Coordination Of Lot-Sizing Decisions In A Game Theoretical Framework Part 2

Master Thesis

Econometrics & Management Science[a] [b]

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Mark van den Hauwe[c]

Thesis committee:
Dr. Wilco van den Heuvel (supervision)
Prof. Dr. Albert P.M. Wagelmans
Dr. Vladimir A. Karamychev

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[a]specialization: *Operations Research and Quantitative Logistics*
[b]Erasmus University Rotterdam, *Erasmus School of Economics*
[c]studentnumber: 201052

Abstract

Individual decision making in a supply chain will often not lead to an outcome in which total profit is maximized. Additional action needs to be taken to increase overall efficiency. We investigate the coordinating effects of two mechanisms: a quantity discount schedule and a holding cost compensation scheme. Each mechanism is analysed in an environment with continuous demand and an infinite planning horizon against the background of non-cooperative game theory. Full coordination is achieved with the first mechanism. The simulation study shows that the performance of the alternative strongly depends on each actor's cost parameters.

Contents

Chapter 1

Introduction

The main purpose of each supply chain is to organize the conversion of raw materials into end-consumer products via various stages of production and distribution. Upstream partners deliver inputs, while downstream partners buy the output. In an arbitrary configuration, the supplier's and buyer's base of different firms may overlap. The mutual relationships cause individual decisions to have system-wide implications. When each actor optimizes his own profit and disregards the effects of his decisions on others, total realized profit may be less than what it could be if all were to collaborate. However, full cooperation may not be possible and, instead, one has to resort to other, less extreme, remedies.

In this thesis we investigate the capabilities of two simple mechanisms to mitigate the negative aspects of anarchistically setting order sizes. For the much studied quantity discount model we provide a new detailed derivation of optimal behaviour by making use of non-cooperative game theory. As an alternative we suggest compensation of downstream holding costs. A customized algorithm is devised to calculate an outcome under the second scheme. In order to get a better understanding of the performance, we numerically evaluate a set of parameter configurations.

The remainder is organized as follows. In Chapter 2 we embed the purpose of the thesis in the more general concept of (counteracting) market inefficiencies. An overview of the problem environment, including assumptions and notation, can be found in Chapter 3. Chapter 4 contains a short description of the game theoretical concepts applied in Chapter 5 for the derivation of sequentially optimal behaviour in the lot-sizing games. We present the results of the simulation study in Chapter 6. In Chapter 7, we look at some directions for future research. Chapter 8 concludes with a summary of our findings.

Chapter 2

Problem description

2.1 Introduction

Supply chain inefficiencies are a wide-ranging problem. In Section 2.2 we distinguish several ways in which myopically taking decisions can be bad for total profit. Optimization of the system by integrating all operations in one organization and two other, less extreme, solution approaches are discussed in Section 2.3. We formulate the goal of the thesis in Section 2.4.

2.2 Different supply chain inefficiencies

Double marginalization In the classic case of double marginalization, an upstream actor incorporates a mark-up above marginal costs in his price. Consequently, a downstream supply chain partner will set a higher price and will include a mark-up of his own. The overall result is an inflated price for consumers and a smaller quantity delivered. The realized total supply chain surplus will be less than in the welfare maximizing outcome, where the pricing decisions are solely based on the marginal costs of production and distribution at each stage of the supply chain (Pepall et al. (2002, pages 437-443)).

Non-optimal levels of advertising & product quality Apart from moving product to the end-consumer, other functions have to be performed in a supply chain. These have to do with the characteristics of a product, how and to what extent it is promoted and the amount of capital invested in developing new ones. For society, the benefits to all, including end-consumers, are relevant. In practice, advantageous aspects, like better informedness of consumers in case of advertising, will only be taken into account by an actor in as far as his profits are increased. Pepall et al. (2002, Chapters 10 & 11) shows that the price-sensitiveness of demand can be an important factor in this respect.

An equally important cause of socially inefficient decision making is free-riding. Less will be spent on promotion and product development, because one hopes someone else will make the necessary investments to increase demand.

Product and/or asset specificity (hold-up problem) Assets in the form of physical equipment or knowledge can be very hard or even impossible to employ in another business relationship as profitably as in the current one. When some supply chain member has acquired such a specific asset, the other party could behave opportunistically after the investment has been made. He may threaten to find another contracting party, unless the present one is willing to adjust the contract terms in the sense that the amount paid for the asset will merely be recovered up till the pay-offs in its second-best use. Rational actors anticipating this might not invest at all in situations where this opportunistic behaviour is not discouraged in some way.

Bullwhip effect Lee et al. (1997, pages 95-98) illustrate the well known logistical issue of the bullwhip effect. In a stochastic environment, deviations from the regular order pattern can be wrongfully interpreted as indications that the demand intensity has changed. When the upstream actor forecasts demand for production and inventory planning decisions, the error is amplified because of extrapolation into the future. Moreover, the resulting change of the order pattern will become more pronounced, when safety stocks are adjusted, if longer lead times apply and in case of higher demand variability.

Other problematic factors are order batching, which may translate small (expected) changes in the amount of product into larger ones upstream, and major undesirable swings in demand caused by buyers engaging in forward buying. Distorted information also spreads when some actor cannot meet demand and rations his supplies based on the size of the incoming orders. In response, customers will exaggerate their required amounts. The consequences are excessive inventories, more difficulties in smoothly organizing production and a necessity to hold available extra production capacity. The more one gets upstream the supply chain, the stronger the distortions will be.

