The Concept of Infinity

From Zeno to Hume

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Abstract

The concept of infinity in modern mathematical analysis, denoted by '∞,' is a foundational cornerstone in various disciplines, including economics, engineering, finance, and physics. Despite its pervasive application, the profound question of whether the mind can genuinely comprehend the notion of infinity is often overlooked. This thesis seeks to elucidate the intellectual and historical pathways through which the mind conceives the concept of infinity. By employing the framework of metaphysical and mathematical concepts of infinity as outlined by Moore (2019), this study will explore these concepts in depth. Furthermore, it will address the criticisms and challenges associated with the concept of infinity.

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I

Introduction

Infinity serves as one of the concepts underpinning theories and models, from natural to social sciences. Modern scientific models are mostly constructed in frame of the small is approximated by the infinitesimal, or the large is approximated by the infinite, to obtain computational tractability (Oppy 2006, 150-151). In classical physics, for example, Newton's laws of motion rely heavily on infinitesimal calculus to describe the continuous nature of motion and change. The debates in cosmology regarding whether space and time are finite or infinite, have persisted for thousands of years, continuing up until the advent of Einstein's theory of relativity, which introduced the possibility of finite but unbounded space.[1] Similarly, in neoclassical economics, infinity is employed into calculus to analyze utility maximization, which are central to rational choice theories that still dominate microeconomics.[2] Along with financial markets, the concept of infinity is implicitly used in models involving continuous compounded interest rates, which are essential for pricing a wide range of financial instruments and for understanding the growth of investments over time. Thus, it seems infinity is not merely a philosophical and mathematical abstract notion anymore, but a crucial element that facilitates the advancement in many modern disciplines. However, throughout pre-modern history, philosophers and mathematicians have grappled with the concept of infinity in dilemmas.[3] The concept of infinity has been predominantly shaped by two distinct clusters of concepts, each offering a different perspective on conceiving infinity. These clusters can be broadly categorized into metaphysics and mathematics (Moore 2019, 1-12). The metaphysical concept highlights the unity and completeness of infinity, while the mathematical concept emphasizes the plurality and boundlessness of infinity. Besides these two concepts, Aristotle provided a third category on studying infinity, also significantly contributing to development from the metaphysical concept to mathematical concept. Although each category provides a unique lens to explore infinity, they still face the paradox and question:

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[1] Debates about infinities in the context of space and time have often focused on the problems of large and small infinities. For instance, they pose questions about the unbounded length of straight lines, the existence of an infinite number of past and future days, and the conceptualization of space and time as being composed of points and instants.

[2] Rational choice theories are a significant part of decision theory studies in economics. The decision theory analyzes the processes by which individuals and firms make choices under conditions of certainty and uncertainty. Topics within this field include game theory, as introduced by Nash (1953), and Bayesian decision theory, as proposed by Wald (1949). Additionally, the concept of infinite decision theory suggests the possibility of infinite utilities. Well-known paradoxes, such as the St. Petersburg Game and the Two-Envelope Paradox, raise questions about the inclusion of infinite utilities in decision theory. These paradoxes have been discussed by Samuelson (1977) and Menzies and Oppy (1994, 185-196).

[3] In this paper, the discussion will focus on the pre-modern development of intellectual concepts of infinity, particularly during the Greek and early modern epochs.

How can the mind grasp the notion of infinity?

This thesis aims to illuminate the historical pathways through which the mind perceives the infinite, offering multiple perspectives of philosophers and mathematicians.

This paper will be divided into four chapters, focusing on the historical development of the concept of infinity. The first three chapters will trace the evolution of the metaphysical concept of infinity, typically associated with the infinitely large and magnificent, to a mathematical concept encompassing both the infinitesimally small and infinite divisibility. This transformation represents a significant intellectual debate surrounding various problems related to infinity.[4] Additionally, the final chapter will explore the different criticisms raised by empiricist thinkers regarding the paradoxical nature of infinity. Chapter II starts with the earliest perspective on infinity, which is in metaphysical discussions during Greek epoch. The key intellectual development, like the Zeno's paradoxes, unintentionally unveiling the early mathematical perspective on infinity, will be introduced. Then, Chapter III discusses the important transition role played by Aristotle, from a metaphysical into a mathematical concept. His responses to Zeno's paradoxes, which systematically states the mathematical concept of infinity within his influential distinction, *actual infinity,* and *potential infinity* will be explained. Along with the development of Aristotelian distinction, Aristotelian concept of *continuity* will be introduced and discussed intensively, which is key step in development of Aristotelian perspective on infinity. The first part of Chapter IV contains an overview of Greek geometricians who are regarded as origins of the mathematical perspective on infinity. The second part of Chapter IV will introduce Leibniz's further contributions on continuity after Aristotle, and his infinitesimal calculus. The criticisms targeting both the metaphysical and mathematical developments of the concept of infinity will be discussed in Chapter V, focusing on the renewed empiricist perspectives of Berkeley and Hume. Particularly, Humean strict finitism, which discusses infinity from the standpoint of empiricist epistemology and philosophy of mind, will be the main context of this chapter, revealing a unique critical perspective on the intellectual developments of infinity. Thus, the first part of the chapter will examine Hume's attack on mathematical infinity, including his theory of minima sensibilia, which is inherited from Berkeley's views. The second part will discuss Hume's criticism of metaphysical infinity.

[4] According to Oppy (2006, 7-19), there are three kinds of problems, such as problem of large and small infinites, problem of denumerable and nondenumerable and problem theoretical and physical.

II

The Metaphysical Concept of Infinity

II.1 The Background of Metaphysical Infinity

Philosophical discussions on infinity about the earlier Greek are about the distinction between *'peras'* and *'to apeiron'*, which have the general meanings of *'the what makes finite'* and *'the what is infinite'*.[5] These concepts are firstly distinguished and applied by Anaximander, a Greek philosopher from the Milesian school. The 'to apeiron', as underlying changeless substratum, is principal source of all natural processes. This undifferentiated, eternal, and boundless 'to apeiron' was not a physical substance, but a metaphysical entity that transcended the familiar elements of earth, air, fire, and water, which were seen by earlier Greek thinkers as the fundamental building blocks of the world:

> It is with reason that they all make [the infinite] a principle; for it can neither exist to no purpose nor have any power except that of a principle. For everything is either a principle or derived from a principle. But the infinite has no principle - for then it would have a limit. Again, it is ungenerated and indestructible and so is a principle. For what comes into being must have an end, and there is an end to every destruction. Hence, as I say, it has no principle but itself is thought to be a principle for everything else and to govern everything . . . And it is also the divine; for it is deathless and unperishing, as Anaximander and most of the natural scientists say. (Barnes 1987, 75)

As contrary to the 'to apeiron' principle, every being must be 'peras' with ends, like destruction, disharmony, and imbalance, which are regarded as chaotic badness by Anaximander who believed all 'peras' must return to 'to apeiron'. Although, Pythagoras, a Greek philosopher and mathematician, adopted the distinction between 'peras' and 'to apeiron', and the interplay between them, the 'to apeiron' symbolizes the chaotic badness, whereas the 'peras' symbolizes the definite goodness of structured principle. Natural numbers aid 'peras' in exerting influence over 'to apeiron', thereby creating a musico-mathematical whole. For example, Pythagoras suggested that the planets and celestial bodies, in their geometric shapes, emit a 'music of the spheres' as they orbit. This inaudible music reflects the cosmic order, with celestial harmony governed by the same natural number ratios that define musical harmony. Therefore, natural numbers act as a bridge between the 'peras' and 'to apeiron', orchestrating an unseen

[5] These interpretations of both 'peras' and 'apeiron' originate from the *Philebus*, where Plato later categorizes them into two classes of metaphysics (Letwin 1981, 188).

symphony. The world of 'peras' is fundamentally composed of natural numbers and their ratios, existing within and juxtaposed against the void space of 'to apeiron'.

