.59%	-0.39	2.82
Single Factor (-1, 1)	0.45%	3.60%	-0.26	4.32
Constant Correlation	0.65%	3.63%	-0.36	2.47
Constant Correlation (-1, 1)	0.50%	3.64%	-0.28	2.47
Black-Litterman				
1/N prior	0.71%	4.87%	-0.17	2.51
MinVar prior	0.60%	3.72%	-0.36	1.35
MinVar constant variance prior	0.57%	3.71%	-0.42	1.66

Notes: This table contains the Sharpe Ratio, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 597 monthly observations, from 1974 to 2024. A 60-month rolling window is used to estimate the inputs used in the

optimization of the portfolios. Data and assets from the Industry Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 5 – Monthly Excess Returns for portfolio strategies: Country Sample, 60-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.72%	4.82%	0.08	0.72
Mean-Variance				
Max Sharpe	0.76%	4.53%	-0.47	3.08
Max Sharpe (-1, 1)	1.07%	9.52%	-0.59	0.81
MinVariance	0.48%	3.76%	-0.60	1.38
MinVariance (-1, 1)	1.21%	5.05%	-0.39	0.33
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.74%	4.49%	-0.46	3.24
Constant Variance (-1, 1)	0.59%	10.59%	-0.81	2.32
Single Factor	0.75%	4.55%	-0.40	2.98
Single Factor (-1, 1)	1.16%	11.10%	-0.51	1.87
Constant Correlation	0.70%	4.64%	-0.34	2.34
Constant Correlation (-1, 1)	1.13%	13.69%	-0.03	1.85
MinVariance				
Constant Variance	0.52%	3.72%	-0.62	1.62
Constant Variance (-1, 1)	0.74%	3.23%	-0.58	0.92
Single Factor	0.48%	3.75%	-0.65	1.46
Single Factor (-1, 1)	0.72%	3.04%	-0.53	0.43
Constant Correlation	0.52%	3.75%	-0.50	0.69
Constant Correlation (-1, 1)	0.57%	3.44%	-0.55	0.57
Black-Litterman				
1/N prior	0.82%	4.61%	0.08	2.75
MinVar prior	0.40%	4.04%	-0.54	0.40
MinVar constant variance prior	0.47%	4.00%	-0.47	0.72

Notes: This table contains the Sharpe Ratio, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 180 monthly observations, from 2009 to 2024. A 60-month rolling window is used to estimate the inputs used in the

optimization of the portfolios. Data and assets from the Country Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 6 – Monthly Excess Returns for portfolio strategies: Regional Sample, 60-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.39%	5.38%	-0.55	2.55
Mean-Variance	0.73%	6.05%	-0.74	4.66
Max Sharpe	0.67%	7.92%	-0.22	1.65
Max Sharpe (-1, 1)	0.45%	4.38%	-0.68	1.80
MinVariance	0.53%	4.25%	-0.69	1.61
MinVariance (-1, 1)	0.73%	6.05%	-0.74	4.66
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.70%	6.04%	-0.74	4.72
Constant Variance (-1, 1)	0.58%	8.29%	-0.06	2.00
Single Factor	0.73%	6.05%	-0.74	4.66
Single Factor (-1, 1)	0.70%	8.30%	-0.04	1.82
Constant Correlation	0.73%	6.05%	-0.74	4.66
Constant Correlation (-1, 1)	0.75%	9.40%	-0.24	2.11
MinVariance				
Constant Variance	0.47%	4.40%	-0.69	1.93
Constant Variance (-1, 1)	0.58%	4.25%	-0.83	1.44
Single Factor	0.45%	4.39%	-0.68	1.80
Single Factor (-1, 1)	0.60%	4.33%	-0.67	1.49
Constant Correlation	0.43%	4.50%	-0.69	1.84
Constant Correlation (-1, 1)	0.62%	4.64%	-0.60	1.12
Black-Litterman				
1/N prior	0.59%	5.45%	-0.42	3.46
MinVar prior	0.45%	4.54%	-0.72	1.64
MinVar constant variance prior	0.44%	4.51%	-0.74	1.74

Notes: This table contains the Sharpe Ratio, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 191 monthly observations, from 2008 to 2024. A 60-month rolling window is used to estimate the inputs used in the

optimization of the portfolios. Data and assets from the Regional Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 7 – Monthly Excess Returns for portfolio strategies: International Sample, 60-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.43%	3.32%	-0.47	1.70
Mean-Variance	0.52%	2.34%	-0.77	1.79
Max Sharpe	0.48%	1.95%	-0.45	1.96
Max Sharpe (-1, 1)	0.26%	1.24%	-0.70	2.40
MinVariance	0.24%	1.00%	-1.42	8.73
MinVariance (-1, 1)	0.52%	2.34%	-0.77	1.79
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.61%	2.52%	-0.67	0.98
Constant Variance (-1, 1)	0.67%	3.34%	-0.19	2.53
Single Factor	0.51%	2.36%	-0.79	1.61
Single Factor (-1, 1)	0.58%	2.40%	-0.22	2.39
Constant Correlation	0.49%	2.41%	-0.85	2.02
Constant Correlation (-1, 1)	0.59%	2.89%	0.34	6.68
MinVariance				
Constant Variance	0.24%	1.47%	-0.83	2.32
Constant Variance (-1, 1)	0.34%	1.18%	-0.50	2.12
Single Factor	0.25%	1.24%	-0.72	2.45
Single Factor (-1, 1)	0.27%	0.94%	-1.63	10.42
Constant Correlation	0.25%	1.24%	-0.80	3.09
Constant Correlation (-1, 1)	0.22%	0.94%	-1.53	10.59
Black-Litterman				
1/N prior	0.29%	1.19%	-0.61	2.17
MinVar prior	0.29%	1.22%	-1.13	5.55
MinVar constant variance prior	0.28%	1.22%	-1.09	5.12

Notes: This table contains the Sharpe Ratio, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 166 monthly observations, from 2010 to 2024. A 60-month rolling window is used to estimate the inputs used in the

optimization of the portfolios. Data and assets from the Regional Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 8 – Monthly Excess Returns for portfolio strategies: Industry Sample, 120-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.77%	4.81%	-0.68	3.11
Mean-Variance				
Max Sharpe	0.71%	5.14%	-0.38	2.51
Max Sharpe (-1, 1)	1.08%	12.38%	-0.20	3.63
MinVariance	0.63%	3.58%	-0.49	1.62
MinVariance (-1, 1)	0.54%	4.11%	-0.21	0.95
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.71%	5.13%	-0.39	2.49
Constant Variance (-1, 1)	1.04%	13.00%	-1.00	10.44
Single Factor	0.70%	5.16%	-0.36	2.36
Single Factor (-1, 1)	1.07%	12.60%	-1.01	10.75
Constant Correlation	0.71%	5.33%	-0.38	2.28
Constant Correlation (-1, 1)	1.14%	13.91%	-1.34	14.78
MinVariance				
Constant Variance	0.64%	3.54%	-0.67	2.67
Constant Variance (-1, 1)	0.54%	3.49%	-0.59	2.26
Single Factor	0.66%	3.47%	-0.62	2.23
Single Factor (-1, 1)	0.50%	3.42%	-0.66	2.94
Constant Correlation	0.67%	3.50%	-0.55	1.79
Constant Correlation (-1, 1)	0.56%	3.51%	-0.52	1.73
Black-Litterman				
1/N prior	0.68%	4.44%	-0.53	1.93
MinVar prior	0.57%	3.78%	-0.49	1.63
MinVar constant variance prior	0.57%	3.78%	-0.49	1.63

Notes: This table contains the Mean returns, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 537 monthly observations, from 1979 to 2024. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the Industry Sample are used.