Disproportionate risk exposure Problems may emerge as well if some supply chain member disproportionately carries the burden of negative economic developments. The exposed actor may refrain from actions beneficial to the entire chain. An example is the 2 echelon 1 period Newsboy problem in Axsäter (2006, pages 284-287). The downstream actor is confronted with stochastic end-consumer demand for which product must be purchased beforehand. While the upstream party is guaranteed to make a fixed profit equal to his margin times the number of units ordered in advance, the other actor's profit may vary considerably, and even be negative. The combination of the unilateral risk exposure, and the fact that the upstream price, including the margin, will be higher than the echelon cost of procurement, results in a smaller than jointly optimal downstream order size.

Production planning externalities Each firm has to schedule its production and value adding activities in order to satisfy demand on its output market. The resulting plan yields an order scheme, which, in turn, serves as the basis for the production and lot-sizing decisions taken at the upstream echelon(s). Because of positive or negative externalities to other actors, the individual planning activities may not minimize total costs in the supply chain.

2.3 Vertical integration & other remedies

Joint ownership by vertically integrating Combining activities in one firm has traditionally been regarded an important solution in all kinds of situations where market exchange is very costly or totally fails at all. Williamson (1971) discerns five classes of characteristics, which he deems important for the attractiveness of the approach. Among these are the monopolistic or oligopolistic contexts where double marginalization is likely to emerge. In Christy & Grout (1994) and Klein et al. (1978), it is indicated that, when there is much product and process/asset specificity, high costs of complex contracting make vertical integration a more preferred safeguard against mutual hold-up.

Despite the wide-ranging spectrum of problems vertical integration can remedy, the construction of a more sizable firm potentially has some major disadvantages. As described in Jeuland & Shugan (1983, page 250), a downstream actor may carry products from other manufacturers to exploit economies of scope, which becomes problematic after integration, a vertical merger may not be allowed by law, or each independent actor carries out his specialized function less efficiently in a larger organization.

Market solutions (closer collaboration) A less extreme approach is increasing the level of mutual cooperation in the supply chain. To counteract the bullwhip effect and coordinate buying practices across different actors, Lee et al. (1997, pages 98-100) put forward electronic data interchange (EDI) and vendor managed inventories (VMI). EDI means that supply chain partners share company specific information. Communicating details about stock levels and market forecasts enables anticipation on sudden drops or surges in demand, thereby decreasing the necessity to keep large safety stocks and making expensive overreactions less likely. Because of the relegation of all inventory related operations to the upstream partner, the VMI mechanism is somewhat more extreme.

Market solutions (aligning incentives) The autonomy of the supply chain members is respected most when coordination takes place by introducing contractual provisions which only influence the operating environment of another actor indirectly. For instance, the two-part tariff, as explained in Pepall et al. (2002, pages 481-483), is particularly suited to confront the problem of double marginalization. A lump sum payment and charging of the marginal cost of production for each unit bought will result in a lower price on the finished goods market and higher total surplus. The construction is typically found in a franchising agreement, and can, under circumstances, alleviate the problem of suboptimal product quality levels as well.

Methods to mitigate free-riding on someone else's (advertising) expenditures, are resale price maintenance and exclusive selling/dealing contracts. With resale price maintenance the price charged to consumers is no longer freely determined by the retailer. Exclusive selling and dealing agreements, on the contrary, restrict the number of downstream or upstream partners. The creation of (local) monopolies ensures that the exclusive supply chain member reaps all the benefits of his efforts without another supply chain member benefiting at his expense (Pepall et al. (2002, Chapter 9)).

For the 2 echelon Newsboy problem, Axsäter (2006, pages 286-287) proposes a buy-back contract. If demand is less than the quantity ordered in advance, the remaining products can be returned to the upstream partner. Adequately setting the wholesale and buy-back price results in an order size maximizing total expected supply chain profit.

Finally, a lot of supply chain efficiencies can be resolved by the profit sharing contract. Because each party gets a predetermined fraction of the pooled profit, operations will shift to the collectively most desired outcome. A major drawback is that it requires quite some monitoring resources. Success depends largely on the truthful revelation of individual revenues and costs.

2.4 Focus of research

Our analysis is restricted to the counteraction of production planning externalities, and more specifically to the negative consequences of anarchistic lot-sizing. As common ownership and a construction like VMI are quite rigorous forms of exerting vertical control, and integration of activities has some major problems of its own, we concentrate on two simple market solutions to align incentives.

The quantity discount schedule, which has been studied extensively in the literature[1], can be used to directly reward the choice for certain order sizes. The second scheme allows

[1] For an extensive review of the literature, see Chapter 3 in 'Coordination of lot-sizing decisions in a game theoretical framework Part 1'

compensation of the other actor's costs of holding inventory. Both are studied in a non-cooperative game theoretical environment. Instead of Pareto efficient (bargaining) outcomes, as in Kohli & Park (1989), Kim & Hwang (1989) and Chakravarty & Martin (1988), we look for subgame perfect Nash equilibria. For each schedule we are interested in the degree to which total profit is enhanced and how the extra surplus, if any, benefits each partner.

Chapter 3

Modelling the supply chain

3.1 Introduction

Before deriving optimal behaviour in the supply chain, we present the underlying assumptions and our notation. Decision making takes place in a 2 actor serial supply chain facing continuous, constant and price-inelastic demand for an infinite period of time. Due to the exclusion of backordering (some of) the deterministic demand and the properties of the delivery and production processes, we can limit each actor's expenses to input prices and lot-sizing costs. Section 3.2 gives further details and an explanation of the strategic and informational aspects. We describe the model parameters, the coordination/lot-sizing decisions and some other notational elements in Section 3.3. Section 3.4 lays down the sequence of decision making.

