Consumption order formulations for lot-sizing problems

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Abstract

An expansion of the lot-sizing problem in order to get formulations for the FIFO, LIFO, LEFO and FEFO consumption orders are given for the uncapacitated lot-sizing problem (ULSP) with detoriation. First the characteristcs of lot-sizing problems are briefly presented and three well known standard ULSP formulations are given. Adjustments of these standard formulations in order to get formulations for the different consumption order models are presented there after. Test results of the ULSP formulations for the four models are given together with a comparison with the EOQ formula.

Contents

1 Introduction

Within companies, supermarkets or other stores there is always the question how much to order or produce to fulfill the customer demand. When less than the demand is procured the customers will be unhappy and when more than the demand is procured the costs will be unnecessarily high. This is why companies want to know the optimal procurement quantity.
The lot-sizing problem considers when and how much of certain products need to be procured such that set up, procurement and holding costs are minimized, while satisfying the customer demand. The main objective is to determine periods when procurement will take place and the quantities that need to be procured. Chapter 2 of this paper will give an introduction to the characteristics of lot-sizing problems and will also give the three standard lot-sizing formulations.
The lot-sizing problem is a well known problem in the literature. One of the first who investigated the problem were Wagner and Whitin [2]. They formulated the most well known basic model for lot-sizing problems. This formulation is one of the three standard formulations discussed in chapter 2.
The most important lot-sizing characteristic that will be discussed in this paper is the deterioration of items. An assumption made is that items deteriorate after a fixed number of periods which depends on the period at which the items are procured. Items are good for consumption as long as they do not reach their expiration date. For example milk, cheese and fruit can not be hold in inventory forever, after a couple of periods those items are rotten and no longer good for consumption. Onal [1] investigated this problem before. He made a general formulation for the lot-sizing problem with detoriation. The focus in this paper will be on four more focussed lot-sizing models with deterioration:

1. The FIFO (first in first out) model, the product first procured will be sold first. FIFO consumption appears in the store if the inventory system is designed as a queue such that as the items are procured, they are placed at the end of the queue.

2. The LIFO (last in first out) model, the product last procured will be sold first. LIFO consumption appears in the store if the inventory system is designed as a stack such that the newly procured items are always put in front of the stack.

3. The FEFO (first expired first out) model, the product that deteriorates first is sold first. To be able to sell the early expiring items, the store should have complete control over which item the customer will buy. For example, the customer must ask the store to get the item from the depot.

4. The LEFO (last expired first out) model, the product that deteriorates last is sold first. LEFO consumption appears if the customers are allowed to choose the items themselves, which is usually the case in stores, they will choose the items with the longest remaining lifetime.

The main purpose & problem statement of this article is to use three well known formulations of the classis lot-sizing problem to find formulations for each of the above four models. In Onal[1] a formulation for lot-sizing problems with detoriation is discussed, but Onal does not give any formulation for one of the four different models. This paper will give formulations for all of the models. In chapter 3 these formulations will be discussed.

In chapter 4 the formulations for all four models will be tested. The formulations will be compared with each other in terms of solving speed, number of variables, number of constraints and number of nonzeros. The program AIMMS is used to test the formulations.

2 Introduction to lot-sizing problems

2.1 Characteristics of lot-sizing problems

The standard lot-sizing problem only takes demand of the customer, set-up costs, inventory costs and unit production costs into account. In this chapter other possible characteristics will be discussed. (Karimi et al. [10] also discussed the characteristics of lot sizing problems).

2.1.1 Planning horizon

The planning horizon can either be finite or infinite. A finite planning horizon means that the production planning has a finite schedule. If the planning horizon is infinite then the production planning has an infinite schedule. This paper only considers finite time horizons.

2.1.2 Number of products

The basic lot-sizing model considers only one item in the production system. But most stores sell more than one product. A multi-item model is more complex than a single item model. Brahimi et al. [11] investigate the single item lot-sizing problem with three different mathematical programming formulations, three of these models will be used in this article (the three standard formulations).

2.1.3 Capacity

In a production system there could be a restriction on the number of items procured or on the number of items in inventory et cetera. If there are no restrictions on capacity then the model is said to be uncapacitated. If there are restrictions on capacity then the model is capacitated. Bahl et al [7] investigated both the capacitated and the uncapacitated model in lot-sizing problems. Karimi et al [10] investigate the capacitated lot sizing problem and gives a review of models and algorithms to this problem. This paper will consider the uncapacitated lot-sizing problem.

2.1.4 Deterioration

The deterioration of items is the most important characteristic discussed in this paper. Deterioration is for instance the case with food and milk. It affects the model and makes the model more complex. Nahmias[4], Friedman and Hoch[5] and Onal [1] already investigated the effect of deterioration on the lot-sizing model. Friedman and Hoch investigated the discrete model, while Nahmias gives a review of the studies to perishable inventory. Onal [1] gives a brief introduction to the investigation of lot-sizing deterioration models with a FIFO, LIFO, LEFO and FEFO order. In chapter 3 the FIFO, LIFO, LEFO and FEFO deterioration lot-sizing models will be investigated further.

2.1.5 Demand

The demand of customers can be static or dynamic. Static demand means that the value does not change over time, while with dynamic demand the demand changes over time. Furthermore, the demand can be deterministic or probabilistic. Deterministic demand means that the demand is known in advance, while with probabilistic demand the demand is not known in advance and must be estimated.
This paper will discuss dynamic deterministic demand. For different demand types, new kind of models need to be formulated

2.1.6 Number of suppliers

When a store has decided to order its quantity it could order his quantity at a single supplier or multiple suppliers (because of capacity restrictions for example).
This could mean that the items of one supplier deteriorate faster than the items of an other supplier. So when there are a multiple number of suppliers the model will be more complex. This paper considers ordering from only one supplier.

2.1.7 Inventory shortage

Inventory shortage means that the demand of a customer in the current period can be fulfilled in future periods (Backlogging cases). Zangwill [3] discusses the effect of backlogging at lot-sizing problems. In this paper backlogging will not be discussed any further.

2.2 First standard formulation

The formulation of Wagner & Whitin [2] is the first standard formulation. This formulation is the most well known standard formulation for lot-sizing problems. The demand per period, the set up costs for procuring in a period and the inventory costs per item per unit of time are taken into account.

Set:

$T = \{1, ..., N\}$ is a set of periods indexed by t.

Parameters:

$d_t =$ Demand in period t

$h_t =$ Holding costs per unit of inventory per period of time t

$s_t =$ Set up costs

$u_t =$ Unit production cost in period t

Variables:

$x_t =$ Number of procured items in period t

$i_t =$ The number of inventory at the end of period t.

$$y_t = \begin{cases} 1 \text{ if there is procurement in period t} \\ 0 \text{ elsewhere} \end{cases}$$

Objective function:

$$\min \sum_{t=1}^{N} (s_t y_t + h_t i_t + u_t x_t) \qquad (1)$$

subject to

$$i_{t-1} + x_t = d_t + i_t \qquad \forall t \qquad (2)$$

$$x_t \leq M y_t \qquad \forall t \qquad (3)$$

$$x_t, i_t \geq 0, \qquad \forall t \qquad (4)$$

$$y_t \in (0, 1) \qquad \forall t \qquad (5)$$

Explanatian of the constraints:
(2) The inventory level in the last period plus the number of procured items in the current period is equal to the demand in the current period plus the inventory level at the end of the current period.
(3) If there is procurement then y_t takes the value of 1. M is a large number equal to the total demand of all the periods.
(4) x_t, i_t are positive.
(5) y_t is a binary variable.

Explanation of the objective function (1):
Minimize the costs of procuring, the costs of holding items in inventory and the set up costs.