Table 9 – Monthly Excess Returns for portfolio strategies: Country Sample, 120-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.41%	4.51%	-0.33	2.87
Mean-Variance				
Max Sharpe	0.54%	4.04%	-0.35	1.27
Max Sharpe (-1, 1)	1.05%	6.97%	-0.17	1.03
MinVariance	0.34%	3.41%	-0.70	1.35
MinVariance (-1, 1)	0.36%	3.38%	-0.77	0.19
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.53%	4.00%	-0.39	1.44
Constant Variance (-1, 1)	0.67%	6.88%	0.02	0.24
Single Factor	0.53%	4.05%	-0.34	1.29
Single Factor (-1, 1)	0.81%	6.49%	-0.17	0.30
Constant Correlation	0.56%	4.13%	-0.30	1.32
Constant Correlation (-1, 1)	0.64%	7.17%	-0.05	0.52
MinVariance				
Constant Variance	0.23%	3.73%	-0.68	1.86
Constant Variance (-1, 1)	0.37%	3.21%	-0.82	1.36
Single Factor	0.23%	3.77%	-0.70	1.71
Single Factor (-1, 1)	0.34%	3.00%	-0.71	0.70
Constant Correlation	0.25%	3.75%	-0.45	0.76
Constant Correlation (-1, 1)	0.22%	3.49%	-0.59	0.81
Black-Litterman				
1/N prior	0.37%	4.00%	-0.51	1.66
MinVar prior	0.43%	3.59%	-0.40	0.73
MinVar constant variance prior	0.43%	3.59%	-0.40	0.73

Notes: This table contains the Mean returns, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 120 monthly observations, from 2014 to 2024. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the Country Sample are used.

Table 10 – Monthly Excess Returns for portfolio strategies: Regional Sample, 120-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.46%	4.33%	-0.27	1.49
Mean-Variance				
Max Sharpe	0.66%	4.82%	-0.38	0.54
Max Sharpe (-1, 1)	0.73%	6.02%	0.14	0.19
MinVariance	0.61%	3.82%	-0.35	1.35
MinVariance (-1, 1)	0.81%	3.85%	-0.55	1.03
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.64%	4.80%	-0.37	0.53
Constant Variance (-1, 1)	0.69%	6.16%	0.13	0.36
Single Factor	0.66%	4.82%	-0.38	0.54
Single Factor (-1, 1)	0.78%	6.23%	0.09	0.29
Constant Correlation	0.64%	4.85%	-0.40	0.62
Constant Correlation (-1, 1)	0.96%	6.99%	0.19	0.21
MinVariance				
Constant Variance	0.55%	3.87%	-0.35	1.88
Constant Variance (-1, 1)	0.72%	3.92%	-0.54	1.42
Single Factor	0.54%	3.85%	-0.34	1.68
Single Factor (-1, 1)	0.76%	4.08%	-0.33	1.57
Constant Correlation	0.55%	3.88%	-0.33	1.72
Constant Correlation (-1, 1)	0.80%	4.30%	-0.34	0.79
Black-Litterman				
1/N prior	0.74%	4.05%	-0.45	1.17
MinVar prior	0.54%	3.82%	-0.34	1.13
MinVar constant variance prior	0.54%	3.82%	-0.34	1.13

Notes: This table contains the Mean return, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 131 monthly observations, from 2013 to 2024. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the Regional Sample are used.

Table 11 – Monthly Excess Returns for portfolio strategies: International Sample, 120-month rolling window.

Portfolio Strategy	Mean Return	Standard Deviation	Skewness	Kurtosis
1/N	0.25%	3.57%	-0.52	1.76
Mean-Variance				
Max Sharpe	0.42%	2.53%	-0.37	0.64
Max Sharpe (-1, 1)	0.38%	1.95%	0.01	1.25
MinVariance	0.14%	1.29%	-0.61	1.70
MinVariance (-1, 1)	0.13%	1.04%	-1.38	7.30
Ledoit-Wolf Shrinkage				
Max Sharpe				
Constant Variance	0.47%	2.83%	-0.41	0.32
Constant Variance (-1, 1)	0.48%	2.83%	-0.15	1.06
Single Factor	0.41%	2.53%	-0.37	0.63
Single Factor (-1, 1)	0.38%	2.16%	0.01	1.71
Constant Correlation	0.44%	2.52%	-0.41	0.83
Constant Correlation (-1, 1)	0.37%	2.31%	-0.04	1.61
MinVariance				
Constant Variance	0.06%	1.61%	-0.75	2.07
Constant Variance (-1, 1)	0.18%	1.29%	-0.35	2.12
Single Factor	0.09%	1.33%	-0.67	2.52
Single Factor (-1, 1)	0.12%	1.01%	-1.83	11.94
Constant Correlation	0.09%	1.35%	-0.73	3.03
Constant Correlation (-1, 1)	0.08%	1.00%	-1.74	12.46
Black-Litterman				
1/N prior	0,18%	1,25%	-0,68	1,79
MinVar prior	0,20%	1,25%	-0,92	3,90
MinVar constant variance prior	0,20%	1,25%	-0,93	4,05

Notes: This table contains the Mean return, Standard Deviation, Skewness, and Kurtosis on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. The descriptive statistics are calculated based on 106 monthly observations, from 2015 to 2024. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the International Sample are used.

Table 12 – Percentage of Negative Returns: 60-months rolling window, all samples and strategies.

Strategies	Industry Sample	Country Sample	Regional Sample	International Sample	Average
Max Sharpe	41,4%	37,8%	37,2%	34,3%	37,7%
Max Sharpe (-1, 1)	46,1%	42,2%	43,5%	37,3%	42,3%
MinVar	38,7%	41,7%	37,2%	38,6%	39,0%
MinVar (-1, 1)	49,9%	35,6%	36,1%	37,3%	39,7%
MinVar, constant variance	37,7%	43,9%	37,2%	41,0%	39,9%
MinVar, single factor	37,9%	41,1%	37,2%	39,2%	38,8%
MinVar, constant correlation	39,7%	38,3%	37,7%	38,0%	38,4%
MinVar, constant variance (-1, 1)	41,2%	38,3%	36,1%	32,5%	37,0%
MinVar, single factor (-1, 1)	41,9%	40,6%	37,7%	39,2%	39,8%
MinVar, constant correlation (-1, 1)	42,2%	40,0%	39,3%	41,0%	40,6%
Max Sharpe, constant variance (-1, 1)	43,9%	43,9%	43,5%	37,3%	42,1%
Max Sharpe, single factor (-1, 1)	41,5%	42,2%	44,0%	35,5%	40,8%
Max Sharpe, constant correlation (-1, 1)	42,4%	44,4%	41,9%	33,1%	40,5%
Max Sharpe, constant variance	41,0%	38,9%	37,2%	35,5%	38,2%
Max Sharpe, single factor	41,2%	38,9%	37,2%	34,3%	37,9%
Max Sharpe, constant correlation	42,5%	39,4%	36,6%	37,3%	39,0%
1/N	40,4%	45,0%	44,0%	38,6%	42,0%
BL 1/N	41,5%	39,4%	39,3%	38,0%	39,6%
BL MinVar, constant variance	41,4%	41,1%	35,6%	38,0%	39,0%
BL MinVar	41,9%	42,8%	36,1%	38,0%	39,7%
Total	41,7%	40,8%	38,7%	37,2%	37,7%

Notes: This table contains the percentage of all returns that were negative on 20 portfolio strategies for each sample (Industry, Country, Regional, and International). The average is an equal-weighted average of all samples. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the International Sample are used. The samples, from left to right have 597, 180, 191 and 166 observations respectively.

Table 13 – Industry Sample (60 and 120 months): Annual Sharpe Ratios.