3.2 Assumptions

Supply chain structure We look at a serial supply chain comprised of a Retailer serving end-consumer demand and a Wholesaler supplying him. Inputs originate from the Manufacturer, an otherwise passive agent in the 2 echelon setting. A serial configuration is relatively simple, and, as observed in Li et al. (1995, page 1456), avoids distraction from the main purpose of the analysis, namely, investigating the effects of measures to coordinate inventory policy decisions. Using multiple heterogeneous retailers, like Drezner & Wesolowsky (1989, pages 41-42) and Chakravarty & Martin (1988, pages 275-277), would already cause a lot of (unnecessary) complications.

Demand, planning horizon & prices The actors' operations are restricted to a single product, which is reasonable as long as no major cost synergies, so-called economies of scope, can be realized by combining several production and order policies. Demand occurs continuously for an infinite period of time at a constant rate not affected by the price charged to end-consumers. The process is completely deterministic. Because we normalize the number of inputs needed for 1 output, the upstream actors face the same demand intensity. Comparable to Khouja (2003, page 986), value adding activities are reflected in increasing prices as one goes downstream. Apart from the price discount fraction associated with the first coordination mechanism, the actual determination of prices is exogenous to our models. A convenient consequence is that actors can restrict their attention to production and inventory related decisions.

Production & delivery rate In the normalized production structure, the Wholesaler and Retailer add value with, for example, promotional activities, tailored packaging and efficient distribution to their consumer(s). Although Munson & Rosenblatt (2001, page 377) show that capacity utilization may strongly influence savings, we impose infinite transformation and delivery rates: the time needed for value addition is negligible and inventories are replenished instantaneously upon arrival of the shipment. Infinite rates are a good approximation in case only a minor part of the resources in the supply chain is used. Introducing capacity constraints does not essentially change the results, but merely causes the mathematical expressions to be less tractable. Inputs are processed on a per order basis.

Backordering We do not allow shortages. Inventories are always maintained at levels such that all orders can be fulfilled entirely at the moment they come in. Excluding postponed delivery, so-called backordering, is reasonable in situations where stock-outs lead to disproportionately large losses.

Lead times The term lead time is used to designate the time between the moment an order is received from a customer and the moment that the product arrives there. Constant lead times are intrinsically linked to a deterministic demand environment without the possibility of backordering. To be absolutely sure that shortages will be avoided, an actor has to possess perfect knowledge about the demand pattern and about the time it takes to replenish him. A simple shift in a previously established order pattern suffices to accommodate for a change in the lead time. We might, therefore, just as well say that the product is delivered immediately upon ordering.

Costs Three categories of costs are assumed to be relevant. The first is the unit price an actor has to pay for acquiring an item from his upstream supply chain partner. The other two cost categories are related to the lot-sizing decisions: the fixed cost of ordering and the inventory holding cost rate. Each cost rate is linear in its driver, i.e., the number of items sold, the number of orders placed, or the stock level. Moreover, each is constant over time.

The order cost does not depend on the number of items in a batch. It covers administrative expenses and the costs of personnel to handle the incoming product. The properties of the delivery and production processes (constant (zero) lead time, infinite rates and conversion on a per order basis) imply that the production cycle is completely synchronized with the order cycle. Consequently, the order cost can be considered to include as well a fixed cost of setting up the value addition process/production.

As inputs are never kept in stock, the holding costs for each actor are limited to inventories of processed product. The components of a holding cost rate can be divided in two groups, depending on whether or not there is a direct relationship with the value (input price) of the product. Examples of value-related components are the interest foregone on the capital tied up and the losses resulting from obsolescence, damage and theft. On the other hand, costs of financing or renting storage space are at best remotely connected to price.

A major disadvantage of incorporating to some extent dependency on the input price is that the social optimum for a quantity discount scenario prescribes a 100 % discount to achieve minimum holding cost rates, see Weng (1995*b*, page 310). Zero prices are, however, not a realistic benchmark and are unacceptable to the actor offering the discount, so that, as formulated in Chakravarty & Martin (1988, page 274), there is "... no incentive for pursuit of the optimal 'social welfare' solution." Compared to the other mechanism, the incentive compatibility problem would lead to an underestimation of the efficiency enhancing effects of a quantity discount scheme. To avoid these peculiarities and to ensure that the most efficient outcome is the same, irrespective of whether or not coordination takes place, we assume price-independent holding cost rates. The coordination terms related to a mechanism drop out when the individual profit functions are aggregated.

Besides the fixed cost of preparing an outgoing shipment, a separate cost for handling the downstream partner's orders is used in Viswanathan & Wang (2003) and Weng (1995*a*). For reasons of mathematical clarity and to limit the number of parameter dimensions, we disregard, in accordance with Munson & Rosenblatt (2001), these *order processing* costs and, more generally, any other expenditure.

Strategic interaction & information The supply chain members cannot collaboratively agree on a joint lot-sizing policy. Unlike Kohli & Park (1989) and Dudek & Stadtler (2005), negotiations and bargaining are ruled out. Each maximizes profit without taking into account the possible beneficial or detrimental effects on others. From Corbett & de Groote (2000), it follows that the amount of available information is crucial for the nature of strategic interaction. We assume that the supply chain has been functioning for quite a while already.

Because of past order patterns, each participant is well-informed about the intensity of end-consumer demand and about the other actor's cost structure. Common knowledge is also the underlying principle with respect to the observability of decisions.