However, existence of $\sqrt{2}$ posed a serious problem for the Pythagoreans, who believed that all aspects of the universe could be explained in terms of natural numbers and their ratios.[6] It is approximately equal to 1.41421356...., and this decimal expansion continues infinitely without repeating, which is categorized later as an irrational number. This characteristic of $\sqrt{2}$ demonstrates a basic form of mathematically infinity, which was also sharply developed in early Greek mathematicians. Their contributions will be discussed in Chapter IV.

II.2 Parmenides

Metaphysically infinity was firstly clearly expressed by Parmenides of Elea, who was a foundational member of the Eleatic school. Parmenides fundamentally disagreed with the Pythagorean conception of void. Parmenides viewed the void as any reference to 'what is not', which can also be interpreted as 'non-being'. He asserted that speaking of the 'non-being' was illogical and inherently incoherent. If something genuinely 'is not', it cannot cause effects, it cannot be interacted with, and it certainly cannot be the foundation upon which the 'peras' world operates. Therefore, his doctrine asserts that all discussions about the world must only involve what can be firmly said to 'be', which he encapsulates in his famous dictum, 'what is, is'. This 'be' posited that all of reality is eternal entity, which he referred to as 'the One'.[7] Like 'to apeiron' from Anaximander, Parmenides envisioned 'the One' as an unchanging and undivided whole without any differentiation. This view led him to introduce dichotomy of the 'Way of Truth' and the 'Way of Opinion'. The 'Way of Truth' reveals that the true nature of 'the One'. In contrast, the 'Way of Opinion' deals with the world of apparent illusions perceived through our senses. However, although there are many connections between Parmenides and Anaximander, 'the One' of Parmenides is finite and defined in a perfect, complete sphere, instead of the boundless and incomplete 'to apeiron' of Anaximander. This holistic notion of infinity can be seen as first clear appearance of metaphysically infinity, which encompasses everything that exists as a part of a unified whole without division. Now, we can formulate Greek definitions of metaphysically infinity:

> Transcendence: Metaphysically infinity is constituted through the distinction between reality and appearance. Both 'to apeiron' and 'the One' can be viewed as transcended reality, against multifarious and mutable illusions in appearance.

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[6] See Muller (2022, 2-3) and Moore (2019, 18-21) for a more elaborate discussion.

[7] Xenophanes of Colophon, a teacher of Parmenides, held views like Parmenides, proposing that the first principle was a unified entity as one. However, he argued that this principle could not be described as finite or infinite, nor as changing or changeless (Barnes 1987, 96-97).

> Completeness: Metaphysically infinity revolves around the complete and perfect unity of being. For Parmenides, infinity is not about endless extension but about being all-encompassing and self-sufficient.

Furthermore, Parmenides revolutionized the understanding of infinity by introducing it with a reformulated concept of continuity, which subsequently inspired many thinkers.[8] In Greek literature before Parmenides, 'suneches' primarily described uninterrupted activities over a period, signifying temporal extension. However, for Parmenides, it became an ontological feature rather than an activity (Sattler 2019, 7). Along with his metaphysical view of infinity, true continuity is a characteristic of what truly exists, which he termed 'to eon.' By excluding any temporal activities and divisions, continuity is consistent and unchanging, aligned with the present and unified nature of 'the One.' Therefore, although infinity and continuity were not formally correlated until Zeno's paradoxes, Parmenides firstly reveals that they share many common grounds together (Sattler 2021, 8-9):

> Homogeneity: Continuity is not divisible because it is entirely homogeneous. This homogeneity is a prerequisite for continuity. Thus, the continuity requires the absence of any internal differences, such as temporal, spatial, ontological, and logical divisions.

> Compactness: Continuity is connected without any gaps. This connection ensures that there is nothing within being that is not being, maintaining complete interconnectedness.

II.3 Paradoxes

Zeno of Elea, another prominent figure within the Eleatic school, ardently supported the doctrines of his mentor, Parmenides, and advocated for the metaphysical view of infinity, and the indivisible continuity. The main aim of his paradoxes is to challenge the concept of atomism, which posited the world comprised of infinite, perpetually moving atoms that aggregate to form distinct entities. These atomistic philosophers, like Democritus, embraced a genuine plurality of existence, suggesting an infinitely many indivisible minima in one finite whole.[9] However, through the paradoxes, Zeno unexpectedly underscored the philosophical conflict between the notion of definite finitude and indefinite plurality, which demonstrated as impossibility of completing an infinite number of steps in a finite time, or impossibility of infinite divisibility in a continuous one. Both paradoxes of *'the runner'*

[8] The intellectual development of the concepts of continuity, particularly in the realms of geometry and topology, is closely intertwined with discussions of infinity. Therefore, studying the concept of continuity necessitates a thorough understanding of topics such as infinitesimally small and infinite divisibility.

[9] The concept of infinitely many indivisible minima can be expressed as infinitely many indivisible elements, such as points, lines, and planes (Levey 2021, 87-88). The idea that infinite assemblies of points compose space and time is an important discussion in the context of infinitesimally small and continuum (Oppy 2006, 91-99). This idea will be extensively discussed through the perspectives of Galileo and Leibniz in Chapters IV.

and *'the arrow'* are in form of 'for argument's sake,' providing logically valid and emerging philosophical arguments, which assuming assertions are true, and then conducting arguments by following the assertions. The principle is in proof by contractions used to establish the truth of assertions by demonstrating that the assumption of its negation leads to a contradiction.[10] Thus, Zeno's paradoxes were designed to illustrate how any attempt to traverse these divisions leads to logical absurdity, thereby opposing the atomism belief in plurality of existence. The paradox of runner was demonstrated by Aristotle:

> The first asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal. (Aristotle *Physics*, 239b11)

Paradox of runner asserts that before reaching a destination, one must cover half the distance, then half of what remains, continuously halving the distances. Then the paradox can also be expressed in an infinite sequence in $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots\}$, where n is nth term. Zeno argued Achilles firstly travelled to the mid-point at $\frac{1}{2}$, then to $\frac{1}{4}$, then to $\frac{1}{8}$ and so on, which indicates travelling an infinite number of finite distances. If the belief in infinite divisibility is true, then Zeno concludes that motion is impossible because it requires an infinite number of finite steps, which ostensibly could never be completed in finite time. Therefore, it is absurd conclusion for mighty Achilles to take an infinite time to travel from A to B.

The paradox of arrow was described by Aristotle:

> The third is … that the flying arrow is at rest, which result follows from the assumption that time is composed of moments …. He says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now, the flying arrow is therefore motionless. (Aristotle *Physics*, 239b30)

The paradox of the arrow argues that motion is impossible, based on the assumption of plurality that time is composed solely of indivisible moments. Here, Zeno posits that an arrow in flight occupies a specific space at a specific moment in time. This indicates specific moment in time is a distinct 'now' in which the arrow cannot move, because it occupies the same space at that moment. Therefore, if all moments in time are such indivisible instants, then the arrow is always at rest during each of moments, leading to the absurd conclusion that it never actually moves.

[10] Similar arguments can also be found in two additional paradoxes: Achilles and the Tortoise and the Stadium (Oppy 2006, 92-94) and (Moore 2019, 3-4). For a modern analysis of the paradox of Achilles and the Tortoise in the context of sequences and series, see Muller (2022, 5-10).

III

Aristotle

III.1 The Background of Actual and Potential Infinity

Aristotle, the eminent philosopher influencing developments of many modern disciplines, had a methodology that includes importance of empirical observation, particularly evident in his natural philosophy where he systematically cataloged and examined the natural world. His empirical philosophy also leads him into a significant different approach to the paradoxes of infinity, which was markedly distinct from the one of his predecessors. Aristotle principally argued that infinity does not exist as a complete entity, but as a potential within finite boundaries. He adhered to a finitism worldview, positing that everything in the world, such as bodies, geometric objects, and numbers, is finite. However, he identified certain processes that could be endlessly iterated, introducing the concept of 'potential infinity'. Furthermore, he juxtaposed potential infinity with 'actual infinity', which would imply being completed in entirety, without any process in time. Before introducing how Aristotle addressed Zeno's paradoxes by applying the concept of coherent potential infinity, instead of the incoherent actual infinity, it is essential to introduce Aristotelian continuity. His continuity is closely related to the development of potential and actual infinity and is crucial for fully grasping these ideas.