Strategies	Sharpe Ratio (60 months)	Sharpe Ratio (120 months)
Max Sharpe	0.53	0.48
Max Sharpe (-1, 1)	0.23	0.30
MinVar	0.59	0.61
MinVar (-1, 1)	0.06	0.46
MinVar, constant variance	0.59	0.63
MinVar, single factor	0.62	0.66
MinVar, constant correlation	0.62	0.66
MinVar, constant variance (-1, 1)	0.46	0.54
MinVar, single factor (-1, 1)	0.44	0.50
MinVar, constant correlation (-1, 1)	0.47	0.55
Max Sharpe, constant variance (-1, 1)	0.29	0.28
Max Sharpe, single factor (-1, 1)	0.39	0.30
Max Sharpe, constant correlation (-1, 1)	0.34	0.28
Max Sharpe, constant variance	0.53	0.48
Max Sharpe, single factor	0.53	0.47
Max Sharpe, constant correlation	0.56	0.46
1/N	0.54	0.56
BL - 1/N	0.51	0.53
BL MinVar, constant variance	0.54	0.52
BL - MinVar	0.56	0.51

Notes: This table contains the Annual Sharpe Ratio on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (597 and 537 observations respectively). Data and assets from the Industry Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 14 – Country Sample (60 and 120 months): Annual Sharpe Ratios.

Strategies	Sharpe Ratio (60 months)	Sharpe Ratio (120 months)
Max Sharpe	0.58	0.46
Max Sharpe (-1, 1)	0.39	0.52
MinVar	0.44	0.35
MinVar (-1, 1)	0.83	0.37
MinVar, constant variance	0.48	0.21
MinVar, single factor	0.44	0.21
MinVar, constant correlation	0.48	0.23
MinVar, constant variance (-1, 1)	0.80	0.39
MinVar, single factor (-1, 1)	0.82	0.39
MinVar, constant correlation (-1, 1)	0.57	0.22
Max Sharpe, constant variance (-1, 1)	0.19	0.34
Max Sharpe, single factor (-1, 1)	0.36	0.43
Max Sharpe, constant correlation (-1, 1)	0.28	0.31
Max Sharpe, constant variance	0.57	0.46
Max Sharpe, single factor	0.57	0.45
Max Sharpe, constant correlation	0.52	0.47
1/N	0.52	0.31
BL - 1/N	0.61	0.32
BL MinVar, constant variance	0.41	0.42
BL - MinVar	0.34	0.44

Notes: This table contains the Sharpe Ratio on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (180 and 120 observations respectively). Data and assets from the Country Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 15 – Regional Sample (60 and 120 months): Annual Sharpe Ratios.

Strategies	**Sharpe Ratio (60 months)**	**Sharpe Ratio (120 months)**
Max Sharpe	0.42	0.48
Max Sharpe (-1, 1)	0.29	0.42
MinVar	0.36	0.56
MinVar (-1, 1)	0.43	0.73
MinVar, constant variance	0.37	0.49
MinVar, single factor	0.36	0.48
MinVar, constant correlation	0.33	0.49
MinVar, constant variance (-1, 1)	0.47	0.64
MinVar, single factor (-1, 1)	0.48	0.65
MinVar, constant correlation (-1, 1)	0.46	0.65
Max Sharpe, constant variance (-1, 1)	0.24	0.39
Max Sharpe, single factor (-1, 1)	0.29	0.43
Max Sharpe, constant correlation (-1, 1)	0.27	0.47
Max Sharpe, constant variance	0.40	0.46
Max Sharpe, single factor	0.42	0.48
Max Sharpe, constant correlation	0.42	0.46
1/N	0.25	0.37
BL - 1/N	0.38	0.63
BL MinVar, constant variance	0.34	0.49
BL - MinVar	0.34	0.48

Notes: This table contains the Sharpe Ratioon 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (191 and 131 observations respectively). Data and assets from the Regional Sample are used. The Sharpe Ratio corresponds to the annualized monthly SP over the whole sample.

Table 16 – International Sample (60 and 120 months): Annual Sharpe Ratios.

Strategies	Sharpe Ratio (60 months)	Sharpe Ratio (120 months)
Max Sharpe	0.77	0,58
Max Sharpe (-1, 1)	0.85	0,68
MinVar	0.72	0,37
MinVar (-1, 1)	0.84	0,45
MinVar, constant variance	0.57	0,13
MinVar, single factor	0.69	0,23
MinVar, constant correlation	0.69	0,23
MinVar, constant variance (-1, 1)	0.99	0,48
MinVar, single factor (-1, 1)	0.98	0,43
MinVar, constant correlation (-1, 1)	0.83	0,28
Max Sharpe, constant variance (-1, 1)	0.69	0,59
Max Sharpe, single factor (-1, 1)	0.84	0,61
Max Sharpe, constant correlation (-1, 1)	0.71	0,55
Max Sharpe, constant variance	0.83	0,58
Max Sharpe, single factor	0.74	0,57
Max Sharpe, constant correlation	0.70	0,60
1/N	0.45	0,24
BL - 1/N	0.85	0,51
BL MinVar, constant variance	0.79	0,54
BL - MinVar	0.81	0,56

Notes: This table contains the Sharpe Ratio on 20 portfolio strategies. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (166 and 106 observations respectively). Data and assets from the International Sample are used.

Table 17 – Industry Sample (60 and 120 months): Monthly Sharpe Ratio Hypothesis Testing

Strategies	Difference to benchmark (60 months)	Difference to benchmark (120 months)
Max Sharpe	0,00	-0,02
Max Sharpe (-1, 1)	-0,09*	-0,07
MinVar	0,01	0,02
MinVar (-1, 1)	-0,14***	-0,03
MinVar, constant variance	0,01	0,02
MinVar, single factor	0,02	0,03
MinVar, constant correlation	0,02	0,03
MinVar, constant variance (-1, 1)	-0,02	0,00
MinVar, single factor (-1, 1)	-0,03	-0,02
MinVar, constant correlation (-1, 1)	-0,02	0,00
Max Sharpe, constant variance (-1, 1)	-0,07	-0,08
Max Sharpe, single factor (-1, 1)	-0,04	-0,07
Max Sharpe, constant correlation (-1, 1)	-0,06	-0,08
Max Sharpe, constant variance	0,00	-0,02
Max Sharpe, single factor	0,00	-0,03
Max Sharpe, constant correlation	0,01	-0,03
1/N	0,00	0,00
BL - 1/N	-0,01	-0,01
BL MinVar, constant variance	0,00	-0,01
BL - MinVar	0,00	-0,01

Notes: This table contains the average difference in the monthly Sharpe ratio of each portfolio strategy versus the 1/N, and its statistical significance. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (597 and 537 observations respectively). Data and assets from the Industry Sample are used. The statistical significance related to the difference between the strategies' Sharpe ratio and the benchmark, 1/N, using Memmel's (2003) method. ***p<.01, **p<.05, ***p<0.10

Table 19 – Regional Sample (60 and 120 months): Monthly Sharpe Ratio Hypothesis Testing

Strategies	Difference to benchmark (60 months)	Difference to benchmark (120 months)
Max Sharpe	0,05*	0,03
Max Sharpe (-1, 1)	0,01	0,02
MinVar	0,03	0,06*
MinVar (-1, 1)	0,05	0,11
MinVar, constant variance	0,03	0,04
MinVar, single factor	0,03	0,03
MinVar, constant correlation	0,02	0,04
MinVar, constant variance (-1, 1)	0,06	0,08
MinVar, single factor (-1, 1)	0,07	0,08
MinVar, constant correlation (-1, 1)	0,06	0,08
Max Sharpe, constant variance (-1, 1)	0,00	0,01
Max Sharpe, single factor (-1, 1)	0,01	0,02
Max Sharpe, constant correlation (-1, 1)	0,01	0,03
Max Sharpe, constant variance	0,04	0,03
Max Sharpe, single factor	0,05*	0,03
Max Sharpe, constant correlation	0,05*	0,03
1/N	0,00	0,00
BL - 1/N	0,04	0,08**
BL MinVar, constant variance	0,03	0,04
BL - MinVar	0,03	0,03

Notes: This table contains the average difference in the monthly Sharpe ratio of each portfolio strategy versus the 1/N, and its statistical significance. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60 and 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios (191 and 131 observations respectively). Data and assets from the Regional Sample are used. The statistical significance related to the difference between the strategies' Sharpe ratio and the benchmark, 1/N, using Memmel's (2003) method. ***p<.01, **p<.05, ***p<0.10

Table 22 – Turnover figures, 120 months rolling window.