3.3 Notation

Environment & decisions The upper part of Table 3.1 presents an overview of all parameters describing the demand process and the actors' cost characteristics. The Retailer receives P_r for each item sold to end-consumers and pays a unit price P_w to the Wholesaler. With P_m the price paid to the Manufacturer, the gross margin per unit is then straightforwardly $P_w - P_m$ for the Wholesaler. In practice, the nature of production will often lead to downstream firms having a smaller order/set-up cost and a larger inventory holding cost rate than their upstream partner. Here, we do not impose $A_r < A_w$ and $h_r > h_w$, thereby maintaining a maximum degree of flexibility in inventory related cost patterns. In fact, this is yet another motivation for not directly linking holding cost rates to prices, which tend to increase as more value is added.

Parameter	Description
D	*Demand intensity per time unit*
P_i	*Price per unit charged by actor* $i \in \{r, w, m\}$
A_i	*Fixed order cost for actor* $i \in \{r, w\}$
h_i	*Holding cost rate for actor* $i \in \{r, w\}$
Variable	
Q	*Retailer order size*
n_w	*Wholesaler lot-sizing multiple*
α_w	*Quantity discount fraction set by Wholesaler*
R_w	*Quantity discount region set by Wholesaler*
$\bar{Q}_w$	*Order breakpoint belonging to* R_w
β_w	*Holding cost compensation fraction set by Wholesaler*

Table 3.1: *Parameters & decision variables*

The first variable in the remainder of the table, Q, represents the amount of product the Retailer orders each time to satisfy end-consumer demand. It forms the basis for the lot-sizing decision upstream. Because prescribing a lot-for-lot policy similar to Monahan (1984) and Khouja (2003) would make the analysis far less interesting, the Wholesaler is allowed to deliver more than once during an order/production cycle. The zero-inventory property (see Axsäter (2006, pages 62 & 226)) ensures that his best course of action is to choose an integer multiple, making his order size equal n_wQ.

Mostly, a quantity discount scheme is of the following form: the domain of all possible order sizes is divided in several regions by specifying a set of order breakpoints. Each region is tagged with a certain price discount. Depending on whether the schedule is of the all unit quantity discount (AQD) or incremental quantity discount (IQD) type, the reduced price is charged for each item sold or merely for those units of the order falling into the associated region. The equivalence of both approaches in coordinating a 2 actor serial supply chain is proved in Weng (1995*a*), Weng (1995*b*), Kohli & Park (1989) and Kim & Hwang (1989). Because a one-to-one transformation from one type to the other exists, there is no need to consider each. And, as observed in Weng (1995*b*, page 307), since an IQD is more complex in nature, it is convenient to restrict attention to AQD schedules.

The Wholesaler, being the dominant actor, sets the terms of the schemes. The discount α_w is a fraction of the original pre-discount price. In accordance with the literature, one breakpoint $\bar{Q}_w$ is set. The discount region variable R_w, explicitly incorporating the flexibility suggested in Munson & Rosenblatt (2001, page 377), lets the Wholesaler determine whether the Retailer order size qualifies for the per unit discount $\alpha_w P_w$ in the region $\left(0, \bar{Q}_w\right]$ (quantity premiums) or $\left[\bar{Q}_w, \infty\right)$ (proper quantity discount schedule). The holding cost compensation (HCC) scheme results in an adjusted holding cost rate $(1 - \beta_w)\, h_r$.

Demand, prices, cost parameters, Q and the order breakpoint are all positive (> 0). The variable n_w is restricted to the set of positive integers $\mathbb{N} = \{1, 2, 3, ...\}$. We have $\alpha_w \geq 0$ and $\beta_w \in [0, 1)$. Negative values are excluded, because otherwise the downstream partner would be penalized, which is contrary to the compensating nature of both mechanisms. To guarantee that an actor continues to pay something for holding inventory, a necessary requirement for an optimal lot-sizing decision to exist, the compensation fraction must be smaller than 1. We do not impose an upper limit on α_w. In the unlikely scenario that total savings from the quantity discount scheme exceed or equal total gross revenue, the coordinating actor wants to select a value equal to or larger than 1. In practice though, this will not occur, as a member making a loss (negative revenue minus costs) will prevent the supply chain from operating.

Profits, optimal actions & bounds To distinguish among profits (Π, Π_r, or Π_w), optimal behaviour and bounds on some variables, we add (multi-element) superscripts. Optimality, lower and upper bounds are denoted by one of the following: $*$, $-$ or $+$. The other set of elements is made up of u, referring to an uncoordinated supply chain, s for the social optimum and a or b conveying the nature of coordination (which is superfluous if an optimum or bound for a coordination variable is described). When relaxing the requirement $n_w \in \mathbb{N}$, notation will instead be based on ν_w.

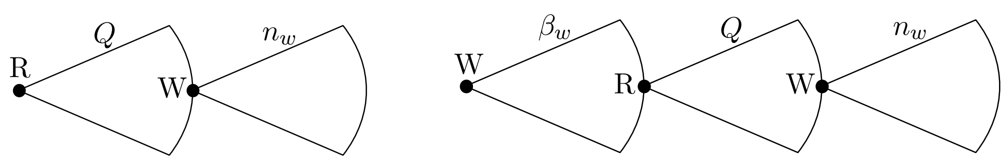
Performance measures The assessment of the capacity to streamline the supply chain is done with the help of the performance measures in Table 3.2. The quantity Δ expresses the total gain in the supply chain as a percentage of the difference between maximum welfare and the total profit resulting under anarchy. The other measures are meant to give an idea about the size of the transfers inciting the Retailer to change his inventory policy, and the benefits of coordination to individual actors. The payment percentage Φ is based on the ratio of the Wholesaler's payment over the efficiency gap. For either echelon, the savings divided by the minimum lot-sizing costs under anarchy, times 100%, gives δ_i. We stick to our system of superscript notation to clarify which mechanism's performance is measured.