2.3 Second standard formulation

If it is useful to know in which period the items procured in a certain period are used to satisfy demand. In order to do this we have to add an extra variable to the model. The advantage of this formulation is that the LP relaxation leads to an optimal solution in which the y variables are integer. This is proven by Krarup and Bilde [12]. This second standard formulation is also known under the name Facility Location Based formulation:

Sets:

$T = \{1, ..., N\}$ is a set of periods indexed by t and i.

Parameters:

$u_t =$ Unit production cost in period t

$h_t =$ Holding costs per unit of inventory per period in period t

$d_t =$ Demand in period t

$s_t =$ The set up costs in period t

$$c_{i,t} = u_i + \sum_{t=i}^{t-1} h_t$$

Variables:

$w_{i,t} =$ The amount procured in period i to satisfy demand in period t

$$y_t = \begin{cases} 1 \text{ if there is procurement in period i} \\ 0 \text{ elsewhere} \end{cases}$$

Objective function:

$$\min \sum_{i=1}^{n} \sum_{t=i}^{n} c_{i,t} w_{i,t} + \sum_{t=1}^{n} s_t y_t \tag{6}$$

subject to:

$$\sum_{i=1}^{t} w_{i,t} = d_t \qquad \forall t \tag{7}$$

$$w_{i,t} \leq d_t y_i \qquad \forall t \text{ and } i \leq t \tag{8}$$

$$w_{i,t} \geq 0 \qquad \forall t \text{ and } i \leq t \tag{9}$$

$$y_t \in 0, 1 \qquad \forall t \tag{10}$$

Explanation of the constraints:
(7) The demand in period t must be satisfied by procurement in the current period or in previous periods.
(8) A restriction that states that if there is procurement in period t the variable y_t takes the value 1.
(9) $w_{i,t}$ is an integer variable
(10) y_t is a binary variable

Explanation of the objective function (6):
Minimize the total costs for procuring, holding and set up.

2.4 Third standard formulation

Evans [8] proposed a shortest path formulation based on a graph representation of the problem, where each node of the graph represents a period, including a dummy period $T+1$ with an arc between each pair of nodes. The arc between nodes t and q represents the option of producing the whole demand from period t through period $q-1$ in period t. The solution of this problem consists of finding a shortest path from node 1 to node $T+1$. A four period example is given in figure 1.

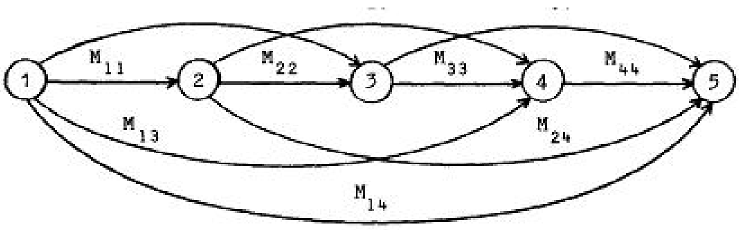

Figure 1: A four period Lot-sizing example as a shortest path problem

The shortest path formulation is as follows:

Sets:

$T = \{1, ..., \text{N}\}$ is a set of periods indexed by t and q

Parameters:

$d_t =$ The demand in period t

$M_{t,q} =$ Total variable production and holding costs of procuring $d_{t,q} = d_t + d_{t+1} + ... + d_{q-1}$ in period t

that is $M_{t,q}= u_q + \sum_{t=q}^{t-1}$

$s_t =$ The set up costs

Variables:

$$y_t = \begin{cases} 1 \text{ if there is production in period t} \\ 0 \text{ otherwise} \end{cases}$$

$Z_{t,q} =$ The fraction of the total demand from period t through period q-1 that is procured in period t

Objective function

$$\min \sum_{t=1}^{T}(s_t y_t + \sum_{q=t+1}^{T+1} M_{t,q} Z_{t,q}) \tag{11}$$

subject to

$$\sum_{t=1}^{T} Z_{1,t} = 1 \qquad 1 \leq q \leq t \leq N \tag{12}$$

$$\sum_{i=1}^{t-1} Z_{i,t} = \sum_{i=t+1}^{T+1} Z_{t,i} \qquad t = 2, ..., T \tag{13}$$

$$\sum_{i=t+1}^{T+1} Z_{i,t} \leq y_t \qquad \forall t \tag{14}$$

$$y_t \in \{0,1\} \qquad \forall t$$

$$Z_{t,q} \geq 0 \qquad \forall t \; \forall q$$

Explanation of the constraints:
(12) There can only be one outgoing arc from period 1
(13) The incoming flow must be equal to the outgoing flow in period i
(14) Makes sure that the binary variable y_t is equal to 1 when there is procurement in period t.

Explanation of the objective function (11):
The costs of producing, holding inventory and variable production costs must be minimized.

t	1	2	3	4	5
s_t	50	50	50	50	50
u_t	0	5	10	0	10
h_t	0	0	0	0	0
v_t	2	5	5	4	5
d_t	20	20	20	20	20

Table 1: Data for every formulation example.

3 Formulations

In this chapter the different formulations that are found for the LIFO, FIFO, FEFO and LEFO models will be proposed. For the LIFO and FIFO model there are two formulations found, while for the FEFO and LEFO models there are four formulations found.
The composotion of each formulation will be as follows:

1. The exact formulation will be shown
2. The constraints which are not considered before will be explained
3. If there is made an adjustment to the objective function, the adjustment will be explained
4. For each formulation there will be an example how the formulation works.

The data that will be used for every example are shown in table 1.
If the manager is free to distribute any item in inventory to the customer, that means there is no constraint on the inventory consumption order. In that case, it is quite easy to determine the optimal procurement strategy:
He would procure 40 items in period 1, to satisfy demand in period 1 and period 2. He would procure 40 items in period 2 to satisfy demand in period 3 and period 5. And he would procure 20 items in period 4 to satisfy the demand in period 4. This would lead to a total cost of:
$50 + 50 + 40 \cdot 5 + 50 = 350$.

3.1 First standard formulation

In this section the four models will be made out of an adjustment to the first standard formulation. Before the formulations for the different models of the lot sizing problem are shown, first the following theorem will be given:
Theorem 1: (See Onal[1])
There exists an optimal solution such that the demand in period t is satisfied by procurement from only one period. It is never satisfied by more procurement of more than one period.

3.1.1 FIFO model first formulation

With theorem 1 it is possible to obtain the following formulation for the FIFO model.

Set:

$T = \{1, ..., N\}$ is a set of periods indexed by t, i and j.

Parameters:

$d_t =$ Demand in period t

$h_t =$ Holding costs per unit of inventory per period of time t

$s_t =$ Set up costs

$u_t =$ Unit production cost in period t

$v_t =$ Expiration date for items procured in period t

Variabelen:

$x_t =$ Number of procured units in period t

$$y_t = \begin{cases} 1 \text{ if there is procurement in period t} \\ 0 \text{ elsewhere} \end{cases}$$

$i_{i,t} =$ The number of inventory at the end of period t procured in period i.