III.2 Aristotelian Continuity

Aristotle provided two characterizations of continuity:

> Continuous are those things whose limit, at which they touch, is one.
>
> Continuous is that which is divisible into what is always further divisible.
>
> (Sattler 2021, 16)

The first is a relational, two-place account where the first object is continuous with the second object if their limits touch and become one. This first characterization seems valid in mathematical realm, which follows rule of *'the parts prior to the whole.'* This rule describes the nature of a continuum when we start by considering individual elements and seek to understand how a continuum can be constructed from these basic units. For example, lines that touch are considered one if they do not intersect or form an angle. Although Aristotle tried to illustrate the first characterization with physical examples, such as different objects glued together to form a single entity while maintaining some distinct characteristics, two solid objects touching do not necessarily merge into one continuous entity.[11] They do not become homogeneous in all respects unless their boundaries form a unified whole. When Aristotle formulated the first characterization and incorporated Parmenides' thought, stating that a continuum was homogeneous, he highlighted the issue of the difference between being continuous and being contiguous in the physical realm.

The second is a one-place that implies any point within a continuum can be divided, and each resulting part will also be a continuum that can be further divided. This property ensures infinite divisibility in a continuum, where the parts of a continuum are of the same whole, following the rule of *'the whole is prior to the parts.'* Aristotle first argued that a continuum can be divided at any point, and all such divisions are equally valid. These arbitrary divisions are epistemic divisions that do not rely on physical differences within the continuum.[12] For example, a plank is regarded as continuous and can potentially be divided into 1-meter or 2-meter intervals as base units, measured with a ruler. Similarly, mathematically speaking, a line can be divided into arbitrary parts as desired. Thus, infinite divisibility is valid in both the physical and mathematical realms, without considering homogeneity, which is presupposed as a whole prior to any possible parts. However, Aristotle explicitly distinguished between the infinite divisibility stated above and being infinitely actually divided. This distinction gives rise to the different concepts between potential and actual infinity. While making infinite divisibility epistemically possible in a continuum, Aristotle also established restrictions for actualizing infinite divisibility into being infinitely divided:

> *From conceptual to physical:* Aristotle explains there are two ways to actualize parts from a continuum, either conceptually or physically. Division from conceptual actualization occurs the example of measuring a plank with a ruler to determine its length. This can involve mentally marking segments of the plank. These marks do not alter the continuum; they can be arbitrary and adjusted anytime, and the plank always remains whole. Division from physical actualization, however, involves cutting the plank into actual pieces. When the plank is physically divided, the original whole is transformed into two individual continuous wholes. In this state, the parts are no longer parts of the original continuum but new, separate entities. Thus, parts remain true parts of the original whole only in a conceptual sense. Once physical division occurs, the continuum is fundamentally altered, and the original whole is lost.
>
> *From possible to impossible:* Theoretically, continua appear to be infinitely divisible, meaning one can keep dividing them conceptually without end. However, this infinite divisibility is restricted in practical terms. For instance, when dividing a plank, there is a limit to how small

[11] See further details of the example in Aristotle *Physics* (227a10-17) and Sattler (2021, 16-17).
[12] See Aristotle *Physics* (231a21-29 and 231b15-18) and Sattler (2021, 17-19).

the pieces can get before it becomes impossible to cut them further with any available tools. Therefore, the physical division cannot continue infinitely.

From temporal to simultaneous: Aristotle firstly rejected the simultaneous actualization of infinite divisibility both theoretically and practically, which implies dividing a continuous whole all at once. He argued that, in practice, not all possible parts of a continuum can be actualized simultaneously because dividing a continuum into infinitely many parts would result in overlapping divisions, necessitating a choice of one division over another. Theoretically speaking, simultaneous actualization encounters Zeno's paradox of dichotomy. In contrast, Aristotle introduced the concept of temporal actualization. The potential for infinite division in a continuum cannot be fully realized, even over an extended period. This infinite divisibility always leaves some potentiality unactualized, which is not completed through an unlimited process of time. Thus, while any potential division can be actualized, it does not imply that all can be actualized either over a limited time or simultaneously.

Through the analysis of Aristotelian continuity, some key ideas of both actual and potential infinity have been fully developed:

(1) If continuum is assumed to be as one-place, it follows the rule of 'the whole is prior to the parts.' This characterization leads clear distinction between infinity of addition and infinity of division, which also makes infinity of division dominates study for Aristotle and later philosophers.

(2) Divisions are allowed in Aristotelian continuum, in either potential infinite divisibility or actual infinitely divided.

(3) Potential infinite divisibility could be actualized into actual infinitely divided but facing the restrictions and paradoxes stated above.

(4) To avoid the restrictions and paradoxes, potential infinite divisibility is utilized, instead of actual infinitely divided.

III.3 Aristotelian Responses to Zeno's Paradoxes

In the paradox of runner, Zeno illustrated it is impossible for Achilles to complete infinitely tasks, which were divided as infinite sequence of halfway intervals everywhere from the path A to B, at the same time. Aristotle criticized the paradox of runner was falling into mistake of applying the actual infinity, which assumes that the infinite sequence needed to be fully actualized simultaneously for motion to occur. In addressing the paradox, he argued dividing the path into infinitely subintervals are constructs of conceptual analysis rather than necessities of physical motion. By introducing potential infinity, he suggested the continuous path could be divided anywhere at any time, instead of being divided everywhere at the same time. Through this reasoning about time, Aristotle argues that the actual infinity of time from start to finish is distinct from the discrete interval divided simultaneously into many subintervals. He suggests that if we consider the discrete interval as needing to be actualized one by one, it would imply that the runner must perform a specific action at the end of each to differentiate one subinterval from the next, like stopping at each midpoint, which would render the run in both theoretically and practically impossible. Therefore, Aristotle concludes that while it is possible to conceptualize a continuous run as being divisible into an infinite sequence of halves, these divisions are theoretical, and do not reflect the finite and physical process of running, which always remains a continuous, undivided action. Aristotle's approach to resolving the paradox of the arrow involves criticizing the foundational assumptions about time and motion posited by Zeno. The paradox of the arrow suggests that an arrow in flight is motionless at any single instant, as it does not occupy more than one position at any one moment, which would seem to imply that motion is impossible. Aristotle refutes this by arguing against the concept of time being composed of indivisible instants. Instead, He again asserts that time, like any other magnitude, is not made up of indivisibles but is instead continuous and can be divided potentially into an infinite number of intervals. Therefore, by rejecting the existence of indivisible moments, Aristotle posits that at any given time, there is always a finite interval during which the arrow is not only occupying space but also changing location, thereby exhibiting motion. If time is not made up of discrete, indivisible instants, the arrow does indeed fly, and the paradox dissolves under the weight of false premises.

IV

The Mathematical Concept of Infinity

IV.1 The Background of Mathematical Infinity

The concept of infinite divisibility in a continuum, as proposed by Aristotle, plays an influential role in addressing Zeno's paradoxes. Aristotle's approach potentially allows for the infinite division of a continuum, rather than actual infinite division. Furthermore, Greek mathematicians, were also among the first to adopt and reflect upon this notion of infinite divisibility. Their mathematical perspective aligned closely with the Aristotelian continuum, asserting that any segment of a continuum could be divided infinitely. However, the mathematical perspective was based on actual infinite division suggests that if a continuum, like a line, continues to be divided into infinite segments until nothing remains. Thus, the Greek mathematicians did not regard the continuum as merely a collection of indivisible points. Although each of these infinitesimal parts has no extension individually, they can be infinitely divided and collectively constitute an extended whole. The mathematical perspective was clearly underscored by a sophisticated understanding of geometry. This was demonstrated through the method of exhaustion, which was found in the work of Eudoxus and Archimedes.[13] For instance, Archimedes used this method to determine the area of a circle by inscribing polygons within it and refining their approximation (Geometric demonstration of the method of exhaustion is presented in Figure 1 on the next page.):

(1) Inscribed Polygon
   1.1. Let C be a circle with radius r.
   1.2. For each natural number n > 2, let P_n be a regular n-sided polygon inscribed in C. P_n can be divided into n congruent triangles with base l_n and height h_n.