Measure	Description
Δ	*Efficiency gain percentage*
Φ	*Payment percentage*
δ_i	*Savings percentage for actor* $i \in \{r, w\}$

Table 3.2: *Performance measures*

3.4 Overview of lot-sizing games

The two types of games are depicted in Figure 3.1. For convenience, we use the holding cost compensation scheme to illustrate the typical set-up of a coordination game. Replacing β_w by the variables α_w and R_w is sufficient to get the schematic overview for the other mechanism. Each figure satisfies the graphical conventions in Watson (2002). A node represents a decision for either the R(etailer) or the W(holesaler), while a pair of branches connected by an arc indicates that the domain for the decision variable at hand consists of an infinite number of elements. Strictly speaking, nodes situated on an arc stand for a multitude of points from which the game may continue: one for each value of the decision variable in the previous stage. Each node in its own information set (the node itself) reflects the assumption that an



(a) Non-coordination

(b) Coordination

Figure 3.1: *Extensive forms for 2 echelon games*

actor taking a decision is fully informed about how the game has proceeded previously. The pay-offs are left out, as these will be established while analysing each game.

To be able to affect downstream behaviour, coordinative action must precede the lot-sizing stages. Contrary to standard practice in the 2 actor quantity discount literature (see for example Lee & Rosenblatt (1986)), we do not combine Wholesaler decision making. Not including the inventory policy decision in the coordination stage better reflects reality where the Retailer order size is observed before choosing an integer multiple. Although the more extensive set-up necessitates the derivation of a rule specifying how the Wholesaler responds to each (possibly irrationally chosen) Q, the difference in approach does not matter for the set of actions actually chosen by the supply chain members. The rewritten first stage optimization problems are constructed such that these also depend on n_w. Assuming a rational Retailer, the initially optimal integer will correspond to the action prescribed by the Wholesaler lot-sizing policy in the third phase of the game.

Chapter 4

Game theory

4.1 Introduction

All optimal behaviour for the actors will be derived using the tools from non-cooperative game theory. In Section 4.2, we briefly explain what is meant by strategies, strategy profiles, subgames and subgame perfect Nash equilibria. After clarifying these, we discuss the technique of backward induction. We end with the concept of an equilibrium path.

4.2 Relevant terminology & concepts

Strategies & strategy profiles Individual behaviour throughout the game is summarized by a strategy, which, in the words of Watson (2002, page 23), gives a complete contingent plan. Combining the individual strategies in a vector gives a strategy profile. Actions must be specified for all information sets belonging to a player. With respect to a representative node located on an arc in Figure 3.1, the player's strategy must thus prescribe an infinite number of actions (one for each actual decision node). Next to theoretical elegance, full contingency has its practical importance. Even though players do not anticipate to ever reach certain parts of the game, expected behaviour at information sets in later stages might be relevant for (optimal) decision making earlier on in the game. Moreover, it provides a means of dealing with an opponent's non-rational behaviour like mistakes (Watson (2002, page 27)).

Two classes of strategies may be discerned. We call a strategy mixed when for some information set an actor puts probability on different values of the decision variable and opts for one of these randomly. A pure strategy is just a special case; at each information set a particular value is selected with probability one, resulting in absolute certainty concerning an actor's decisions. In Chapter 5 we limit ourselves to the last strategy type.

Subgames Before clarifying our equilibrium concept, we note how each game can be subdivided in different subgames. Watson (2002, page 141) describes a subgame as the tree structure initiated by a decision node x where neither x nor any of its successors are part of an information set containing nodes that are not successors of x. The most comprehensive subgame is the game itself. In Figure 3.1, every decision node starts a new subgame.

Subgame perfect Nash equilibria In our games we will be looking for Nash equilibria. In general, these are defined as strategy profiles wherein each actor's strategy maximizes his profit given the other actors' strategies: each actor plays a best response. Sequential decision making necessitates a refinement of the Nash equilibrium concept, the *subgame perfect* Nash equilibrium, which incorporates the notion of sequential rationalizability in extensive form games. Citing Watson (2002, page 143), the idea behind subgame perfection is "..that a solution concept should be consistent with its own application from anywhere in the game where it can be applied." Upon entering a new subgame, the prescribed strategy must remain optimal in the sense that a party does not wish to deviate from it.

Backward induction To find subgame perfect Nash equilibria in pure strategies, we use backward induction as explained in Watson (2002, page 139). We start with the subgames in the last stage to determine the best action, which depends on how the game has evolved up till that point. Bearing in mind this characterization of optimal behaviour, we proceed in a similar manner with the preceding stage. The process of moving backwards, while anticipating subsequent profit maximizing behaviour, continues until the beginning of the game (the first decision node) is reached. Combining the optimal decision rules in all stages gives us the desired Nash strategy profile(s).

Equilibrium paths The resulting sequence of optimal actions constitutes an equilibrium path. In many games, a multitude of paths exists. Since our primary interest is in better aligning the supply chain, we let actors aim for an outcome with maximum social welfare in case of more than one solution.