$b_{i,t} =$ The number of available items at the beginning of period t procured in period i

$$k_{i,t} = \begin{cases} 1 \text{ if procurement from period i is available in period i} \\ 0 \text{ else} \end{cases}$$

$$o_{i,t} = \begin{cases} 1 \text{ if procurement from period i is the oldest available in period t} \\ 0 \text{ else} \end{cases}$$

$$p_{i,t} = \begin{cases} i \text{ if procurement from period i is available in period t} \\ M \text{ elsewhere} \end{cases}$$

Objective function:

$$\min \sum_{t=1}^{N}(s_t y_t + h_t i_t + u_t x_t) + \sum_{i=1}^{N}\sum_{t=1}^{N} k_{i,t} \qquad (15)$$

subject to

$$x_t \leq M y_t, \qquad \forall t$$

$$i_t = \sum_{i=1}^{t} i_{i,t} \qquad \forall t$$

$$i_{t-1} + x_t = d_t + i_t, \qquad \forall t \qquad (16)$$

$$i_{i,t} \leq x_i, \qquad \forall t\ \forall i \qquad (17)$$

$$\sum_{i=1}^{v_t} I_{i,t} = i_t \qquad \forall t \qquad (18)$$

$$b_{i,t} = I_{i,t-1} \qquad \forall t\ \forall i\ t \neq i \qquad (19)$$

$$b_{i,t} = x_t \qquad \text{for } t = i \qquad (20)$$

$$b_{i,t} \leq M k_{i,t}, \qquad \forall t\ \forall i \qquad (21)$$

$$b_{i,t} - d_t o_{i,t} = I_{i,t}, \qquad \forall t\ \forall i \qquad (22)$$

$$p_{i,t} = i k_{i,t} + (1 - k_{i,t})M \qquad \forall t\ \forall i$$

$$p_{i,t} - (1 - o_{t,i})M \leq p_{j,t} \qquad \forall t\ \forall i \text{ and} j \neq i \qquad (23)$$

$$i_{i,t}, x_t, b_{i,t} \geq 0, \qquad \forall t\ \forall i$$

$$y_t, k_{i,t}, o_{i,t} \in \{0, 1\}, \qquad \forall t\ \forall i$$

Explanations of the constraints:
(16) The total inventory of the last period plus the procurement in the current period must be equal to the demand in the current period plus the total inventory at the end of the current period. This is almost the same constraint as (2).
(17) The inventory level at the end of period t coming from period i can not be larger than the total procurement in period i.
(18) The sum over the inventory from period i to the expiration date of products procured in period i must be equal to the sum over the total inventory procured in period i. This constraint makes sure the model holds to the deteroration.
(19)&(20) The number of available items at the beginning of period t coming from period i is equal to the inventory level at the end of the last period plus the items procured in period t ($b_{t,t}= x_t$).
(21) This constraint makes sure that $k_{i,t}$ is 1 if there are products available from period i at the beginning of period t.
(22) The number of oldest available items in period t minus the demand in period t is equal to the inventory level at the end of period t.
(23) $o_{i,t}$ must be 1 for the oldest available procurement. Only for the lowest $p_{i,t}$ $0_{i,t}$ can take the value 1. Because of constraints (15), (18), (19) and (20) the $\sum_{i=1}^{t} o_{i,t}$=1. Because of Theorem 1 this constraint makes sure the formulation follows a FIFO model.

Explanation of the objective function(15):
It is the almost the same objective function as (1), only now the binairy variable for available

procurement is taken into account. In order to make sure $k_{i,t}$ is not always 1 (according to constraint (21) that could be possible) $k_{i,t}$ is taken into the objective function. Then the variable will be minimized and will only be 1 when there is available procurement.

Example:
The data is from table 3.1. Beneath all the variables $i_{t,i}$, x_t, $b_{i,t}$, $k_{i,t}$, $p_{i,t}$ and $o_{i,t}$ are shown:

$i_{i,t}$=

	1	2	3	4	5
1	20	0	0	0	0
2	0	60	0	0	0
3	0	40	0	0	0
4	0	20	0	0	0
5	0	0	0	0	0

x_t= 40 60 0 0 0

$b_{i,t}$=

	1	2	3	4	5
1	40	0	0	0	0
2	20	60	0	0	0
3	0	60	0	0	0
4	0	40	0	0	0
5	0	20	0	0	0

$k_{i,t}$=

	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0
5	0	1	0	0	0

$p_{i,t}$=

	1	2	3	4	5
1	1	0	0	0	0
2	1	2	0	0	0
3	0	2	0	0	0
4	0	2	0	0	0
5	0	2	0	0	0

$o_{i,t}$=

	1	2	3	4	5
1	1	0	0	0	0
2	1	0	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0
5	0	1	0	0	0

As can be seen above the total procurement in period 1 is 40 items, used to satisfy the demand in period 1 and 2. The total procurement in period 2 is equal to 60 and is used to satisfy the demand in period 3, 4 and 5. As also can be seen is that the model follows a FIFO model, the oldest available procurement is sold first. In period 2 there is procurement from period 1 and from period 2 available and the procurement from period 1 is sold. The total costs of the example will be: 50+50+60·5= 400.

3.1.2 LIFO model first formulation

The second formulation is also an adjustment to the first standard formulation in order to get a LIFO model.

Set:

$T = \{1, ..., N\}$ is a set of periods indexed by t, i and j.

Parameters:

The same parameters as in the FIFO model.

New Variables:

$$o_{i,t} = \begin{cases} 1 \text{ if procurement from period i is the youngest available procurement in period t} \\ 0 \text{ else} \end{cases}$$

$$p_{i,t} = \begin{cases} i \text{ if procurement from period i is available procurement in period t} \\ 0 \text{ elsewhere} \end{cases}$$

Objective function:

$$\min \sum_{t=1}^{N} (s_t y_t + h_t \sum_{i=1}^{N} i_{i,t} + u_t x_t)$$

subject to

$$x_t \le M y_t, \qquad \forall t$$

$$i_t = \sum_{i=1}^{t} i_{i,t} \qquad \forall t$$

$$i_{t-1} + x_t = d_t + i_t, \qquad \forall t$$

$$i_{i,t} \le x_i, \qquad \forall t \ \forall i$$

$$\sum_{i=1}^{v_t} I_{i,t} = i_t \qquad \forall t$$

$$b_{i,t} = I_{i,t-1} + x_{t=i}, \qquad \forall t \ \forall i$$

$$b_{i,t} \le M k_{i,t}, \qquad \forall t \ \forall i$$

$$b_{i,t} - d_t o_{i,t} = I_{i,t}, \qquad \forall t \ \forall i$$

$$p_{i,t} = i k_{i,t} \qquad \forall t \ \forall i$$

$$p_{i,t} + (1 - o_{i,t}) M \ge p_{j,t} \qquad \forall t \ \forall i \text{ and} j \ne i \qquad (24)$$

$$i_{i,t}, x_t, b_{i,t} \ge 0, \qquad \forall t \ \forall i$$

$$y_t, k_{i,t}, o_{i,t} \in (0, 1), \qquad \forall t \ \forall i$$

Explanation of the new constraints:

(24) Instead of (23) now $o_{i,t}$ must be 1 for the "highest" available i.

Example:

The data is from table 3.1. It is not necessarily to show all the variables again, that is why only the variables $b_{i,t}$, x_t and $o_{i,t}$ are shown:

$b_{i,t}$=

	1	2	3	4	5
1	20	0	0	0	0
2	0	60	0	0	0
3	0	40	0	0	0
4	0	20	0	20	0
5	0	20	0	0	0

x_t= 20 60 0 20 0

$o_{i,t}$=

	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	1	0	0	0
4	0	0	0	1	0
5	0	1	0	0	0

As can be seen above in period 1 there is a procurement of 20 items used to satisfy the demand in period 1. There is a procurement of 60 items in period 2 to satisfy the demand in period 2,3 and 5. And in period 4 there is a procurement of 20 items to satisfy the demand in period 4. As can be seen this example follows a LIFO model. In period 4 there is procurement available from period 2 and from period 4. The procurement in period 4 is used to satisfy that demand, because that is the most recent available procurement. The total costs of this example will be $50+50+60 \cdot 5+50 = 450$.