Strategies	Industry Sample	Country Sample	Regional Sample	International Sample	Average
Max Sharpe	3,2%	3,4%	2,0%	2,5%	2,8%
Max Sharpe (-1, 1)	7,7%	7,2%	3,1%	1,6%	4,9%
MinVar	83,3%	60,8%	96,4%	14,2%	63,7%
MinVar (-1, 1)	12,5%	11,6%	5,8%	4,7%	8,7%
MinVar, constant variance	11,6%	12,8%	6,2%	2,6%	8,3%
MinVar, single factor	12,5%	11,1%	8,8%	2,8%	8,8%
MinVar, constant correlation	52,8%	45,4%	16,2%	9,6%	31,0%
MinVar, constant variance (-1, 1)	50,1%	39,5%	31,4%	11,2%	33,1%
MinVar, single factor (-1, 1)	47,4%	30,6%	21,1%	8,1%	26,8%
MinVar, constant correlation (-1, 1)	20,2%	15,5%	2,8%	9,8%	12,1%
Max Sharpe, constant variance (-1, 1)	382,4%	189,2%	122,0%	43,1%	184,2%
Max Sharpe, single factor (-1, 1)	18,9%	14,6%	3,6%	8,5%	11,4%
Max Sharpe, constant correlation (-1, 1)	19,1%	15,4%	2,8%	10,6%	12,0%
Max Sharpe, constant variance	20,4%	15,1%	3,8%	10,8%	12,5%
Max Sharpe, single factor	343,6%	163,3%	73,0%	30,8%	152,7%
Max Sharpe, constant correlation	335,6%	137,2%	98,7%	31,9%	150,8%
1/N	307,6%	133,2%	71,3%	28,1%	135,0%

BL - 1/N	13,8%	17,4%	5,4%	1,9%	9,6%
BL MinVar, constant variance	8,1%	10,2%	7,1%	0,9%	6,6%
BL - MinVar	8,2%	11,1%	8,1%	0,9%	7,1%

Notes: This table contains the average monthly turnover of 20 portfolio strategies over the entirety of each sample (Industry, Country, Regional, and International). The average is an equal-weighted average of all samples. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 120-month rolling window is used to estimate the inputs used in the optimization of the portfolios. Data and assets from the International Sample are used. The samples, from left to right have 537, 120, 131 and 106 observations respectively.

Table 23 – Exposure to Risk Factors: Industry Sample, selected strategies, 60-months.

Factors	Max Sharpe	Max Sharpe (-1, 1)	MinVar	MinVar (-1, 1)	MinVar, constant variance (-1, 1)	Max Sharpe, constant variance (-1, 1)	1/N	BL 1/N
Equity								
Alpha	0.0%	0.5%	0.0%	-0.3%	0.0%	0.9%	-0.02%	0.0%
Mkt Excess	1.00***	0.58***	0.72***	0.44***	0.61***	0.75***	1.05***	0.94***
SMB	-0.07	-0.50***	-0.10***	-0.22**	-0.22***	-0.81***	0.26***	-0.04
HML	-0.30***	-0.90***	0.08**	-0.06	-0.05	-1.26***	0.09***	-0.17***
RMW	0.40***	0.06	0.15***	0.17	0.13**	0.38*	0.27***	0.21***
CMA	0.28***	0.44	0.28***	0.44**	0.40***	0.03	0.11***	0.18***
R-squared	0.70	0.11	0.70	0.08	0.48	0.21	0.96	0.76
Bond								
Alpha	0.4%***	0.4%	0.3%***	0.00%	0.2%	0.6%	0.3%***	0.3%***
10 year	-0.20	-0.22	0.17	-0.43	0.07	-0.50	-0.02	-0,18
S&P500	0.97***	0.53***	0.65***	0.34***	0.52***	0.74***	1.05***	0.92***
10y – 2y	0.62**	1.02	0.02	0.78	0.16	1.84*	-0.07	0.54***
R-squared	0.67	0.06	0.64	0.07	0.45	0.10	0.90	0.75
Volatility								
Constant	0.01***	-0.01	0.01***	0.00	0.01***	0.02***	0.01***	0.01***
VIX	-0.12***	-0.11***	-0.09***	-0.04*	-0.07***	-0.11***	-0.14***	-0.12***

R-squared	0.32	0.04	0.34	0.02	0.19	0.04	0.42	0.35

Notes: these regressions do not all share the same period, as risk factor data comes from different sources. The equity model is the Fama-French 5 model (1974-2024), and its data is from the Kenneth French Library. The bond model (1974-2023) is based on Daniel et al. (2014) with data from CRSP. VIX (1990-2024) data is from CBOE. MinVar refers to the minimum variance portfolio, while Max Sharpe refers to the mean-variance optimization that attempts to maximize the Sharpe Ratio. In the Black-Litterman part, the three portfolios mentioned (1/N, MinVar, and MinVar constant variance) indicate the weights used as the reference prior, which in the original Black and Litterman (1992) corresponded to the market weights. (-1, 1) and (0, 1) correspond to the weight constraints for each asset, with the first number representing the minimum and the second, the maximum. Constant Variance, Constant Correlation, and Single Factor refer to the different methods of Ledoit and Wolf's (2003a) shrinkage of the covariance matrix. A 60-rolling window is used to estimate the inputs used in the optimization of the portfolios (597 observations, from 1974-2024). Data and assets from the Industry Sample are used. ***p<.01, **p<.05, ***p<0.10