(2) Area of the inscribed polygon
   2.1. Area of each triangle is $\frac{1}{2} l_n h_n$
   2.2. Area of P_n is $\frac{1}{2} n l_n h_n$

(3) Approximation
   3.1. When n approaches infinity, nb_n equals the circumference of $C = 2\pi r$.
   3.2. h_n equals to r.

[13] Refer to Kline (1972, Chapters 3 and 4) and Moore (2019, pp. 27-30) for additional graphical demonstrations.

(4) Area of circle in approximation

4.1. As n becomes infinite, $\frac{1}{2}nl_nh_n$ equals to $\frac{1}{2}\cdot 2\pi r \cdot r$, through substitutions from the questions above.

4.2. Area of circle is approximated into πr^2.

The key point made is that as the number of sides of the inscribed polygon increases toward infinity, the polygon P_n constitutes a deformed circle C', which is increasingly approximates the circle C:

Assume C *with* $r = 1$

Circumference of C is 2π. Central angle of each triangle formed by the polygon sides is $\theta = \frac{2\pi}{n}$. Thus, area of each triangle is $\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}$, formed by base of $2\sin\frac{\pi}{n}$ and height of $\cos\frac{\pi}{n}$. Total Area of the $C' = n \cdot (\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n})$ approximates to area of C, while n increases infinitely.

Assume $n = \{4, 6, 8, 10, \dots\}$

The area of $C' = \{2, 2.59807, 2.82842, 2.93892, \dots\}$ approaches to the area of C in $\pi = 3.14159$.

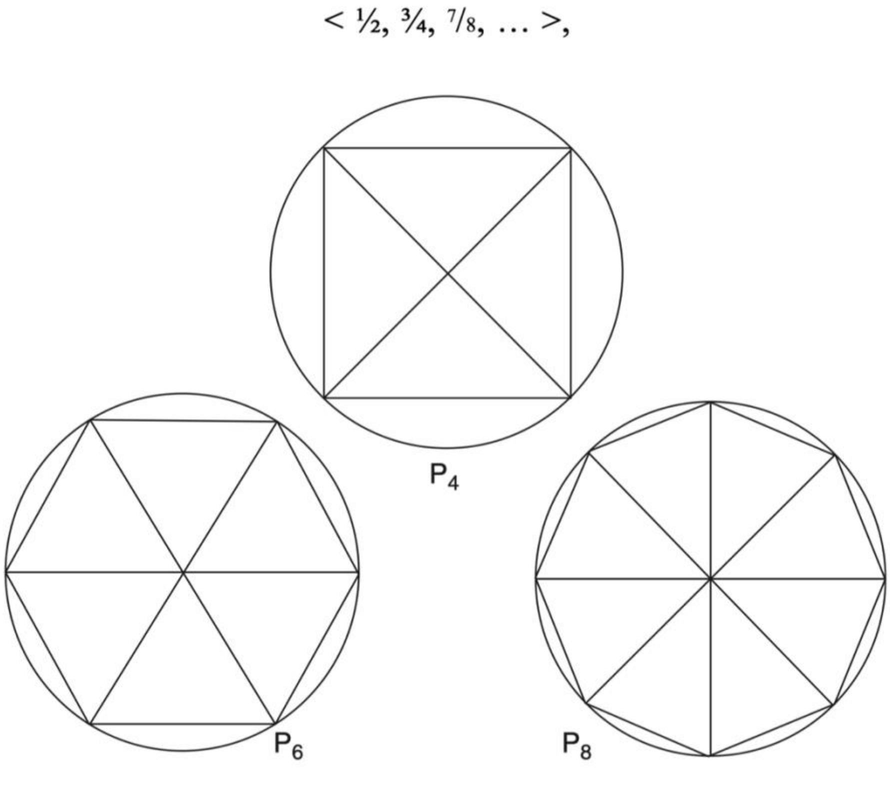

Figure 1: Method of Exhaustion. Source: Moore, *The Infinite* (2019), 28.

Through the method of exhaustion, infinite divisibility was allowed in geometrical magnitudes. The circle was approximated by polygons with an increasing number of sides.[14] However, the circumference of the circle can only be expressed as infinitely approaching the perimeter of the polygon. This raises the question: is C'equivalent to C? Alternatively, one might ask: is it legitimate to assume a polygon with infinitely many infinitesimally small sides? If we assume such a polygon, do its sides have a length of zero or not? This dilemma also touches on *mathematical generalization*.[15] Its perspective posits that there is always one unit larger or smaller than the previous unit within finite quantities, such as natural numbers. Therefore, it is logically valid to generalize finite quantities to the concept of infinity because there are infinitely many successive finite units.

The questions related to mathematical generalization were inherited by later thinkers, such as Leibniz, who stands as one of the towering figures in the history of philosophy and mathematics. Among his most significant achievements are the formulation of the *law of continuity* and the development of *infinitesimal calculus*. These advancements are built upon the Greek concept of mathematical infinity and Aristotelian actual infinity. Besides, Leibniz's development of infinitesimal calculus represents a significant application of these concepts, laying the foundational stones for modern mathematical analysis. The topic of the IV.2 exploits the 'law of continuity,' which continues the Aristotelian discussion of continuity and underpins his entire approach to infinitesimal calculus. The law of continuity states that "Nothing takes place suddenly, and it is one of my great and best confirmed maxims that *nature never makes leaps*."[16] By complementing the Greek mathematical infinity, the law states if there is a supposed transition in continuous motion, we may take it for granted that general rules will always succeed, and that what succeeds in the finite will also succeed in the infinite. This principle was crucial for justifying the use of infinitesimals in calculus, as it posits that the behavior of quantities remains consistent whether they are finite or infinite. The topic of the IV.3 introduces Leibnizian infinitesimal calculus, addressing the dilemmas like those faced by Greek mathematicians. Thus, this section delves into the challenges encountered in the application of infinitesimal calculus and highlights Leibniz's response to them.

IV.2 Law of Continuity

In Leibniz's dialogue *"Pacidius Philalethi,"* Leibniz firstly draws on the Aristotle's relational, two-place characterization of continuity. It posits the distinction between continuous and contiguous, where contiguous things have touching boundaries without gaps, and continuous things are connected by a

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[14] In a similar discussion found in Aristotle *Physics* (207b27-34), he noted that the subjects of geometric study were consistently finite lines, divided at a limited number of points, rather than infinite space investigation.
[15] See more paradoxes and examples given by Moore (2019, 68-72).
[16] As Leibniz explains in his *New Essays on Human Understanding*, and as mentioned by Look (2020, *Stanford Encyclopedia of Philosophy*, 3.6).

common boundary. If the contiguous rule of 'the parts prior to the whole' is assumed, then this applies to changes in qualities and place, indicating that natural transitions involve intervals of motion, space, time, and matter, separated by boundary points. Without worrying Parmenides' homogeneity, these intervals touch at their boundaries but do not coincide, which can be seen as contiguous composition of the continuum, but not strictly continuous. Furthermore, Leibniz also asserts that if these boundaries are indivisible elements, not extended parts, then two conclusions are developed in the context of two-place characterization of continuity.

The impossibility of existing any gaps in continuum

Leibniz discusses the problem of assigning positions within a continuum, assuming a two-place characterization of continuity. For example, if a moving point x travels continuously from position A to position C, then at the next moment, x must occupy a position immediately after A. This fundamental and universal assumption is usually called *principle of general order*, in Leibnizian continuity, which must always remain valid.[17] Therefore, Leibnizian continuity rejects any leaps of indivisible elements, even if the skipped space is considered infinitely small. He treats such leaps as excruciating:

(1) The point x instantaneously jumps to the distant location without passing through the intermediate points.