3.1.3 LEFO model first formulation

The third formulation is an adjustment to the first standard formulation in order to get a LEFO model.

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t, i and j.
Parameters:
The same parameters as in the FIFO model.
New Variables:

$$o_{i,t} = \begin{cases} 1 \text{ if procurement from period i is the last expired available procurement in period t} \\ 0 \text{ else} \end{cases}$$

$l_{i,t} =$ The expiration date of items procured in period i and available in period t

Objective function:

$$\min \sum_{t=1}^{N} (s_t y_t + h_t \sum_{i=1}^{N} i_{i,t} + u_t x_t) + \sum_{i=1}^{N} \sum_{t=1}^{N} k_{i,t}$$

subject to

$$x_t \leq M y_t, \qquad \forall t$$

$$i_t = \sum_{i=1}^{t} i_{i,t} \qquad \forall t$$

$$i_{t-1} + x_t = d_t + i_t, \qquad \forall t$$

$$i_{i,t} \leq x_i, \qquad \forall t \; \forall i$$

$$\sum_{i=1}^{v_t} I_{i,t} = i_t \qquad \forall t$$

$$b_{i,t} = I_{i,t-1} + x_{t=i}, \qquad \forall t \; \forall i$$

$$b_{i,t} \leq M k_{i,t}, \qquad \forall t \; \forall i$$

$$b_{i,i} - d_t o_{i,t} = I_{t,i}, \qquad \forall t \; \forall i$$

$$l_{i,t} = v_i k_{i,t} \qquad \forall t \; \forall i \qquad (25)$$

$$l_{i,t} + (1 - o_{i,t})M \geq l_{j,t}, \qquad \forall t \; \forall i \text{ and} j \neq i \qquad (26)$$

$$l_{i,t}, i_{i,t}, x_t, b_{i,t} \geq 0, \qquad \forall t \; \forall i$$

$$y_t, k_{i,t}, o_{i,t} \in (0, 1), \qquad \forall t \; \forall i$$

Explanation of the new constraints:
(25) The expiration date of available items in period t is the expiration date of items procured in period i times the binary variable if items procured in period i are available in period t
(26) This constraint makes sure the model follows a LEFO model, because $o_{i,t}$ must be 1 for the product with the latest expiration date. Furthermore it is almost the same constraint as constraint (23)

Example:
The data is from table 3.1. Beneath the variables x_t, $b_{i,t}$, $l_{i,t}$ and $o_{i,t}$ are shown:
x_t= 20 80 0 0 0

$b_{i,t}$=

	1	2	3	4	5
1	20	0	0	0	0
2	0	80	0	0	0
3	0	60	0	0	0
4	0	40	0	0	0
5	0	20	0	0	0

$l_{i,t}$=

	1	2	3	4	5
1	2	0	0	0	0
2	0	5	0	0	0
3	0	5	0	0	0
4	0	5	0	0	0
5	0	5	0	0	0

$o_{i,t}$=

	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0
5	0	1	0	0	0

As can be seen the procurement in period 1 is 20, used to satisfy the demand in period 1. The procurement in period 2 is 80, used to satisfy the demand in period 2,3,4 and 5. The demand of every period is satisfied by the available product with the longest remaining lifetime in that period, so this formulation follows a LEFO model. The total costs of this example are $50+50+80 \cdot 5= 500$.

3.1.4 FEFO model first formulation

The last adjustment to the first standard formulation will be an adjustment to make a formulation that follows the FEFO model.

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t, i and j.
Parameters:
The same parameters as in the FIFO model.
New Variables:

$$o_{i,t} = \begin{cases} 1 \text{ if procurement from period i is the first expired available procurement in period t} \\ 0 \text{ else} \end{cases}$$

$l_{i,t}$ = The expiration date of items procured in period i and available in period t

Objective function:

$$\min \sum_{t=1}^{N} (s_t y_t + h_t \sum_{i=1}^{N} i_{i,t} + u_t x_t)$$

subject to

$$x_t \leq M y_t, \qquad \forall t$$

$$i_t = \sum_{i=1}^{t} i_{i,t} \qquad \forall t$$

$$i_{t-1} + x_t = d_t + i_t, \qquad \forall t$$

$$i_{i,t} \leq x_i, \qquad \forall t \ \forall i$$

$$\sum_{i=1}^{v_t} I_{i,t} = i_t \qquad \forall t$$

$$b_{i,t} = I_{i,t-1}, \qquad \forall t \ \forall i \ \& \ t \neq i$$

$$b_{i,t} = x_t \qquad \text{for t=i}$$

$$b_{i,t} \leq M k_{i,t}, \qquad \forall t \ \forall i$$

$$b_{i,t} - d_t o_{i,t} = I_{i,t}, \qquad \forall t \ \forall i$$

$$l_{i,t} = v_i k_{i,t} + (1 - k_{i,t}) M \qquad \forall t \ \forall i \qquad (27)$$

$$l_{i,t} - (1 - o_{i,t}) M \leq l_{j,t} \qquad \forall t \ \forall i \text{ and } i \neq j \qquad (28)$$

$$i_{i,t}, x_t, b_{i,t} \geq 0, \qquad \forall t \ \forall i$$

$$y_t, k_{i,t}, o_{i,t} \in (0, 1), \qquad \forall t \ \forall i$$

Explanation of the new constraints:
(27) This constraint is almost the same as (25) only now with a slight adjustment in order to follow a FEFO model. If procurement in period i is not available in period t the expiration date of those "items" will be M(with M=N+1). Constraint (28) explains why this is necessarily.
(28) For the lowest expiration date $o_{i,t}$ will be 1. This is why the expiration date of non available items will be a large number M, otherwise they have the lowest expiration date.

Example:
The data is from table 3.1. Beneath the variables x_t, $b_{i,t}$, $l_{i,t}$ and $o_{i,t}$ are shown:
x_t= 40 40 0 20 0

$b_{i,t}$=

	1	2	3	4	5
1	40	0	0	0	0
2	20	0	0	0	0
3	0	40	0	0	0
4	0	20	0	20	0
5	0	20	0	0	0

$l_{i,t}$=

	1	2	3	4	5
1	2	0	0	0	0
2	2	0	0	0	0
3	0	5	0	0	0
4	0	5	0	4	0
5	0	5	0	0	0

$o_{i,t}$=

	1	2	3	4	5
1	1	0	0	0	0
2	1	0	0	0	0
3	0	1	0	0	0
4	0	0	0	1	0
5	0	1	0	0	0

As can been seen the total procurement in period 1 is 40, used to satisfy demand in period 1 and 2. The total procurement in period 2 is 40, used to satisfy demand in period 3 and 5 and the total procurement in period 4 is 20 used to satisfy the demand in period 4. In every period the demand is satisfied by available items with the lowest expiration date, so the formulation follows a FEFO model. The total costs of this example will be: $50+50+40 \cdot 5+50 = 350$.

3.2 Second standard formulation

The second standard formulation is the Facility Location based formulation. In this section the second standard formulation is adjusted in order to get four formulations for the FIFO, LIFO, LEFO and FEFO model.

3.2.1 FIFO model second formulation

The first adjustment is made to get a formulation for the FIFO model. Two variables are added to the Facility Location based formulation to force an FIFO consumption order. The new formulation is defined in the following way:

Sets:

$T = \{1, ..., N\}$ is a set of periods indexed by t and i.