Chapter 5

Nash equilibria for lot-sizing games

5.1 Introduction

In this chapter we analyse the sequentially rational behaviour for the different lot-sizing games. The subgame perfect pure Nash equilibria are formally described, and, where needed, an algorithmic environment is included to explain the main steps in the calculation of an equilibrium path. The assumption of one-time interaction implies that a player's actions remain the same for the entire infinite time horizon. The same order cycles will be repeated ad infinitum. Without any discounting for the time value of money, each actor simply maximizes average profit per time unit. In Section 5.2 we study the anarchy situation. After deriving the joint policy that maximizes total supply chain profit in Section 5.3, we introduce quantity discounts in Section 5.4, and holding cost compensation in Section 5.5.

5.2 Non-cooperative outcome

Pay-offs The total gross margin for the Retailer is $(P_r - P_w)D$. Subtracting his average lot-sizing costs yields

$$\Pi_r^u(Q) = (P_r - P_w)D - A_r\frac{D}{Q} - h_r\frac{Q}{2} \tag{5.2.1}$$

The costs of ordering equal the fixed order cost A_r times the average number of orders per time unit $\frac{D}{Q}$. If no shortages are allowed, it is most efficient to replenish when stocks have been depleted. Because of infinite delivery and production rates, the inventory level instantaneously becomes Q upon arrival of the replenishment order, and next, diminishes to 0 again at the constant demand rate. On average $\frac{Q}{2}$ is kept in stock during a typical order cycle. Multiplication by h_r gives the holding cost term.

The difference $(P_w - P_m)$ is the basis for the total gross margin per time unit in the Wholesaler non-coordination profit. The calculation of ordering costs makes use of A_w and the order size $n_w Q$. Similar to the Retailer, the product of the holding cost rate and the average stock level is the last component of

$$\Pi_w^u(Q, n_w) = (P_w - P_m) D - A_w \frac{D}{n_w Q} - h_w \frac{(n_w - 1) Q}{2} \tag{5.2.2}$$

Under our assumptions regarding shortages, the delivery process and the production technology, it is shown in Chiang et al. (1994, pages 156-157) and Joglekar (1988, Appendix) that the total inventory of converted product held upstream during each order cycle is $((n_w - 1) Q + (n_w - 2) Q + \cdots + Q) \frac{Q}{D} = \frac{(n_w - 1) n_w Q^2}{2D}$. Dividing by the length of an order cycle $\frac{n_w Q}{D}$, gives the average $\frac{(n_w - 1) Q}{2}$.

Wholesaler lot-sizing Using backward induction, we first take a look at the last stage in Figure 3.1(a). For a fixed Q, the Wholesaler has to solve

$$\begin{aligned} \max_{n_w} \Pi_w^u(Q, n_w) = (P_w - P_m) D - A_w \frac{D}{n_w Q} - h_w \frac{(n_w - 1) Q}{2} \\ \text{subject to:} \\ n_w \in \mathbb{N} \end{aligned} \tag{5.2.3}$$

The concave Wholesaler profit is maximized at

$$n_w^{*u}(Q) = \min \{ n_w : \Pi_w^u(Q, n_w + 1) \le \Pi_w^u(Q, n_w) \, | n_w \in \mathbb{N} \}$$

In terms of the problem parameters we get

$$n_w^{*u}(Q) = \min \left\{ n_w : \frac{2 A_w D}{h_w Q^2} \le n_w (n_w + 1) \, | n_w \in \mathbb{N} \right\}$$

Like Munson & Rosenblatt (2001, pages 375-376), an explicit functional form is obtained by rearranging the terms of the condition somewhat and applying the quadratic formula to $(n_w)^2 + n_w - \frac{2 A_w D}{h_w Q^2} = 0$. Rounding up the positive (non-integer) solution gives the integer of interest. Alternatively, with $\Pi_w^u(Q, 0) = -\infty$, we can describe the best Wholesaler lot-sizing action as

$$n_w^{*u}(Q) = \max \{ n_w : \Pi_w^u(Q, n_w - 1) \le \Pi_w^u(Q, n_w) \, | n_w \in \mathbb{N} \}$$

Combining both decision rules gives

$$n_w^{*u}(Q) = \begin{cases} \left\lceil -\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + \dfrac{2A_w D}{h_w Q^2}} \right\rceil \\ \text{or} \\ \left\lfloor \dfrac{1}{2} + \sqrt{\dfrac{1}{4} + \dfrac{2A_w D}{h_w Q^2}} \right\rfloor \end{cases} \tag{5.2.4}$$

More demand and a higher fixed order cost force the Wholesaler to save on the costs of placing orders by increasing his integer multiple. Smaller integers, which lower the average quantity of product in store, become more attractive if the holding cost rate or the Retailer order size increases. As required, the minimum of each rule is 1. If the square root term times 2 is an odd number, no rounding is needed. Instead of a unique lot-sizing decision, two successive integers will be optimal. The smallest follows from the upper entier expression.

Retailer order size The Wholesaler lot-sizing rules are irrelevant for the Retailer. End-consumer demand is fulfilled most efficiently by solving, at the node initiating the game, the standard economic order quantity (EOQ) problem

$$\begin{aligned} &\max \Pi_r^u(Q) = (P_r - P_w)\, D - A_r \frac{D}{Q} - h_r \frac{Q}{2} \\ &\text{subject to:} \\ &\qquad Q > 0 \end{aligned} \tag{5.2.5}$$

As the objective function is concave in Q, it suffices to set the derivative with respect to Q equal to zero. The solution to the first order condition equals

$$Q^{*u} = \sqrt{\frac{2A_r D}{h_r}} \tag{5.2.6}$$

The influence of the parameters resembles the effects of A_w, D and h_w on $n_w^{*u}(Q)$.