(2) The point x simultaneously occupies all intermediate positions, effectively expanding to fill the entire space between A and C.

Leibniz later reformulated the principle of general order into abstract terms, which became the foundation for his infinitesimal calculus:

> "When the difference between two cases, in *what is given* or presupposed, can be made less than any given quantity, it is necessary likewise that the difference in what is sought, or in the consequences that result from what is presupposed, is made less than any given quantity. Or to speak more commonly: *when the cases (or givens) continually approach one another, so that one finally passes over into the other, the same must happen in the corresponding consequences or results (or what is sought)*." (Berlin: Akademie-Verlag 1923, 6.4.2032)

The incoherent existence of indivisible elements in continuum

[17] The principle of general order is a core concept of Leibniz's law of continuity (Levey 2021, 148-151).

Leibniz's treatment of neighboring intervals as 'minima' brings forth the debate on whether there are minimal discrete quantities serving as pre-assigned units of measure in the continuum.[18] One of the most famous arguments in this context comes from Galileo, who posited that decomposing a continuum into an infinite powder of points is not inevitable.[19] In *"Pacidius Philalethi,"* Leibniz counters this claim using the *'diagonal paradox'* to argue that the continuum cannot be composed of indivisible elements, such as points.[20] In the diagonal paradox, Leibniz presents a geometric demonstration involving a rectangular parallelogram LNPM and its diagonal NM. He explains that the number of points on the sides LM and NP of the parallelogram is equal, as they can be connected by parallel lines. These parallel lines also intersect the diagonal NM, thereby placing the points on the sides into a one-to-one correspondence with the points on the diagonal. This arrangement suggests an equality between the lengths of LM, NP, and NM. However, in any parallelogram, the diagonal NM is longer than either of its sides, each of which constitutes only a part of the diagonal. This leads to the paradoxical conclusion that the entire length of the diagonal would be equal to only a part of itself, contradicting Euclid's principle that the whole is greater than the part. Therefore, Leibniz concludes that continuous lines cannot be composed of infinite points.

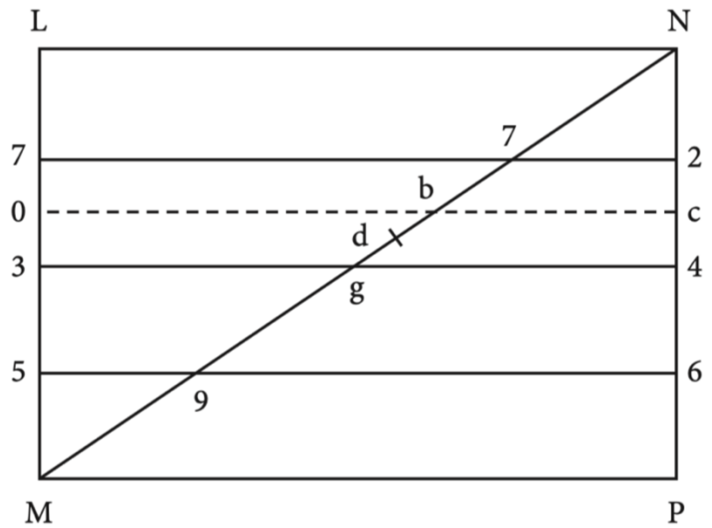

Figure 2: Diagonal Paradox. Source: Levey, *The History of Continua: Philosophical and Mathematical Perspectives* (2021), 130.

In the face of the diagonal paradox, the theory based on the relational, two-place characterization of continuity and indivisible elements in the continuum encounters a dead end, much

[18] The argument of 'minima' is like the argument of the atomistic philosophers stated in Chapter I.
[19] Galileo accepts that any actual continuum will contain an infinite number of indivisible elements (Levey 2021, 89-93).
[20] For more primary sources, see those provided by Levey (2021, 127-135).

like the one faced by Galileo. Consequently, the concept of one-place characterization, ‘the whole is prior to the parts,’ reappears with Leibnizian new hypotheses. Leibniz introduces these two additional hypotheses designed to avoid the issues inherent in the theory of indivisible elements.

Hypothesis of mode

All forms of indivisible elements such as points and lines, are not intrinsic elements but rather modes that are results of dividing the continuum. For example, Leibniz’s character Charinus in *“Pacidius Philalethi,”* discusses that the indivisible points only come into existence through the act of division and do not pre-exist in the continuum. This view reinforces the Aristotelian one-place characterization of continuity following the rule of ‘the whole is prior to the parts.’ Leibniz proposed that a continuous whole possesses indeterminate parts that can be divided in infinite ways. He illustrates continuity with example of the arithmetical unit, which underscores the one-place characterization. The number 1 can be divided into a finite or infinite number of fractional series, such as:

$$\{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\}$$

Or

$$\{\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\cdots\}$$

Or

$$\{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\}$$

Leibniz argues that it is misleading to consider any division as the true subdivision of the unit. Instead, he asserts that the unit does not inherently possess a determinate substructure; rather, its parts are only potential. Thus, Leibniz’s perspective on continuity asserts that the parts of a continuous whole are indeterminate and can be conceptualized in various ways. This view allows for the existence of many possible pairs of co-integrating parts, if they overlap at least at a boundary, without being fixed in a single division. This epistemic approach reflects the Aristotelian concept of potential infinity, which also allows for conceptual actualization. Consequently, Leibniz argues that a continuum cannot be composed of discrete points. Instead, points, lines, and surfaces should be understood as modes of the continuum. The continuum itself is not merely a relation between adjacent parts but a general structural property of a whole, encompassing all possible ways of dividing it into parts.

Hypothesis of non-uniformity

Furthermore, Leibniz's views on the modes of a continuum are intricately connected to his concept of non-uniformity. He describes non-uniformity as a novel structure of motion, employing the metaphor of a folded cloth to illustrate this idea. According to Leibniz, a continuum, much like a cloth, is not constructed from indivisible elements but is instead composed of folds within folds. This conceptualization is vividly depicted in the dialogue *"Pacidius Philalethi,"* where Charinus elucidates that the structure of motion is actually divided into an infinity of different motions, each influenced by external forces at every moment. These external forces are non-uniformly driven by different levels of accelerations, which are expressed in infinitely many extended subintervals densely ordered throughout the full interval. Definite locations can be assigned to the moving body at the boundaries between the adjacent subintervals, marking their ends and beginnings.

Thus, Leibniz contends that it is logically valid to conceptualize a continuum as mode, which treats motion across an interval as divided into finer subintervals of motion, continuing ad infinitum. Consequently, this fractal-like structure serves two critical purposes. Firstly, it eschews the resolution of a continuum into indivisible parts resulting the diagonal paradox, thereby opposing Galileo's concept of the continuum as an aggregate of indivisible points. Secondly, it addresses Zeno's paradoxes by endorsing the simultaneous actualization of infinite divisibility as logically valid. Then, actual infinity appears less problematic for Leibniz. This stance contrasts sharply with the Aristotelian view, which deems such simultaneous actualization in a continuum as both logically and physically absurd.