Parameters:

$$c_{i,t} = u_i + \sum_{t=i}^{t-1} h_t$$

d_t = Demand in period t

s_t = The set up costs in period t

v_t = Expiration date of products procured in period t

Variables:

$w_{i,t}$ = The amount procured in period i to satisfy demand in period t.

$$z_{i,t} = \begin{cases} \text{1 if procurement in period i satisfies demand in period t} \\ \text{0 elsewhere} \end{cases}$$

x_t = The number of items procured in period t

$$y_t = \begin{cases} \text{1 if there is procurement in period t} \\ \text{0 elsewhere} \end{cases}$$

a_t = The period used by period t to satisfy the demand in period t

Objective function:

$$\min \sum_{i=1}^{n} \sum_{t=i}^{n} c_{i,t} w_{i,t} + \sum_{t=1}^{n} s_t y_t + \sum_{i=1}^{n} \sum_{t=1}^{n} z_{i,t} \tag{29}$$

subject to:

$$\sum_{i=1}^{v_t} w_{t,i} = d_t \qquad \forall t \tag{30}$$

$$w_{i,t} \leq d_t y_i \qquad \forall t \text{ and i} \leq \text{t}$$

$$w_{i,t} \leq d_t z_{i,t} \qquad \forall t\ \forall i \tag{31}$$

$$a_t = \sum_{i}^{N} (z_{i,t} i) \qquad \forall t \tag{32}$$

$$a_t \geq a_{t-1} \qquad \forall t \tag{33}$$

$$a_t, w_{i,t}, x_t \geq 0 \qquad \forall t\ \forall i$$

$$z_{i,t}, y_t \in 0, 1 \qquad \forall t\ \forall i$$

Explanation of the restrictions:
(30) The demand in period t must be satisfied by the amounts procured in periods i, ... , v_t and used in period t.
(31) $z_{i,t}$ must be 1 if procurement from period i satisfies the demand in period t.
(32) The period used by period t to satisfy demand in period t. This constraint holds because the demand is satisfied by procurement from only one period (Theorem 1).
(33) This constraint makes sure the formulation follows a FIFO model, because the demand in period t can not be satisfied by procurement from an earlier period than the demand in period t-1.

Explanation of the objective function (29):
The objective function is almost equal to (6). The only adjustment made is that $\sum_{i=1}^{n} \sum_{t=1}^{n} z_{i,t}$ is added to the objective function. Without this adjustment $z_{i,t}$ could always be 1 and would not violate constraint (31), now $z_{i,t}$ is minimized and so he will only be 1 when there is no other choice.

Example:
See table 3.1 for the data for this example. Beneath the variables $w_{i,t}$, $z_{i,t}$, x_t and a_t are shown.

$w_{i,t}$=

	1	2	3	4	5
1	20	0	0	0	0
2	20	0	0	0	0
3	0	20	0	0	0
4	0	20	0	0	0
5	0	20	0	0	0

$z_{i,t}=$

	1	2	3	4	5
1	1	0	0	0	0
2	1	0	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0
5	0	1	0	0	0

$x_t=$ 40 60 0 0 0

$a_t=$ 1 1 2 2 2

As can be seen above in period 1 there is a procurement of 40 items, used to satisfy demand in period 1 and 2. In period 2 there is a procurement of 60 items, used to satisfy demand in period 3, 4 and 5. This example follows a FIFO model, that is because of the constraint $a_t \geq a_{t+1}$ (33). That means that the period used to satisfy demand is in the current period is higher or equal to the period that satisfied the demand in the previous period. The total costs of this example will be: 50+ 50+ 60·5 = 400, which is equal to the total costs of the first formulation of the FIFO model.

3.2.2 LIFO model second formulation

The second adjustment made to the Facility location based formulation is made in order to get a formulation for the LIFO model.

Set:

$T = \{1, ..., N\}$ is a set of periods indexed by t, i and j.

Parameters:

The same parameters as in the FIFO model

New Variables:

$b_{i,t}$ = The number of items procured in period i that are available in period t

$$k_{i,t} = \begin{cases} 1 \text{ if there are items available procured in period i in period t} \\ 0 \text{ elsewhere} \end{cases}$$

Objective function:

$$\min \sum_{t=1}^{N} s_t y_t + \sum_{i=1}^{N} \sum_{t=1}^{N} c_{i,t} w_{i,t} + \sum_{i=1}^{n} \sum_{t=1}^{n} (z_{i,t} + k_{i,t}) \tag{34}$$

subject to

$$\sum_{i=t}^{v_t} w_{i,t} = d_t, \qquad \forall t$$

$$w_{i,t} \leq d_t y_i, \qquad \text{for } 1 \leq i \leq t \leq \text{N}$$

$$w_{i,t} \leq M z_{i,t}, \qquad \forall t \forall i$$

$$b_{i,t} = \sum_{j=t \& j \geq i}^{N} w_{j,t}, \qquad \forall t \text{ and i} \geq \text{t} \tag{35}$$

$$b_{i,t} \leq M k_{i,t}, \qquad \forall t\ \forall i \tag{36}$$

$$\sum_{j=1}^{N} (z_{j,t} i) \geq k_{i,t} i, \qquad \forall t\ \forall i \tag{37}$$

$$w_{i,t}, x_t, w_t, z_t \geq 0, \qquad \forall t\ \forall i$$

$$y_t, k_{i,t} \in \{0, 1\} \qquad \forall t\ \forall i$$

Explanation of the new constraints:

(35)The number of available items in period t coming from period i is the sum over the number of procured items in period i in and used in future periods.

(36) $k_{i,t}$ must be 1 if there is procurement from period i available in period t.

(37) This constraint makes sure that the model follows the LIFO model, because in period t the demand must be satisfied by the most recent available procurement. This constraint holds for the LIFO model because of Theorem 1, without that theorem the demand could be satisfied by procurement from more than one period.

Explanation of the objective function (34) :

The objective function is almost the same as (6) and (29) only now $k_{i,t}$ is also added to the objective function. This has the same reason as why $z_{i,t}$ is added to the objective function. Without the adjustment made to the objective function $k_{i,t}$ could always be 1 without violating constraint (35), and it must be only 1 when there is procurement available from period i in period t. With the adjustment to the objective function $k_{i,t}$ is minimized and will only be 1 when it is needed to be.

Example:

For this example the data of table 3.1 is used. Beneath the variables $w_{i,t}$, $b_{i,t}$, $z_{i,t}$ and $k_{i,t}$ are shown:

$w_{t,i}=$

	1	2	3	4	5
1	20	0	0	0	0
2	0	20	0	0	0
3	0	20	0	0	0
4	0	0	0	20	0
5	0	20	0	0	0

$z_{t,i}=$

	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	1	0	0	0
4	0	0	0	1	0
5	0	1	0	0	0

$b_{t,i}=$

	1	2	3	4	5
1	20	0	0	0	0
2	0	60	0	0	0
3	0	40	0	0	0
4	0	20	0	20	0
5	0	20	0	0	0

$k_{i,t}=$

	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	1	0	0	0
4	0	1	0	1	0
5	0	1	0	0	0

As can be seen in the first table above, period 1 procures 20 items used to satisfy demand in period 1. Period 2 procures 60 items, used to satisfy demand in period 2, 3 and 5. And period 4 procures 20 items, used to satisfy demand in period 4. As can be seen this example follows a LIFO model, because the products procured last are sold first, in period 4 the products procured in period 4 are choosen above the products procured in period 2. The total cost of this example are $50 + 50 + 5 \cdot 60 + 50 = 450$, which is equal to the total costs of the first LIFO model.