Equilibrium strategies Our findings are summarized in Proposition 5.2.1. We observe that $Q^{*u} \times n_w^{*u}(Q)$ actually describes an infinite number of Nash equilibria. There is an unlimited number of (irrational) Retailer order sizes for which the Wholesaler can randomly choose among one of two optimal lot-sizing multiples returned by $n_w^{*u}(Q)$.

Proposition 5.2.1. *All subgame perfect pure Nash equilibria in the non-coordination game are given by the strategy profiles $Q^{*u} \times n_w^{*u}(Q)$ satisfying Equations (5.2.6) and (5.2.4).*

Proof. See foregoing. Since we do not neglect any optimal decision at some decision node, backward induction guarantees that all subgame perfect pure Nash equilibria are covered. □

Substitution of Q^{*u} in $n_w^{*u}(Q)$ results in the Wholesaler's action on the equilibrium path. A unique outcome exists when $\lceil \nu_w^{-u} \rceil = \lfloor \nu_w^{+u} \rfloor$ in

$$n_w^{*u} = \left\lceil \nu_w^{-u} \right\rceil \text{ or } \left\lfloor \nu_w^{+u} \right\rfloor \tag{5.2.7}$$

where:

$$\nu_w^{-u} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{A_w h_r}{A_r h_w}} \tag{5.2.8}$$

$$\nu_w^{+u} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{A_w h_r}{A_r h_w}} \tag{5.2.9}$$

The actors' profits become

$$\Pi_r^{*u} = (P_r - P_w)\, D - \sqrt{2 A_r h_r D} \tag{5.2.10}$$

$$\Pi_w^{*u} = (P_w - P_m)\, D - \left(\frac{A_w / n_w^{*u}}{2 A_r} + \frac{h_w \left(n_w^{*u} - 1\right)}{2 h_r} \right) \sqrt{2 A_r h_r D} \tag{5.2.11}$$

The parameter D does not appear in the expression for n_w^{*u}. Therefore, the realized lot-sizing costs upstream are scaled by the same constant as those for the Retailer if demand changes. By construction, $\Pi_w^u\left(Q^{*u}, \nu_w^{-u}\right) = \Pi_w^u\left(Q^{*u}, \nu_w^{-u} + 1\right) = \Pi_w^u\left(Q^{*u}, \nu_w^{+u}\right)$ holds. Since the concave function $\Pi_w^u\left(Q^{*u}, n_w\right)$ has its unrestricted maximum at $n_w = \nu_w^{*u}$, we get

$$\Pi_w^u\left(Q^{*u}, \nu_w^{-u}\right) = \Pi_w^u\left(Q^{*u}, \nu_w^{+u}\right) \le \Pi_w^{*u} \le \Pi_w^u\left(Q^{*u}, \nu_w^{*u}\right) \tag{5.2.12}$$

with:

$$\nu_w^{-u} < \nu_w^{*u} < \nu_w^{+u}$$

where:

$$\nu_w^{*u} = \sqrt{\frac{A_w h_r}{A_r h_w}} \tag{5.2.13}$$

$$\Pi_w^u\left(Q^{*u}, \nu_w^{*u}\right) = (P_w - P_m)\, D - \left(\sqrt{\frac{A_w h_w}{A_r h_r}} - \frac{h_w}{2 h_r} \right) \sqrt{2 A_r h_r D} \tag{5.2.14}$$

5.3 Fully optimized supply chain

Joint profit To maximize total supply chain profit, we sum each actor's profit and decide on Q and n_w simultaneously. A combination of the approach in Goyal (1976) and the procedure to determine the individually optimal lot-sizing multiple(s) is used to solve:

$$\max \Pi^s(Q, n_w) = (P_r - P_m) D - (A_r + A_w/n_w) \frac{D}{Q} - (h_r + h_w (n_w - 1)) \frac{Q}{2}$$

$$\text{subject to:} \tag{5.3.1}$$

$$Q > 0 \qquad n_w \in \mathbb{N}$$

Optimal collaborative Retailer action Given a particular $n_w \in \mathbb{N}$, we have the system-wide equivalent of Problem (5.2.5). With A_r replaced by $A_r + A_w/n_w$, and h_r appearing instead of the joint holding cost rate $h_r + h_w (n_w - 1)$, incurred for an average item moving through the supply chain, the more comprehensive solution becomes

$$Q^{*s}(n_w) = \sqrt{\frac{2(A_r + A_w/n_w) D}{h_r + h_w (n_w - 1)}}$$

The difference between $Q^{*s}(n_w)$ and Q^{*u} in Equation (5.2.6) concisely illustrates the potential for inefficient decision making under anarchy.