IV.3 Infinitesimal Calculus

Leibnizian law of continuity can be seen as an early precursor to modern concept of limits in calculus. This principle allowed Leibniz to address the paradoxes risen from actual infinity. By accepting to conceptualize motion as a continuous process involving infinitesimal steps, Leibniz demonstrated that it is indeed possible to move from one extended subinterval to another, as the sum of an infinite series of infinitesimal distances can converge to a finite distance. Leibniz firstly shows this idea through his arithmetic progression transferring the complexity of geometry into a more manageable arithmetical form. In his work, *"De Quadratura Arithmetica,"* infinitesimals are regarded as a convenient shorthand, representing relationships among finite quantities that could be made arbitrarily small. While this is aligned with the Archimedes' method of exhaustion, Leibniz presents an exact expression of the circle's area using an infinite series, in $\{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots\}$ equals to $\frac{\pi}{4}$. A crucial aspect of this Leibnizian approach is interpretation of infinitesimals and infinite quantities as syncategorematic terms.[21] This means that such terms do not denote special categories of non-finite magnitudes but rather

[21] Syncategorematic terms are usually used in language expressions. For example, these are words that do not have a standalone meaning but contribute to the meaning of other words or expressions in a sentence. They typically function as grammatical or logical connectors rather than bearing content on their own.

impose systems of logical and mathematical relations on other terms within a statement. When Leibniz refers to a difference as ‘infinitely small,’ he implies that the difference can be made smaller than any finite value, rather than positing the existence of a fixed infinitesimal quantity. The modern method of using partial sums to address Zeno’s paradox of dichotomy can be seen as deriving from the Leibnizian approach.[22]

Assume s_n expressed as partial sums, for each natural number n > 1:

$$s_1 = \frac{1}{2} = 0.50000000$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = 0.75000000$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.87500000$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.93750000$$

$$\vdots$$

$$s_{20} = 0.99999905$$

$$\vdots$$

$$s_{25} = 0.99999997$$

$$\vdots$$

$$s_n = 1$$

When many terms of the series are added sufficiently, such that n approaches infinity, the partial sums can be made as close to 1 as desired. Thus, it seems logically valid to speculate that the sum of this infinite series is 1. Consequently, infinite addition is ordinarily defined from its finite input by relying on the principle of general order. This method expressed by Leibniz as “the rules of the finite are found

[22] For an analysis of partial sums, see Muller (2022, 8-9).

to succeed in the infinite" is a fundamental principle applied in his development of infinitesimal calculus.[23] In his work *"Cum prodiisset,"* Leibniz employs the method and principle of general order to elucidate expression of 'dy as the element of y' and '$\frac{dy}{dx}$' in his infinitesimal calculus. To illustrate this, he uses a parabola AY described by its equation $x^2 = ay$:

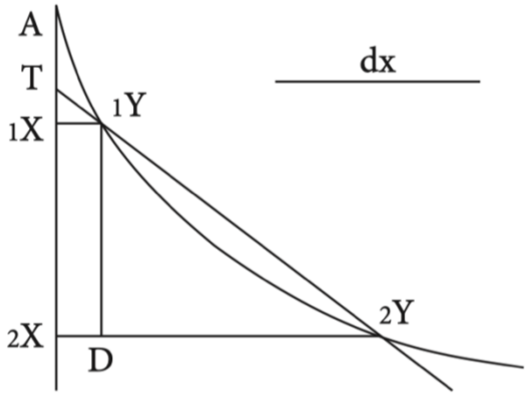

Figure 3: Principle of General Order in Infinitesimal Calculus (In the following steps, $_1X$ will be expressed as X_1). Source: Levey, *The History of Continua: Philosophical and Mathematical Perspectives* (2021), 151.

I. Both dx and dy are defined as finite differences

$$dx = |AX_1 - AX_2|$$

$$dy = |X_1Y_1 - X_2Y_2|$$

II. Derivation of $\frac{dy}{dx}$

$$x^2 = ay \xRightarrow{reformulate} y = \frac{x^2}{a}$$

Assume y moves with dy, while x moves with dx:

$$y + dy = \frac{(x + dx)^2}{a} = \frac{x^2 + 2xdx + dx^2}{a}$$

[23] The expression from the letter Leibniz wrote to Varignon, along with more primary sources, is provided by Levey (2021, 153-154).

Subtracting y in left side and $\frac{x^2}{a}$ *in right side of the equation:*

$$dy = \frac{2xdx + dx^2}{a}$$

$$\frac{dy}{dx} = \frac{2x + dx}{a}$$

As dx *approaches infinitely small quantity in 0:*

$$\frac{dy}{dx} = \frac{2x}{a}, where\ dx = 0$$

Leibniz applies the finite operations of dx and dy to find the fraction of $\frac{dy}{dx}$. Then, the principle of general order is applied to transform dx into an infinitely small quantity, while still behaving like a finite difference that can be made arbitrarily small. Therefore, the principle of general order supports the transition from finite differences to infinitesimally small differences, allowing the speculation of the infinitude rule through the finitude rule. The dilemma of regarding dx as both zero and non-zero is reminiscent of Archimedes' method of exhaustion, which questions the legitimacy of assuming a polygon with sides that are infinitesimally small, yet neither exactly zero nor non-zero. Faced with this dilemma, Leibniz posits that as dx approaches an infinitely small quantity, it indeed becomes a fictional entity in a transitional state between something and nothing. However, the principle of general order justifies treating infinitesimals as these fictional quantities by arguing that the properties and operations valid for finite quantities should also apply to infinitesimals. Leibniz supports the argument by appealing to the *principle of sufficient reason*, which asserts that every mathematical concept must be governed by reason.[24] Therefore, it suggests that infinitesimals, while fictional and paradoxical, are rational and coherent extensions of finite quantities.

[24] Leibniz's principle of sufficient reason, which states that nothing happens without a reason or cause, is an essential concept in his metaphysics and mathematics. For a basic introduction, refer to Look (2020, *Stanford Encyclopedia of Philosophy*, 3.4).

V

Critique

V.1 The Background of Empiricist Criticisms

Although the syncategorematic terms of Leibnizian infinitesimals imply that it is possible to conceive the mathematical concept of infinity through the rule of finitude, Leibniz rejects the notion of actual infinite quantities or numbers, considering them paradoxical. Instead, he uses infinitely small numbers as a technical convenience in calculus. Therefore, his methodological principle asserts that in continuous transitions, the ultimate case is of the same kind as preceding cases, allowing general reasoning to apply directly to limiting cases. This principle supports the mathematical fictions, such as viewing a circle as an infinite polygon or equality as an infinitely small difference. This Leibnizian philosophy is regarded as one of rationalism, wherein despite experiential and imaginative limitations, humans have an innate, clear idea of the infinite.[25] This idea is fundamental to the understanding that infinity can be arrived at by a priori means through the exercise of the sufficient reason, characteristic of mathematics.

However, many thinkers still criticize Leibnizian infinitesimals. For example, empiricists insist that it is only through experience that we can come to know such truths about infinity. Berkeley, an Anglican bishop of Cloyne, emerged as one of the most prominent critics of infinitesimal calculus during its early development. One of Berkeley's primary criticisms targets Leibnizian instrumental use of infinitesimals, treating them as mere technical conveniences without a rigorous foundation. Similar paradoxical arguments, as stated in Chapter IV, rest on a vague and incoherent notion of an infinitesimal difference, which is defined as neither quite nothing nor quite something. Furthermore, central to this Berkeley's critique is his rejection of *abstract general ideas*.[26] He asserts that such ideas, which are supposed to represent non-specific concepts, like infinity and abstract mathematical entities, are inconceivable. For example, abstract general ideas of infinity, such as infinite divisibility and infinitesimals, cannot be related with any experiences in nature. He illustrates the challenge the impossibility of conceiving the abstract general ideas through imagining an abstract triangle that fits no specific criteria, such as being neither isosceles, scalene, nor equilateral:

[25] For an introduction to rationalism and empiricism, refer to Moore (2019, 73-80) and Markie and Folescu (2023, *Stanford Encyclopedia of Philosophy*, 1.1-1.2). For a historical introduction, refer to Jacquette (2001, 13-22).
[26] See more about Berkeley's critique and his influence on Hume's critique in Jacquette (2001, 28-39).

> If any man has the faculty of framing in his mind such an idea of a triangle as is here described, it is in vain to pretend to dispute him out of it, nor would I go about it. All I desire is, that the reader would fully and certainly inform himself whether he has such an idea or no. And this, methinks, can be no hard task for any one to perform. What more easy than for any one to look a little into his own thoughts, and there try whether he has, or can attain to have, an idea that shall correspond with the description that is here given of the general idea of a triangle, which is, *neither oblique, nor rectangle, equilateral, equicrural, nor scalenon, but all and none of these at once?* (Berkeley, *Treatise*, *Works*, Vol II, 33)

Besides, Berkeley offers an alternative to abstract general ideas through his theory of *representative generality.* According to this theory, the mind entertains only particular ideas that represent other ideas within the same category. Thus, geometry requires only an unlimited process, akin to the Aristotelian notion of potential infinity, rather than actual infinite divisibility within it. For example, consider a succession of lines, each representing half the extension of the previous one. No single line is infinitely divided in thought, but rather, the representative generality involves an unlimited succession of divided lines, extending beyond the limits of immediate perception. While the theory of representative generality is used to represent general concepts through specific, perceivable instances, Berkeley also introduces concept of *minima sensibilia* to ensure that these instances are the smallest units of perception.[27] Minima sensibilia is considered an alternative theory to infinitary theory of Euclidean geometry, and significantly influences Hume's *strict finitism.* The Humean strict finitism is regarded as a major opponent of both infinite divisibility and infinitesimals in mathematical infinity, as well as metaphysical infinity.