3.2.3 LEFO model second formulation

In this subsection the second formulation for the LEFO model will be given. This formulation is an adjustment of the Facility location based formulation. There are 2 more variables added to the formulation in comparison to the second LIFO model formulation. Both variables are meant to keep track on the expiration date of procured items. The formulation is as follows:

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t and i.
New Parameters:
There are no new parameters in this model
New Variables:
$a_{i,t}$ = The expiration date of items procured in period i and available in period t
l_t = The expiration date of the used items in period t

Objective function:

$$\min \sum_{t=1}^{N} s_t y_t + \sum_{i=1}^{N}\sum_{t=1}^{N} c_{i,t} w_{i,t} + \sum_{i=1}^{N}\sum_{t=1}^{N} k_{i,t}$$

subject to

$$\sum_{i=t}^{v_t} w_{i,t} = d_t \qquad \forall t$$

$$w_{i,t} \leq d_t y_i \qquad 1 \leq i \leq t \leq N$$

$$w_{i,t} \leq M z_{i,t} \qquad \forall t, \forall i$$

$$b_{i,t} = \sum_{i=t}^{N} w_{i,t}, \qquad \forall t \text{ and } i \geq t$$

$$b_{i,t} \leq M k_{i,t}, \qquad \forall t, \forall i$$

$$a_{i,t} = v_i k_{i,t} \qquad \forall t, \forall i \qquad (38)$$

$$l_t = \sum_{i=1}^{N} (v_i z_{i,t}) \qquad \forall t \qquad (39)$$

$$l_t \geq a_{i,t} \qquad \forall t, \forall i \qquad (40)$$

$$w_{i,t}, x_t, l_t, a_{i,t} \geq 0, \qquad \forall t, \forall i$$

$$k_{i,t}, z_{i,t}, y_t \in \{0, 1\} \qquad \forall t, \forall i$$

Explanation of the constraints:
(38) The expiration date of products available in period t who are procured in period i is equal to the expiration date of items in period i times the binairy variable if those items are available in period t.
(39) The expiration date of the products that are used in period t who are procured in period

i is equal to the expiration date of items in period i times the binary variable if those items are used in period t. This constraint holds because of Theorem 1. Without Theorem 1 the demand could be satisfied by more than one period and the constraint would be no longer sufficient.
(40) This constraint makes sure the model follows a FEFO model: The items used in period t must have the largest remaining lifetime. So the used items expiration date must be the largest available expiration date in the period.

Example:
The data of table 3.1 is used for this example. Beneath the variables $w_{i,t}$, $a_{i,t}$ and l_t are shown:

$w_{t,i}$=

	1	2	3	4	5
1	20	0	0	0	0
2	0	20	0	0	0
3	0	20	0	0	0
4	0	20	0	0	0
5	0	20	0	0	0

$a_{t,i}$=

	1	2	3	4	5
1	2	0	0	0	0
2	0	5	0	0	0
3	0	5	0	0	0
4	0	5	0	0	0
5	0	5	0	0	0

l_t= 2 5 5 5 5

In period 1 there is a procurement of 20 items, used to satisfy the demand in period 1. In period 2 there is a procurement of 80 items and used to satisfy the demand in period 2, 3, 4 and 5. As can be seen this example follows a LEFO model. The expiration date in every period is the highest available expiration date in that period.
The total costs os this example will be 50+50+80·5= 500. This is equal to the total costs of the example of the first formulation for a LEFO model.

3.2.4 FEFO model second formulation

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t and i.
New Parameters:
There are no new parameters in this model
New Variables:
$a_{i,t}$ = The expiration date of items procured in period i and available in period t
l_t = The expiration date of the used items in period t

Objective function:

$$\min \sum_{t=1}^{N} s_t y_t + \sum_{i=1}^{N} \sum_{t=1}^{N} c_{i,t} w_{i,t} + \sum_{i=1}^{N} \sum_{t=1}^{N} k_{i,t}$$

subject to

$$\sum_{i=t}^{v_t} w_{i,t} = d_t \qquad \forall t$$

$$w_{i,t} \leq d_t y_i \qquad 1 \leq i \leq t \leq N$$

$$w_{i,t} \leq M z_{i,t} \qquad \forall t, \forall i$$

$$b_{i,t} = \sum_{i=t}^{N} w_{i,t} \qquad \forall t \text{ and } i \geq t$$

$$b_{i,t} \leq M k_{i,t}, \qquad \forall i, \forall t$$

$$a_{i,t} = v_i k_{i,t} + (1 - k_{i,t}) M \qquad \forall t, \forall i \qquad (41)$$

$$l_t = \sum_{i=1}^{N} v_i z_{i,t} \qquad \forall t$$

$$l_t \leq a_{i,t} \qquad \forall t, \forall i \qquad (42)$$

$$w_{i,t}, b_{i,t}, a_{i,t}, l_t \geq 0, \qquad \forall t, \forall i$$

$$y_t, k_{i,t} \in \{0, 1\} \qquad \forall t \; \forall i$$

Explanation of the constraints:
(41) This restriction is almost the same as (38) in the LEFO model. Only now if there are no items from period i available for period t, the value is no longer 0 but now M (where M is equal to the highest expiration date+1). Constraint (42) explains this.
(42) Because of (41) this constraint makes sure this formulation follows a FEFO model. Period t chooses the available item with the lowest expiration date to fulfill the demand in the period. Without constraint (41) the item with the lowest expiration date is an item that is not available, their expiration date would be zero. But now their expiration date is M.

Example:
This example will make use of the data available in table 3.1. The above formulation will give the following results for $w_{i,t}$, $a_{i,t}$ and l_t:

$w_{i,t}$=

	1	2	3	4	5
1	20	0	0	0	0
2	20	0	0	0	0
3	0	20	0	0	0
4	0	0	0	20	0
5	0	20	0	0	0

$a_{i,t}$=

	1	2	3	4	5
1	2	6	6	6	6
2	2	5	6	6	6
3	6	5	6	6	6
4	6	5	6	4	6
5	6	5	6	6	6

l_t= 2 2 5 4 5

The second table is the remaining lifetime of products procured in period i and available in period t. If there are no products available from period i in period t, then the "expiration date" of those products is 6 (M). As can be seen the products with the lowest expiration date will be chosen to sell first. So this example follows an FEFO model.
In period 1 there is a procurement of 40 items, used to satisfy demand in period 1 and 2. In period 2 there is a procurement of 40 items, used to satisfy demand in period 3 and 5. At last in period 4 there is procurement of 20 items, used to saitsfy demand in period 4 itself.
The total cost will be 50+50+40·5+50= 350, which is the same as in the first formulation for the FEFO model.

3.3 Third standard formulation

It is not possible to make an adjustment of the third standard formulation in order to get a LIFO, FIFO, FEFO of LEFO model. This is because the third standard formulation is allready formulated according to an order. That order cannot be changed towards a LIFO , FIFO, FEFO or LEFO model.

3.4 Fourth formulation

There is a third possibility to get a formulation for the FEFO and LEFO model. This is because the FEFO and LEFO model both have optimalty propertys. This means that in the optimal solution both models have a characteristic.
Theorem 2 (See Onal[1]): There exists an optimal solution to the lot sizing problem with no consumption order constraints, where the items are distributed in a FEFO manner.
Theorem 3 (See Onal[1]): For the LEFO model the zero inventory property (ZIO) holds. The ZIO property means that there is only procurement in a period when there is no inventory left from the previous periods.
Formulations for the FEFO and LEFO model with optimality propertys will be discussed in this section.

3.4.1 LEFO model 1

First the LEFO model, this formulation is an adjustment of the first standard formulation with the optimality property taken into account.