APPENDIX B – Figures

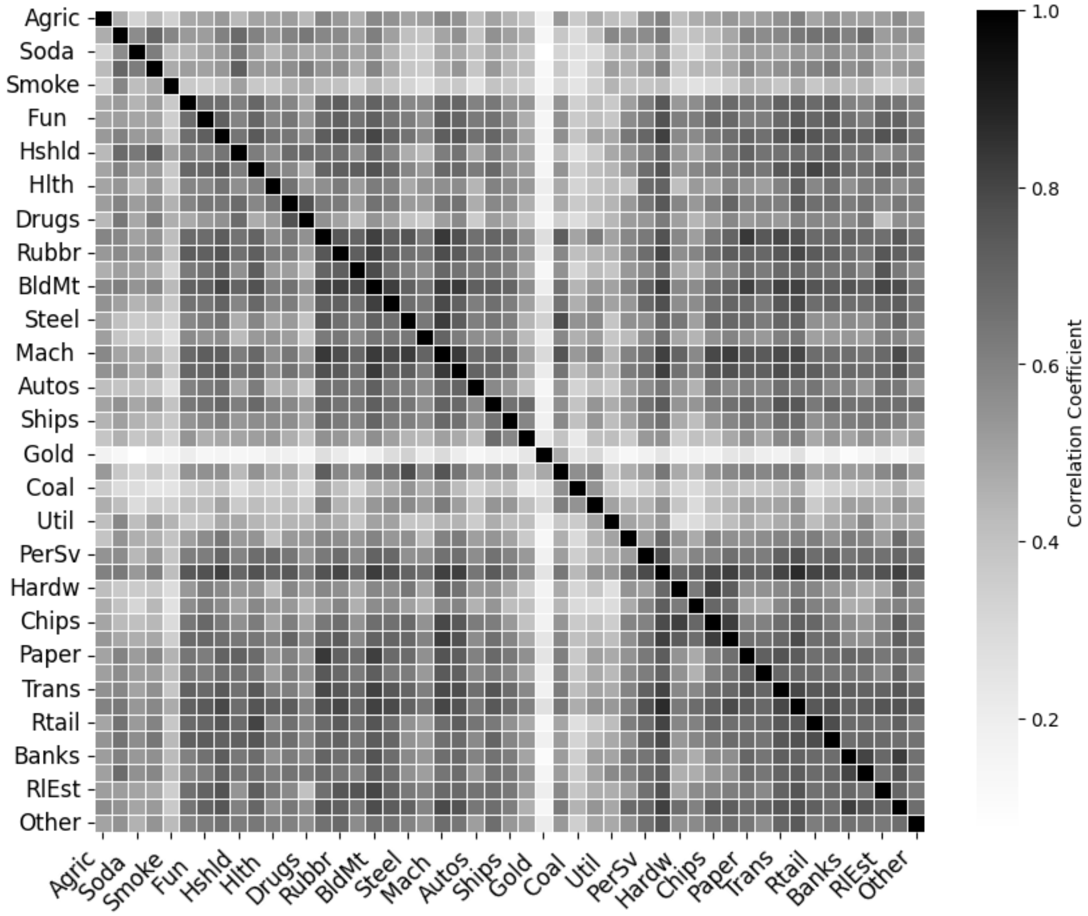

Figure 1 – Return Correlations: Industry Sample, 1969-2024

Notes: Monthly return correlations for the assets in the period of July 1969 to March 2024 (657 months). The assets represent the 49 Industry portfolios of the Kenneth French Data Library.

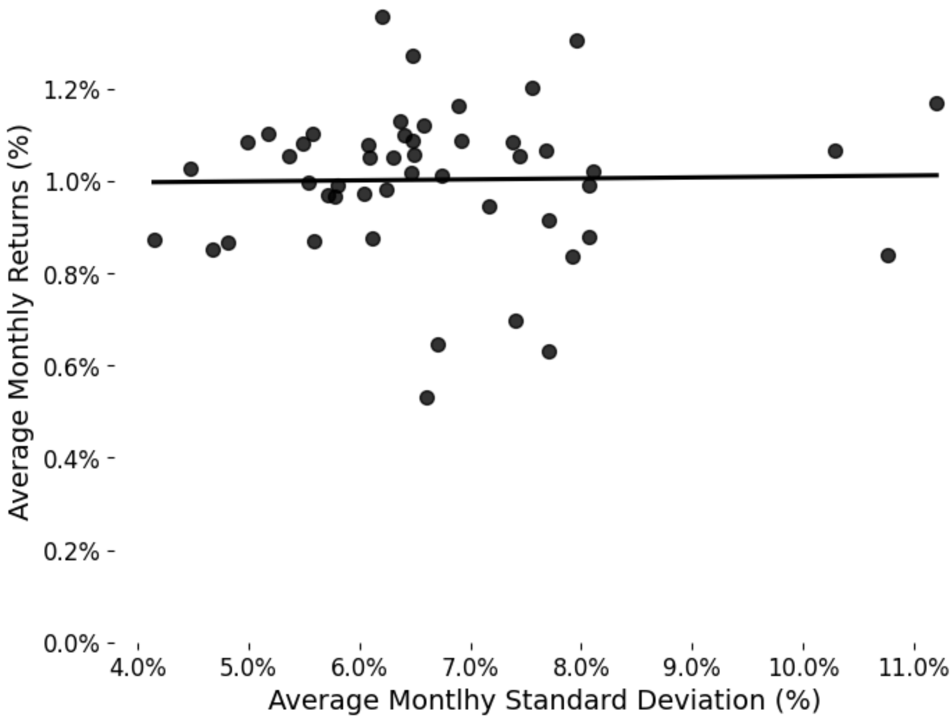

Figure 2 – Average Return vs. Std. Deviation: Industry Sample, 1969-2024

Notes: average and standard deviations of monthly returns over the whole sample period of July 1969 to March 2024 (657 months). The plots represent the 49 Industry portfolios of the Kenneth French Data Library. The black line represents the linear trend.

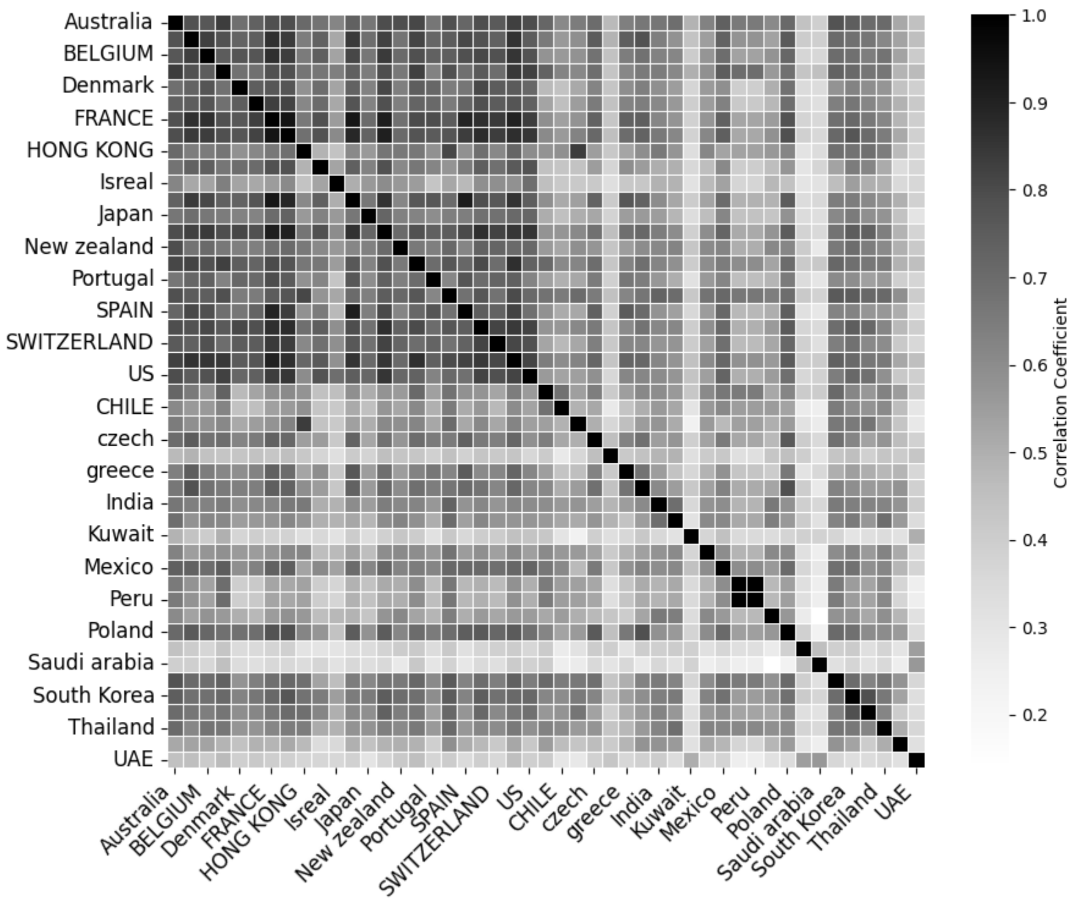

Figure 3 – Return Correlations: Country Sample, 2004-2024

Notes: Monthly return correlations for the assets in the period of May 2004 to April 2024 (240 months). The assets represent 47 national indexes (24 emerging and 23 developed) from Bloomberg.