V.2 Critique of Mathematical Infinity

Hume, renowned for his skeptical and empirical approach to a range of philosophical fields, shares Berkeley's skepticism about infinity. However, Hume does not follow Berkeley's critique of rejecting abstract general ideas. Instead, Hume is heavily influenced by Berkeley's concept of minima sensibilia, which developed from the theory of representative generality mentioned earlier. *Hume devised two experiments, the inkspot experiment and the grain of sand thought experiment, to challenge the notions of infinite divisibility and infinitesimals in mathematical infinity.* Alongside these criticisms, Hume also presents arguments supporting his doctrine of minima sensibilia as the irreducible constituents of extension. The inkspot experiment is considered a fundamental argument, interpreted as demonstrating that vision cannot perceive sense impressions of infinitely divisible extension:

[27] Berkeley is credited with originating the phrase "minima sensibilia." However, there is debate about whether Berkeley treats his minima sensibilia as sensible indivisibles that are extended, or as sensible indivisibles that are extensionless. I will follow Jacquette's decision, which argues that Berkeley's minima sensibilia are extensionless (Jacquette 2001, 116).

> Put a spot of ink upon paper, fix your eye upon that spot, and retire to such a distance, that at last you lose sight of it; 'tis plain, that the moment before it vanish'd the image or impression was perfectly indivisible.' Tis not for want of rays of light striking on our eyes, that the minute parts of distant bodies convey not any sensible impression; but because they are remov'd beyond that distance, at which their impressions were reduc'd to a minimum, and were incapable of any farther diminution. (Hume, *Treatise*, 29-30)

In the experiment, a small spot of ink is placed on paper and fixed by an observer's gaze. As the observer gradually retreats, the spot remains visible until a critical distance is reached, at which point the spot disappears. The critical observation here is that the moment before the spot vanishes, the visual impression of the ink spot is indivisible. This implies that the image perceived by the observer has reached a limit of resolution beyond which it cannot be reduced further without disappearing entirely. Here are three steps to understand the arguments from the inkspot experiment:

(1) The inkspot experiment underscores the limitations of sensory impressions, particularly human visual perception. The indivisibility of the image at the point of disappearance suggests that there is a quantifiable limit to how small a visual stimulus can be before it ceases to exist in our perceptual field.

(2) Hume's *copy principle*, essential to his account of the origin of ideas, must be introduced alongside the inkspot experiment. The copy principle states that impressions of sensation are the ultimate experiential source of every idea. Consequently, the limitations of sense impressions of finite and finitely divisible extensions are also limitations of the ideas that are mental copies of these impressions. Therefore, there is no method by which ideas can compensate for the limitations and defects of the originating sense impressions of which they are merely copies.

(3) Consequently, the inkspot experiment suggests that finite minds cannot derive an idea of infinitely divisible extension from finitely divisible sense impressions. There is a minimal smallness for the extended objects of sense perception, meaning they are not infinitely divisible. Therefore, our adequate ideas of extension can only derive from this minimal smallness, also known as minima sensibilia.

The grain of sand thought experiment is meant to demonstrate that reflection, through the agency of imagination, also reveals the impossibility of infinite divisibility:

> ... the imagination reaches a minimum, and may raise up to itself an idea, of which it cannot conceive any sub-division, and which cannot be diminished without a total annihilation. When you tell me of the thousandth and ten thousandth part of a grain of sand, I have a distinct idea of these numbers and of their different proportions; but the images, which I form in my mind to represent the things themselves, are nothing different from each other, nor inferior to that image, by which I represent the grain of sand itself, which is suppos'd so vastly to exceed them. What consists of parts is distinguishable into them, and what is distinguishable is separable. But whatever we may imagine of the thing, the idea of a grain of sand is not distinguishable, nor separable into twenty, much less into a thousand, ten thousand, or an infinite number of different ideas. (Jacquette 2001, 46)

In the thought experiment, one can conceive of numerical distinctions such as the thousandth or ten thousandth part of a grain of sand. However, the mental images representing these subdivisions are indistinguishable from the image of the entire grain of sand. Thus, as Hume argues, when the imagination reaches its minimal limit, it constructs an idea that cannot be further subdivided. This indivisible idea is the smallest unit of mental representation, beyond which any attempt at division leads to the annihilation of the concept. The limitations of imagination indicate the inability to imagine infinitely small subdivisions within a minimal unit, which can only be perceived as minima sensibilia.

Humean Minima Sensibilia in Continuity

The inkspot argument introduces Humean minima sensibilia as atomic constituents of the idea of extension, echoing the Aristotelian first characterization of continuity. The rule of 'the parts prior to the whole' always emphasizes a relational, two-place account, raising the problem of forming a continuum. For example, according to the first characterization mentioned in Chapter III, if indivisibles are not in direct continuous contact, they cannot constitute continuous extension. Furthermore, there is a *paradox of extension* stating that if two indivisibles are touching on a line, the left side of one indivisible touches the right side of the other. However, indivisibles do not have left and right sides, since to do so, they would have to be divisible into parts. Since Hume shares the presupposition of 'the parts prior to the whole' with the Aristotelian first characterization of continuity, he also needs to address the puzzle of how the finitely indivisible parts can be put together to constitute extension. Thus, Hume solves this puzzle by defining minima sensibilia as *sensible, extensionless indivisibles.*[28] This definition can be interpreted through these three concepts individually:

[28] Hume also attempts to avoid Bayle's trilemma through this definition. Bayle's trilemma is discussed in detail by Jacquette (2001, 22-28 and 114-117).

Sensible: Sensible means that these smallest units are sensory perceptions, possessing phenomenal properties such as color and tangibility.

Extensionless: To avoid the paradox of extension, Hume characterizes extension as the property of an object being divisible into at least two spatial parts, which are divided into left and right halves. Therefore, Hume defines extension simply as implying divisibility. If something is capable of division, its indivisible parts must be extensionless.

Indivisible: Indivisible refers to the minima as the smallest units.

Therefore, Hume addresses the direct contact issue shared by the Aristotelian first characterization of continuity by asserting that there is no inherent difficulty in imagining such contact, using his definition of sensible, extensionless indivisibles. Firstly, Hume argues that two such extensionless indivisibles can be juxtaposed without possessing left- or right-hand parts and without interpenetrating or annihilating each other. Secondly, the term ‘sensible’ implies that when two extensionless indivisible points, which can be thought of as tangible or colored, come into contact, their union results in a composite object that is both distinguishable and divisible into its constituent parts. Each part retains its distinct existence despite their contiguity. For example, Hume clarifies his point by suggesting that imagining these points as being of different colors can help prevent the idea of their merging into a single entity. He uses the example of a blue point and a red point lying next to each other without either color being annihilated or the two merging into a new, indistinct color. Thus, this illustrates that the points can remain separate and identifiable even when they are contiguous. Nevertheless, this definition of sensible, extensionless indivisibles allows Hume to provide a more intuitive and phenomenally grounded understanding of continuity. Consequently, Antony Flew humorously refers to Hume as the ‘Father of Pointillisme,’ indicating that Humean continuity can sometimes be seen as akin to the dots in a pointillist painting.[29]

[29] For more details, refer to Jacquette (2001, 117) and Flew (1976, 257-269).