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t and i.
Parameters:
The same parameters as in the first formulation of the LEFO model.
New Variables:
Almost the same variables as in the first formulation of the LEFO model.
Only without $o_{i,t}$ and $l_{i,t}$
Objective function:

$$\min \sum_{t=1}^{N}(s_t y_t + h_t \sum_{i=1}^{N} i_{i,t} + u_t x_t) + \sum_{i=1}^{N}\sum_{t=1}^{N} k_{i,t}$$

subject to

$$x_t \leq M y_t \qquad \forall t$$

$$\sum_{i=1}^{t-1} I_{i,t-1} + x_t = d_t + \sum_{i=1}^{t} I_{i,t} \qquad \forall t$$

$$I_{i,t} \leq x_i \qquad \forall t\ \forall i$$

$$\sum_{i=1}^{v_t} I_{i,t} = \sum_{i=1}^{t} I_{i,t} \qquad \forall t$$

$$b_{i,t} = I_{i,t-1} i \qquad \forall t\ \forall i\ i \neq t$$

$$b_{i,t} = x_t \qquad \text{for i=1}$$

$$b_{i,t} \neq M k_{i,t} \qquad \forall t\ \forall i$$

$$b_{i,t} - d_t k_{i,t} = I_{i,t} \qquad \forall t\ \forall i \qquad (43)$$

$$b_{i,t}, i_{i,t}, x_t \geq 0 \qquad \forall t\ \forall i$$

$$k_{i,t}, y_t \in 0 \qquad \forall t\ \forall i$$

Explanation of the new constraints:
(43) Because of Theorem 3 there is no procurement as long as there is inventory from a previous period available. This means that at the beginning of a period there is only procurement from one period available. So the available procurement is used to satisfy the demand in the current period.

3.4.2 LEFO model 2

Now the second formulation for the LEFO model. This formulation is an adjustment of the Facility Location based formula and the optimality characteristic of the LEFO model is taken into account when formulating this formulation.

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t and i.
New Parameters:
There are no new parameters in this model
New Variables:
Almost the same variables as in the second formulation of the LEFO model
Only $a_{i,t}$ and l_t are not used in this formulation

Objective function:

$$\min \sum_{t=1}^{N} s_t y_t + \sum_{i=1}^{N} \sum_{t=1}^{N} c_{i,t} w_{i,t} + \sum_{i=1}^{N} \sum_{t=1}^{N} k_{i,t}$$

subject to

$$\sum_{i=t}^{v_t} w_{i,t} = d_t \qquad \forall t$$

$$w_{i,t} \leq d_t y_i \qquad 1 \leq i \leq t \leq N$$

$$b_{i,t} = \sum_{i=t}^{N} w_{i,t}, \qquad \forall t \text{ and } i \geq t$$

$$k_{i,t} \leq M k_{i,t}, \qquad \forall t, \forall i$$

$$\sum_{i=1}^{N} k_{i,t} = 1 \qquad \forall t \tag{44}$$

$$w_{i,t}, b_{i,t} \geq 0, \qquad \forall t, \forall i$$

$$k_{i,t}, y_t \in \{0, 1\} \qquad \forall t, \forall i$$

Explanation of the new constraints:
(44) Because of theorem 3 model there is only procurement from one period available in a period. So the sum over $k_{i,t}$ must be equal to 1.

3.4.3 FEFO model 1

Now the first formulation for the FEFO model with the optimality property will be proposed. This formulation is an adjustment of the first standard formulation of Wagner and Whitin.

Set:

$T = \{1, ..., N\}$ is a set of periods indexed by t and i.

Parameters:

The same parameters as in the first formulation of the FEFO model.

New Variables:

Only the variables x_t, y_t and $i_{i,t}$ are used.

Objective function:

$$\min \sum_{t=1}^{N} (s_t y_t + h_t \sum_{i=1}^{N} i_{i,t} + u_t x_t)$$

subject to

$$x_t \leq M y_t \qquad \forall t$$

$$\sum_{i=1}^{t-1} i_{i,t-1} + x_t = d_t + \sum_{i=1}^{t} i_{i,t} \qquad \forall t$$

$$i_{i,t} \leq x_i \qquad \forall t \ \forall i$$

$$\sum_{i=1}^{v_t} I_{i,t} = \sum_{i=1}^{t} i_{i,t} \qquad \forall t$$

$$i_{i,t}, x_t \geq 0 \qquad \forall t \ \forall i$$

$$y_t \in \{0, 1\} \qquad \forall t$$

Explanation:

Because of Theorem 2, the only difference needed with the original Wagner & Whitin formulation is that this formulation takes the deterioration of items into account.

3.4.4 FEFO model 2

Set:
$T = \{1, ..., N\}$ is a set of periods indexed by t and i.
New Parameters:
There are no new parameters in this model
New Variables:
Only the variables $w_{i,t}$ and y_t are used.

Objective function:

$$\min \sum_{t=1}^{N} s_t y_t + \sum_{i=1}^{N} \sum_{t=1}^{N} c_{i,t} w_{i,t}$$

subject to

$$\sum_{i=t}^{v_t} w_{i,t} = d_t \qquad \forall t$$

$$w_{i,t} \leq d_t y_i \qquad 1 \leq i \leq t \leq N$$

$$w_{i,t} \geq 0, \qquad \forall t, \forall i$$

$$y_t \in \{0, 1\} \qquad \forall t, \forall i$$

Explanation:
Because of Theorem 2, the only difference needed with the original Facility Location based formulation is that this formulation takes deteroriation into account.

4 Testing

In this chapter all the models of the previous chapter will be tested upon solving speed, gapsize, number of constraints, number of variables and number of nonzeros.

4.1 Generating data

First the data must be generated in order to test the models. Every model will be tested on 120 problem instances. These are divided in 4 different time horizons: 25 periods, 50 periods, 100 periods and 150 periods. And in 3 different average set up costs: between 0 and 50, between 100 and 200 and between 300 and 500. On every combinination of the above characteristics the model is tested 10 times.
The set up costs, the unit costs per item procured, the holding costs, the expiration date and the demand in a period are generated in the following way, where U is an uniformly distributed random number between 0 and 1:
Set up costs = 50U for the first division (the setup costs are between 0 and 50), 100U+100 for the second division(the setup costs are between 100 en 200) and 200U+300 for the last division (the setup costs are between 300 and 500).
Unit costs= 10U, the unit costs are between 0 and 10 per time unit and per item unit.
Expiration date= t+10U, the items detoriate within 10 periods of time.
Holding costs= 5U, the holding costs are between 0 and 5 per time unit.
Demand= 100U, the demand is between 0 and 100 per unit of time.

4.2 Economic order quantity formula

The Economic order quantity(EOQ) formula is originally found in the beginning of the 20th century. It is a formula to determine the optimal order quantity, the higher the set up costs are the higher the order quantity will be. And the higher the holding costs are, the lower the order quantity will be. The EOQ formula is:

$$Q = \sqrt{\frac{2DF}{h}}$$

Where:

- Q= Quantity order
- D= Demand
- F= Fixed costs (Set up costs)
- h= Holding costs
- $\frac{D}{Q}$= the number of orders per period.

The formulation has an important limitation the setup costs, holding costs and the demand are assumed to be constant over time. In the models spoken of in this article the costs and the demand is not constant over time. But with this formulation it is possible to approximate the number of orders in a period and to compare the results with the testresults of the formulations. The EOQ formula estimates 0.64 orders per period. Where the FEFO model estimates 0.50 orders per period, the LEFO model 0.37 orders per period, the LIFO model 0.52 orders per period and the FIFO model 0.40 orders per period. The testresults show that the models order less often than the EOQ formula. The reason herefore could be that the demand and the costs are not constant over time, or because the formula does not have constraints on consumption orders.