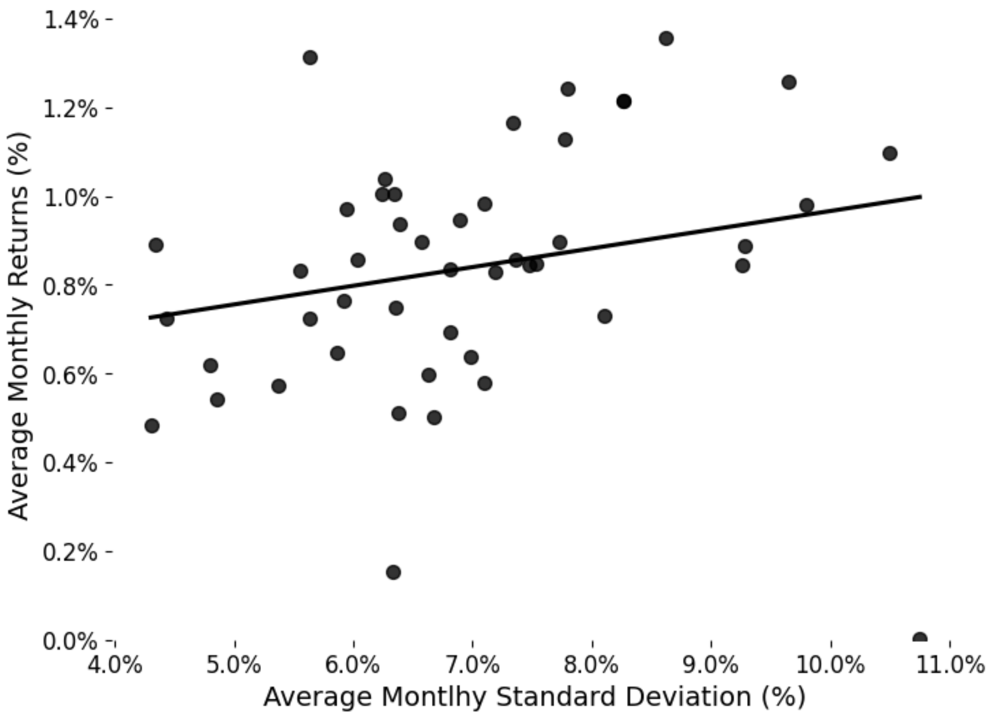

Figure 4 – Average Return vs. Std. Deviation: Country Sample, 2004-2024

Notes: average and standard deviations of monthly returns over the whole sample period of May 2004 to April 2024 (240 months). The points represent 47 national indexes (24 emerging and 23 developed) from Bloomberg. The black line represents the linear trend.

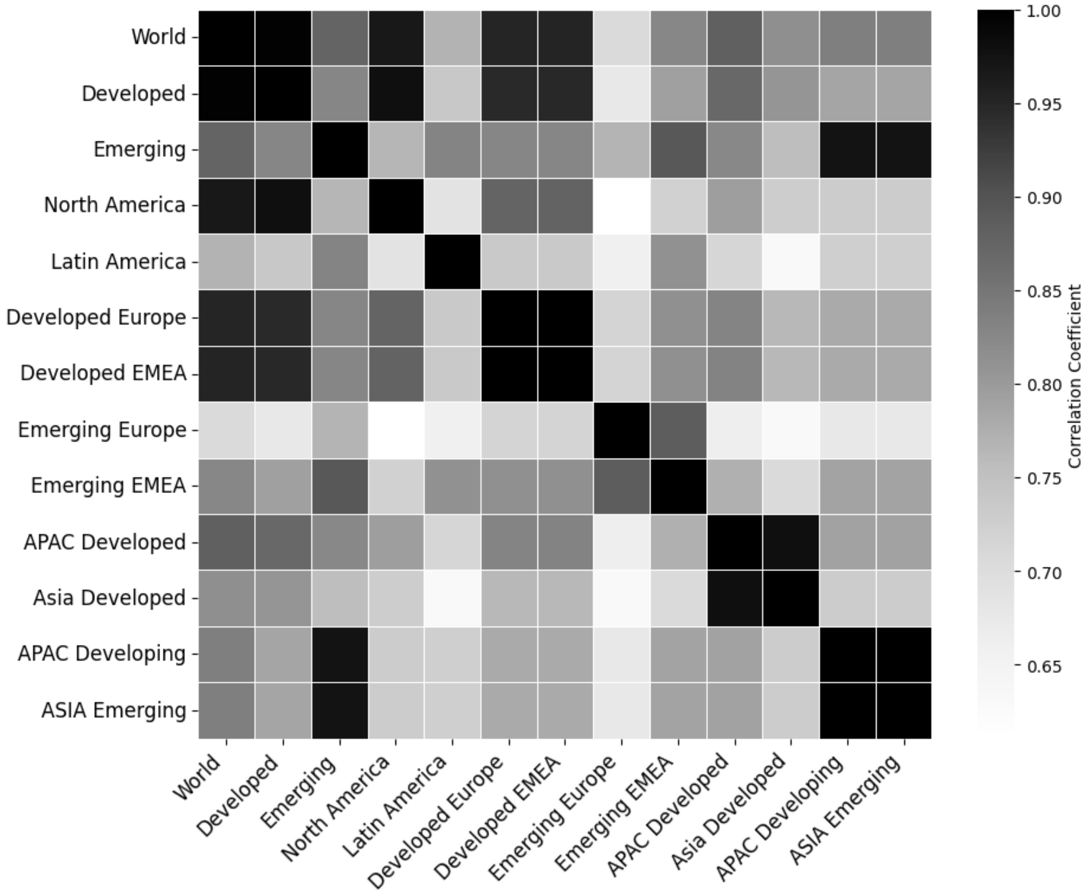

Figure 5 – Return Correlations: Regional Sample, 2006-2024

Notes: Monthly return correlations for the assets in the period June 2003 to April 2024 (251 months). The assets represent 13 indexes from Bloomberg mentioned in Table 2 from Bloomberg.