Figure 4: Famous Pointillist Painting *"A Sunday on La Grande Jatte – 1884"* by Seurat. Source: *Art Institute of Chicago*.

V.3 Critique of Metaphysical Infinity

To introduce Hume's critique of metaphysical infinity, it is essential to provide an overview of its intellectual developments after the Greek philosophers. Following the Greek epoch, the concept of metaphysical infinity expanded into theological discussions concerning the nature of God, the universe, and existence. These discussions typically focused on the concept of the infinitely large, contrasting with mathematical discussions of the infinitesimally small. In theological contexts, infinity often symbolized the divine, reflecting profound concepts related to the infinite nature of a divine being and the vastness of creation. Spinoza, one of the most renowned philosophers of the 17th century, built upon the fundamentals of metaphysical infinity introduced by Parmenides, developing them into a more sophisticated philosophical system characterized by monism or pantheism. Like many of his predecessors, including medieval scholastic thinkers and Descartes, Spinoza intertwined discussions of God and infinity. However, he argued that infinity must be understood in metaphysical terms, reminiscent of Parmenides' ideas.[30] For example, Spinoza posited that God or Nature is an absolute, indivisible, unified whole, whose essence as a complete entity encompasses all existence. Spinoza's metaphysical understanding of infinity also distinguishes between absolute infinity and what he terms 'infinity in its own kind.' Absolute infinity, which only God possesses, encompasses all reality. In

[30] Refer to Moore (2019, 73-76) and Nadler (2024, *Stanford Encyclopedia of Philosophy*, 2).

contrast, the weaker notion of infinity applies to entities in appearance, such as the relatively infinite sense of space and time.

Later, Hume addressed the development of metaphysical infinity in his work *"Dialogues Concerning Natural Religion,"* through the philosophical dialogue among Demea, Cleanthes, and Philo.[31] He provided a critical perspective on the concept, contributing to the ongoing philosophical discourse on metaphysical infinity:

> ***Demea***
> Demea's stance is rooted in mysticism, reflecting a transcendent distinction of metaphysical infinity like that of the Greek epoch, but within a more theological context. He claims that God is utterly unknown and incomprehensible, contending that God can only be described as an infinite and universal being. This perspective aligns with the scholastic view that God's nature transcends human understanding and defies anthropomorphic descriptions. Thus, Demea holds that God's attributes are so vastly different from anything humans can experience that any attempt to understand God through reason is futile.
>
> ***Cleanthes***
> Cleanthes suggests the existence of an infinite intelligent designer, God. By drawing an analogy between natural objects and human artifacts, Cleanthes argues that their similar structures imply similar causes. He concludes that God, like a human mind but vastly superior, is the designer of the universe. While his argument includes the idea of God as an infinite being, his empirical approach aligns him more closely with Spinoza's arguments. For example, Cleanthes rejects the notion that God's nature is entirely incomprehensible. He believes that, while human understanding of God may be limited, it is still possible to gain knowledge of God through reason and observation of the natural world.

Through his characters, Demea and Cleanthes, Hume presents the complex entanglement between Christian theologies and the concept of metaphysical infinity. While their thoughts do not fully align with the philosophies of traditional scholastics, Descartes, or Spinoza, they reflect the framework of intellectual discussions on metaphysical infinity that developed after the Greek epoch. The character called Philo represents the center of Hume's skeptical critique on the development of metaphysical infinity:

[31] There are also introductions to each character in Morris and Brown (2023, *Stanford Encyclopedia of Philosophy*, 8).

Philo

Philo leans towards agnosticism, suggesting that human reason is limited and may not be capable of definitively proving or disproving the existence of God. He often highlights the limitations of human knowledge and the potential for error in metaphysical speculations. For example, in responding to Demea's mysticism, Philo raises questions about the nature of the divine cause, questioning whether it is a single, all-powerful, and all-good being or a complete collection of independent entities. He argues that humans, due to their limited experience and knowledge, cannot justify any definitive claims about the nature or attributes of the divine cause. Philo also criticizes the design argument for relying on a flawed analogy. He argues that we lack direct experience of the world's creation and cannot infer its origin from our limited experience with human artifacts. In dialogue, he points out the vast differences between the universe and human creations, noting that the universe's scale and complexity make any comparisons inadequate.[32]

[32] Some texts, such as Part II (144), in Hume's *Dialogues Concerning Natural Religion.*

VI

Recapitulation

The goal of this thesis is to investigate how mind conceives the infinite by tracing the historical evolution of different concepts of infinity. Thus, the central question raised in the Introduction is: '*How can the mind grasp the notion of infinity?*' In Chapter II, the concepts of infinity by Greek philosophers are examined, particularly from a metaphysical perspective. Anaximander and Pythagoras were among the first to study the early metaphysical distinction between 'peras' and 'to apeiron.' By the time of Parmenides, the concept of metaphysical infinity had developed into a well-defined framework characterized by transcendence, completeness, and discussed the key concepts of continuity. Parmenides' view of metaphysical infinity not only laid important foundations for later discussions on the topic, but also contributed to the development of mathematical infinity. Mathematical infinity emerged in Zeno's paradoxes, which were intended to support Parmenides' metaphysical infinity. These paradoxes centered on the question of whether infinite divisibility was possible within a continuum, touching on geometry and mathematics. Before mathematical infinity was formally introduced, Aristotle offered his unique perspective on paradoxical infinite divisibility by distinguishing between actual and potential infinity. In Chapter III, Aristotle's unique distinction is discussed in detail, particularly in relation to his arguments on continuity. This distinction provided a third perspective on infinity, different from both earlier metaphysical infinity and later mathematical infinity. It also inspired further investigations into mathematical infinity by suggesting that a continuum could be potentially infinitely divisible. Chapter IV explores how Aristotle's claims, and the work of Greek geometers laid the groundwork for Leibniz's development of mathematical infinity. Leibniz continued discussions on continuity, which are now regarded as fundamental principles of his infinitesimal calculus. However, despite principles like the law of continuity implying the possibility of conceiving the mathematical concept of infinity through finite rules, Leibniz rejected the notion of actual infinite quantities or numbers, considering them paradoxical. Instead, he used infinitely small numbers as a technical convenience in calculus. In Chapter V, this perceived flaw in Leibniz's approach was highlighted by empiricists such as Berkeley and Hume. They insisted that only through experience can the mind understand truths about infinity. Berkeley criticized Leibnizian infinitesimals as vague and incoherent, defining them as neither quite nothing nor quite something. Hume further illustrated his theory of minima sensibilia through two experiments, concluding that only a finite number of sensible, extensionless indivisibles exist in a continuum. Consequently, he argued that the mind, limited by its finite experiences and the finite nature of its ideas and images, cannot conceive the infinite. This empiricist view was also reflected in Hume’s criticisms of metaphysical infinity through his agnostic character Philo.

Furthermore, the investigation of mind on the concept of infinity is ongoing and remains the subject of many successive debates. One of the critical moments in the modern development of mathematical infinity was Cauchy’s response to Berkeley’s criticism, which led to a precise definition of limits. This formalized the mathematical concept of infinity within modern analysis. By defining limits with precise epsilon-delta arguments, Cauchy addressed the criticisms regarding the ambiguous use of infinitesimals and made the concept of infinity mathematically rigorous. This development allowed mathematicians to handle infinite processes and quantities with newfound precision, enhancing the modern notions of continuity, derivatives, and integrals (Moore 2019, 63-68). The greatest modern influence from metaphysical infinity was to inspire Cantor’s groundbreaking work on set theory, which introduced the notion of different cardinalities of infinity. For example, his investigation into the nature of sets led him to discover that the set of real numbers is uncountably infinite, meaning its size is strictly greater than the size of the set of natural numbers, which is countably infinite. This distinction was formalized through his famous diagonal argument and the concept of bijections. Cantor showed that even within the realm of the infinite, there is a structure and hierarchy (Moore 2019, 109-129). Additionally, Cantor’s work on the development of the continuum hypothesis and transfinite numbers further expanded the understanding of mathematical infinity (Muller 2022, 25-34).

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