4.3 Results

The results of testing every model 120 times will be published and discussed in this section. But first an explanation of how the results are published. The models are tested on different time horizons and on different setup costs. The different time horizons make a significant difference on the solving time, the number of variables, the number of constraints and the number of nonzeros of every model. Therefore the results are published for every time horizon seperately. The different set up costs only make a difference for the solving speed when the model is tested on 150 periods. But the difference is relativilly small, in order to get a significant difference in solving speed the time horizon has to be significant larger than 150 periods. The problem on testing the models with a significant large time horizon is that the memory needed, in order to solve the problem, is too large. Therefore the models are not tested on more than 150 periods and the results will not be published for the different setup costs seperately.
Table 2 contains the results of the FIFO models, table 3 the results of the LIFO models, table 4 the results of the FEFO models and table 5 contains the results of the LEFO models. For every table holds that the number of variables, constraints, nonzeros, the gapsize and the solving speed for every model and every time horizon are averages on 30 tests.

Table 2: Test results FIFO model

Model	#variables	#constraints	#nonzeros	Gapsize	Solving speed
FIFO 1st 25	3201	18848.30	55063.50	30%	0.27
FIFO 1st 50	12651	137698.17	407418.70	30%	2.03
FIFO 1st 100	50301	1050398.30	3128960.13	30%	14.87
FIFO 1st 150	112951	3488098.27	10414322.20	30%	17.14
FIFO 2nd 25	1301	1346.13	4902.40	0%	0.05
FIFO 2nd 50	5101	5188.40	19182.63	0%	0.32
FIFO 2nd 100	20201	20349.60	75716.93	0%	3.07
FIFO 2nd 150	45301	45345.60	146577.40	0%	9.60

Table 3: Test results LIFO model

Model	#variables	#constraints	#nonzeros	Gapsize	Solving speed
LIFO 1st 25	3176	18823.23	54601.50	29%	0.31
LIFO 1st 50	12601	137648.17	405868.00	29%	2.24
LIFO 1st 100	50201	1050298.30	3123328.10	29%	16.18
LIFO 1st 150	91503	312213.00	10402019.00	29%	33.56
LIFO 2nd 25	2526	3171.30	25878.40	0%	0.11
LIFO 2nd 50	10051	12588.40	176133.63	0%	0.58
LIFO 2nd 100	40101	50149.60	1287117.93	0%	3.45
LIFO 2nd 150	67650	112545.6	4184928.00	0%	12.65

The black lighted models are the best formulations in their category for table 2 and 3. As can be seen the second FIFO formulation and the second LIFO formulation are the best formulations. They have the lowest solving time, the lowest gapsize, the lowest number of variables, constraints and nonzeros. The higher the time interval the longer the solving time and the more variables, constraints and nonzeros are needed. The gapsize only depends on which formulation is used, the gapsize for the first formulation is 30%, while the gapsize for the second formulation 0% is. That could be expected, because if the second formulation is solved with a LP relaxation the y variables will be integer in the optimal solution.

Also in table 4 and 5 are the black lighted formulations the best formulations in their category. As can be seen for the FEFO model 4.2 is the best formulation and for the LEFO model is 4.1 the best formulation. They have the lowest solving time, the lowest number of variables, constraints and nonzeros.

But the fourth FEFO model and the fourth LEFO model take Theorem 2 and Theorem 3 into account. The best FEFO and LEFO model without any knowledge in front are green highlighted in the table. That are the second FEFO and LEFO model.

As can be seen the higher the time interval is, the longer the solving time is and the more variables, constraints and nonzeros are used. The gapsize only depend on which formulation is used. For the first FEFO formulation the gapsize is 29

Table 4: Test results FEFO model

Model	#variables	#constraints	#nonzeros	Gapsize	Solving speed
FEFO 1st 25	3176	18823.23	54601.50	29%	0.38
FEFO 1st 50	12601	137648.17	405868.00	29%	3.14
FEFO 1st 100	50201	1050298.30	3123321.97	29%	20.59
FEFO 1st 150	112801	3487948.27	10402203.47	29%	53.11
FEFO 2nd 25	3176	3801.00	11859.27	0%	0.08
FEFO 2nd 50	12601	15088.40	57453.67	0%	0.31
FEFO 2nd 100	50201	60149.60	312366.27	0%	1.41
FEFO 2nd 150	112801	135195.6	889931.00	0%	4.52
FEFO 4.1 25	676	998.03	3332.73	29%	0.06
FEFO 4.1 50	2601	3872.90	13301.53	29%	0.26
FEFO 4.1 100	10201	15248.83	53255.77	29%	1.22
FEFO 4.1 150	22801	45448.27	142479.50	29%	1.24
FEFO 4.2 25	351	647.20	1417.97	0%	0.02
FEFO 4.2 50	1326	2543.90	5362.53	0%	0.03
FEFO 4.2 100	5151	10069.03	20674.93	0%	0.10
FEFO 4.2 150	22651	22695.6	79271.73	0%	0.41

Table 5: Test results LEFO model

Model	#variables	#constraints	#nonzeros	Gapsize	Solving speed
LEFO 1st 25	3176	18823.23	54582.37	30%	0.27
LEFO 1st 50	12601	137648.17	405824.27	30%	2.06
LEFO 1st 100	50201	1050298.30	3123234.23	30%	13.31
LEFO 1st 150	Memory too small				
LEFO 2nd 25	3176	3821.13	12740.13	0%	0.13
LEFO 2nd 50	12601	15138.40	61096.17	0%	0.48
LEFO 2nd 100	50201	60249.60	327030.20	0%	3.52
LEFO 2nd 150	112801	135337.00	922908.20	0%	15.60
LEFO 4.1 25	1926	2573.23	7726.50	30%	0.06
LEFO 4.1 50	7601	10148.17	30868.00	30%	0.21
LEFO 4.1 100	30201	40298.30	123328.10	30%	0.80
LEFO 4.1 150	67801	90448.27	277259.50	30%	2.10
LEFO 4.2 25	2526	2572.20	10252.03	0%	0.08
LEFO 4.2 50	10051	10143.90	51141.17	0%	0.26
LEFO 4.2 100	40101	40269.03	287165.60	0%	1.20
LEFO 4.2 150	67651	67845.60	765571.70	0%	3.35

5 Conclusions and recommendation on further research

In this section the conclusion of this article will be given and recommendation on further possible research.

5.1 Conlusions

The main purpose of this article was to use three well know formulations of the classic lot-sizing problem to find formulations for the FIFO (first in first out), LIFO (last in last out), FEFO (first expired first out) and LEFO (last expired first out) models. As can be seen in chapter 4, for the first and second formulation all the models are found. For the third formulation none of the models are found, this is because the third formulation is allready formulated according to an order.// The adjustments made to the second standard formulation are the best formulations for the FIFO and LIFO model. The adjustment to the second standard formulation with use of Theorem 2 is the best FEFO formulation. The adjustment to the first standard formulation with use of Theorem 3 is the best LEFO formulation. If there is no information about the optimality propertys then an adjustment to the second formulation is the best for the FEFO and LEFO model.

5.2 Recommendation on further research

In chapter 2 the characteristics of lot-sizing problems are spoken of. The most important characteristic that can be taken into account for further research is the capacity. There could be a restriction upon the number of items procured in a period or on the number of items in inventory. In this article the demand is satisfied from only one period, for example by procurement in the previous period. When there are restrictions on capacity this is no longer the case. It is possible that the demand in the current period is higher than the inventory capacity in the previous period, then it is no longer possible to fulfill the demand in the current by only procurement in the previous period. This is what is makes more difficult to find formulations for capacitated lot-sizing prblems.

Furthermore it could be useful to investigate formulation for multiple products and multiple suppliers.

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