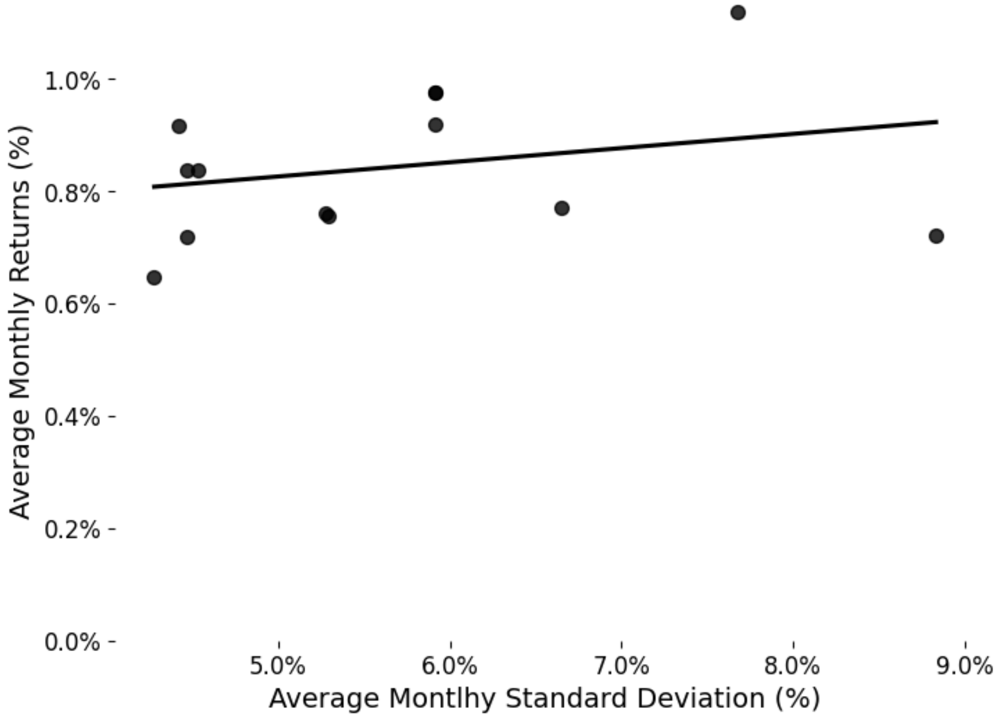

Figure 6 – Average Return vs Std. Deviation: Regional Sample, 2003-2024

Notes: average and standard deviations of monthly returns over the whole sample period of June 2003 to April 2024 (251 months). The points represent 13 indexes from Bloomberg mentioned in Table 2. The black line represents the linear trend.

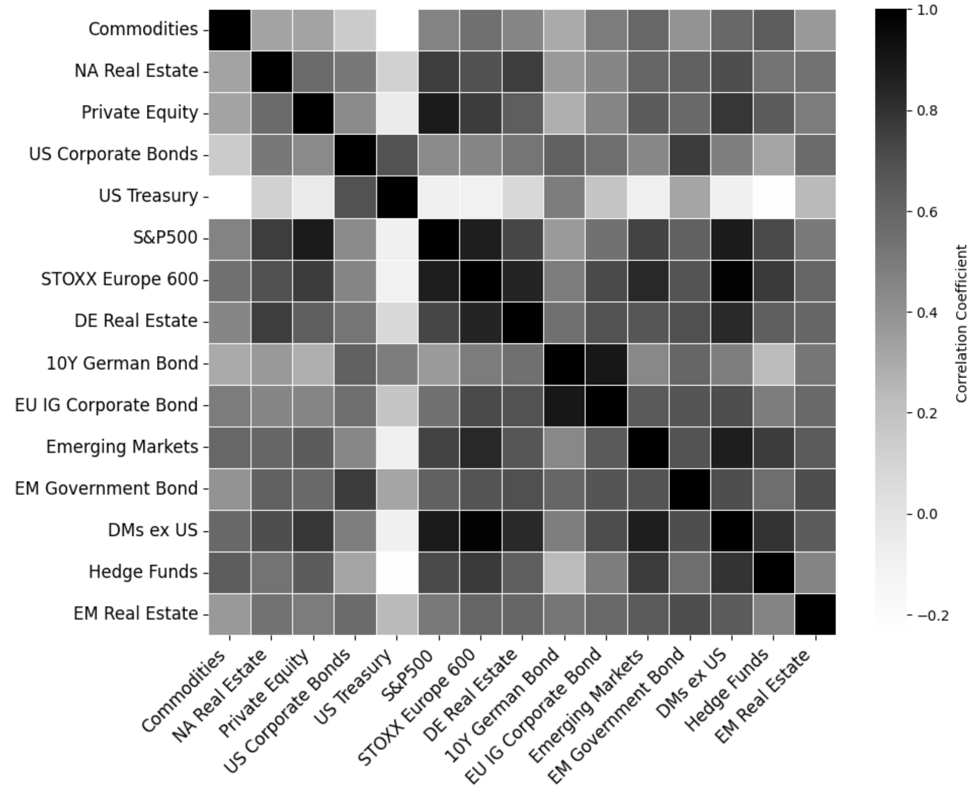

Figure 7 – Return Correlations: International Sample, 2005-2024

Notes: Monthly return correlations for the assets in the period July 2005 to April 2024 (226 months). The assets represent 15 indexes from Bloomberg and other sources mentioned in Table 3.

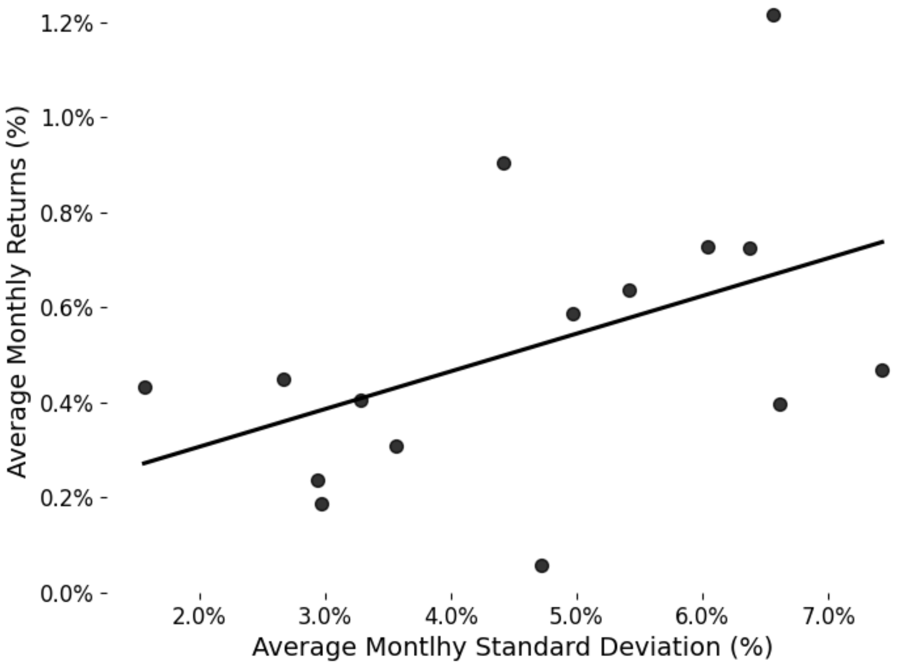

Figure 8 – Average Returns vs Std. Deviation: International Sample, 2005-2024

Notes: average and standard deviations of monthly returns over the whole sample period of July 2005 to April 2024 (226 months). The points represent 15 indexes from Bloomberg and other sources mentioned in Table 3. The black line represents the linear trend.

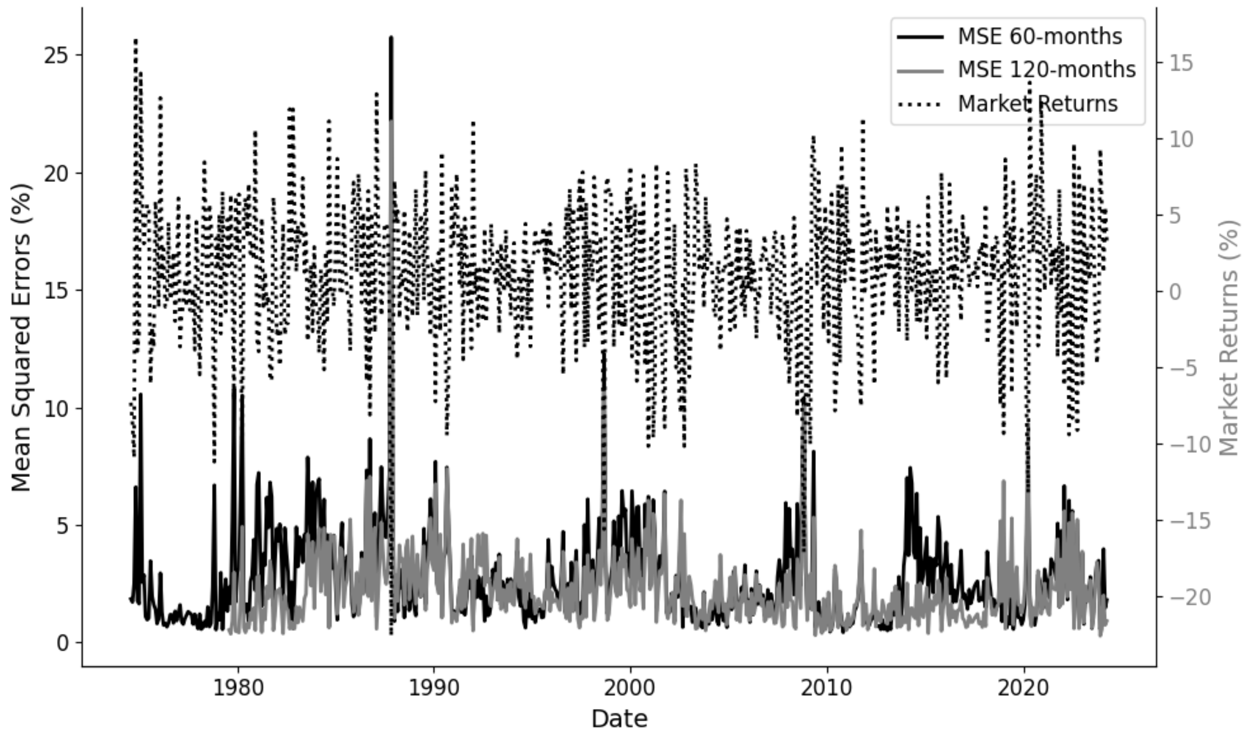

Figure 11 – Industry Sample: MSE for the 60 and 120 rolling samples

Note: The graph shows a series of Mean Squared Errors (%) for the estimates of the sample mean returns with the 60-month and 120-month rolling windows, from July 1974 to March 2024. The data is from the Industries Sample, 49 industry portfolios from the Kenneth French Data Library. For the calculation of the error, the realized returns of the next month after the estimation date are used.

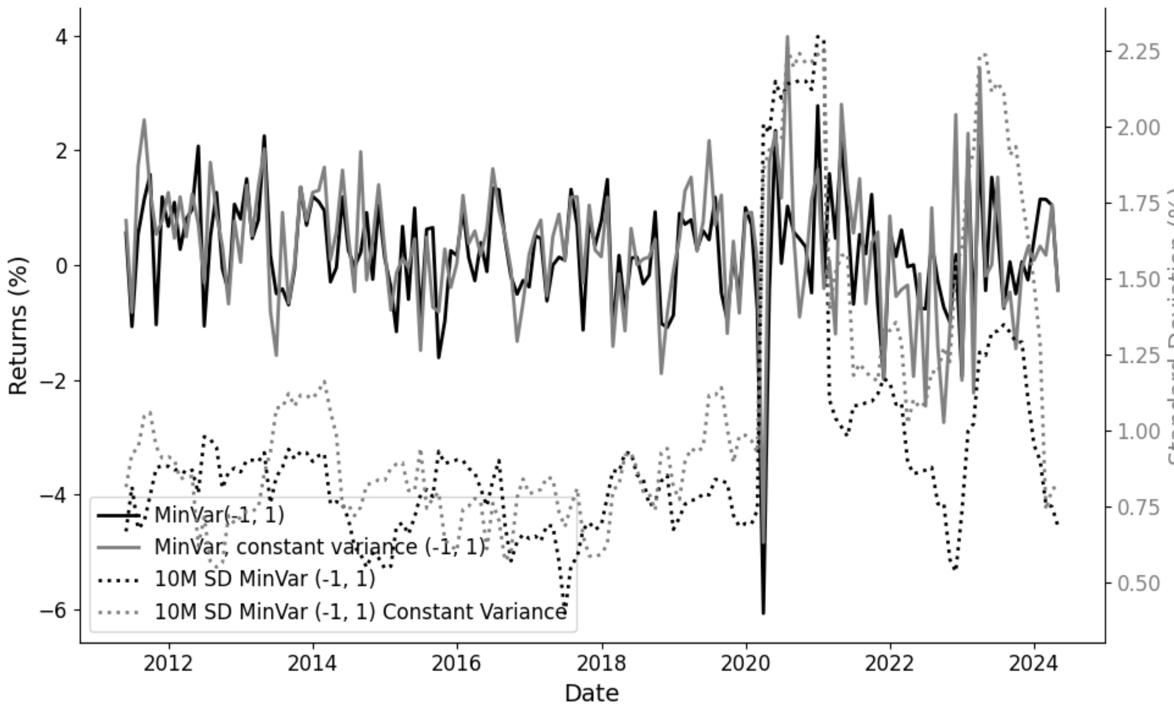

Figure 18 – International Sample: Monthly Returns and 10-month rolling SDs for MinVar (-1, 1) and Minvar (-1, 1) constant variance.

Notes: The period is 2011 to 2024. The graph shows the monthly returns (left axis) and 10-month rolling standard deviations (right) for the MinVar (01, 1) and MinVar (-1, 1) constant variance portfolios. The strategy's estimation window was 60 months.

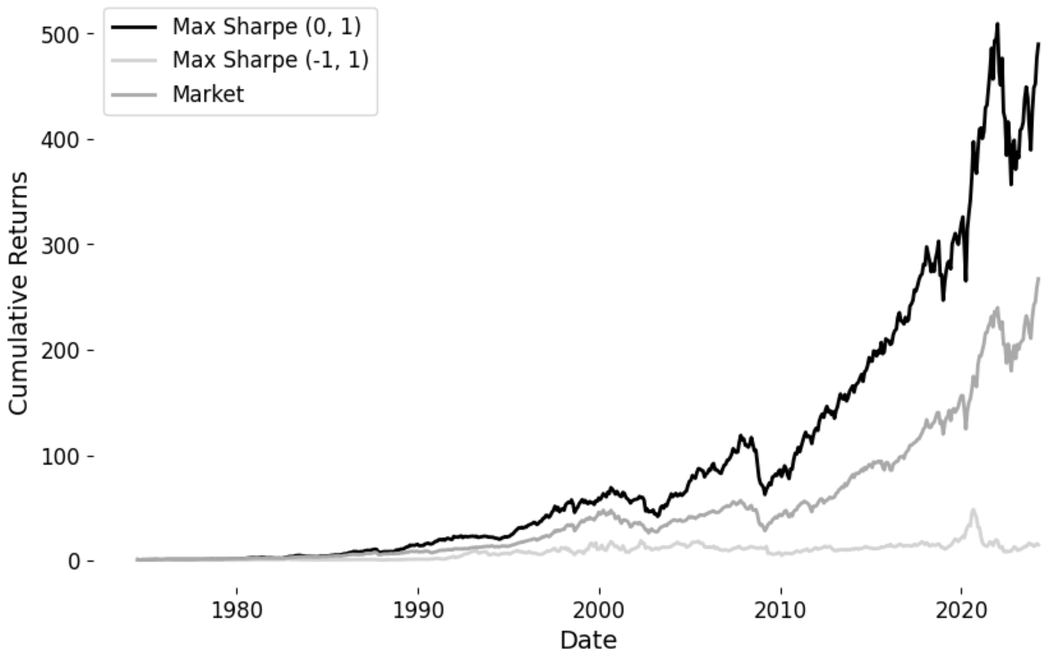

Figure 19 – Industry Sample: Max Sharpe performance versus the Market, 60-month window.

Notes: cumulative returns for the Max Sharpe (0, 1), Max Sharpe (-1, 1) and Market portfolios from 1974 to 2024. The market portfolio is from the Kenneth French Data Library.