Optimal collaborative Wholesaler action Insertion of $Q^{*s}(n_w)$ results in a single variable objective function $\Pi^s(n_w) = \Pi^s(Q^{*s}(n_w), n_w)$ and a reduced problem

$$\max \Pi^s(n_w) = (P_r - P_m) D - \sqrt{2(A_r + A_w/n_w)(h_r + h_w (n_w - 1)) D}$$

$$\text{subject to:}$$

$$n_w \in \mathbb{N}$$

As mere inspection does not reveal the behaviour of $\Pi^s(n_w)$, we relax the domain restriction from $n_w \in \mathbb{N}$ to $n_w \geq 1$ and take the derivative

$$\frac{\partial \Pi^s(n_w)}{\partial n_w}_{|n_w > 1} = -\frac{\sqrt{D}\left(A_r h_w + A_w (h_w - h_r) / (n_w)^2\right)}{\sqrt{2(A_r + A_w/n_w)(h_r + h_w (n_w - 1))}}$$

In case $h_r > h_w$, we solve $A_r h_w + A_w (h_w - h_r) / (n_w)^2 = 0$ by setting n_w equal to

$$\bar{\nu}_w^s = \sqrt{\frac{A_w (h_r - h_w)}{A_r h_w}}$$

The derivative is positive below and negative beyond $\bar{\nu}_w^s$. Comparable to the non-coordination setting, after defining $\Pi^s(0) = -\infty$, the optimal lot-sizing integer(s) is(are) characterized by

$$n_w^{*s} = \begin{cases} \min\left\{n_w : \Pi^s\left(n_w+1\right) \leq \Pi^s\left(n_w\right) | n_w \in \mathbb{N}\right\} \\ \text{or} \\ \max\left\{n_w : \Pi^s\left(n_w-1\right) \leq \Pi^s\left(n_w\right) | n_w \in \mathbb{N}\right\} \end{cases}$$

The minimum and maximum integer are found by rounding up or down the respective positive solutions to $(n_w)^2 + n_w - \frac{A_w(h_r-h_w)}{A_r h_w} = 0$ and $(n_w)^2 - n_w - \frac{A_w(h_r-h_w)}{A_r h_w} = 0$. Some further adjustment is necessary to take into account $h_r \leq h_w$, for which, with $\frac{\partial \Pi^s(n_w)}{\partial n_w}_{|n_w>1} < 0$, a lot-for-lot policy is optimal. Choosing the maximum of an arbitrary constant in the set $(0,2]$ ($[0,2)$) and the term $\frac{A_w(h_r-h_w)}{A_r h_w}$ makes that all scenarios are included in

$$n_w^{*s} = \begin{cases} \left\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \max\left\{1, \frac{A_w\left(h_r-h_w\right)}{A_r h_w}\right\}}\right\rceil \\ \text{or} \\ \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \max\left\{1, \frac{A_w\left(h_r-h_w\right)}{A_r h_w}\right\}}\right\rfloor \end{cases} \tag{5.3.2}$$

Retailer orders are combined in a single batch to a larger extent when ordering is relatively expensive for the Wholesaler ($\frac{A_w}{A_r}$ large) and when the cost structures favour holding inventory upstream ($\frac{h_r-h_w}{h_w}$ large). Smaller ratios let the optimum move towards a collaborative lot-for-lot policy. Again, two successive integers may be optimal.

Welfare maximizing outcome A solution to the joint optimization problem is comprised of n_w^{*s}, and Q^{*s}, which is calculated by substituting n_w^{*s} into $Q^{*s}\left(n_w\right)$. With these values, profits for each echelon are

$$\begin{aligned} \Pi_r^{*s} &= \left(P_r - P_w\right) D - \left(\frac{A_r}{2\left(A_r + A_w/n_w^{*s}\right)} + \frac{h_r}{2\left(h_r + h_w\left(n_w^{*s}-1\right)\right)}\right) \\ &\quad \cdot \sqrt{2\left(A_r + A_w/n_w^{*s}\right)\left(h_r + h_w\left(n_w^{*s}-1\right)\right) D} \\ \Pi_w^{*s} &= \left(P_w - P_m\right) D - \left(\frac{A_w/n_w^{*s}}{2\left(A_r + A_w/n_w^{*s}\right)} + \frac{h_w\left(n_w^{*s}-1\right)}{2\left(h_r + h_w\left(n_w^{*s}-1\right)\right)}\right) \\ &\quad \cdot \sqrt{2\left(A_r + A_w/n_w^{*s}\right)\left(h_r + h_w\left(n_w^{*s}-1\right)\right) D} \end{aligned}$$

Because Q^{*s} does not have to coincide with the individually optimal Q^{*u}, the inequality $\Pi_r^{*s} \leq \Pi_r^{*u}$ holds. By definition, we have $\Pi^{*s} = \Pi_r^{*s} + \Pi_w^{*s} \geq \Pi^{*u}$, and thereby $\Pi_w^{*s} \geq \Pi_w^{*u}$. As in the previous section, a change in demand merely scales the downstream order size and the resulting lot-sizing costs for both actors.

5.4 Quantity discount schedule

Overview of profits When a quantity discount scheme is introduced, the lot-sizing stages are preceded by the Wholesaler's decision on α_w and R_w. Both coordination variables appear in the extended gross margin expression for the Retailer. If the downstream order quantity is in the discount region, the indicator variable $1_{\{Q\in R_w\}}$ takes the value 1 and the Wholesaler charges $(1-\alpha_w)P_w$. Otherwise, no discount is granted and the regular unit price applies. The remaining terms of Equation (5.2.1) do not change in

$$\Pi_r^a(Q,\alpha_w,R_w) = \left(P_r - \left(1-\alpha_w 1_{\{Q\in R_w\}}\right)P_w\right)D - A_r\frac{D}{Q} - h_r\frac{Q}{2}$$

Incorporating the quantity discount policy similarly in Equation (5.2.2) gives

$$\Pi_w^a(Q,n_w,\alpha_w,R_w) = \left(\left(1-\alpha_w 1_{\{Q\in R_w\}}\right)P_w - P_m\right)D - A_w\frac{D}{n_w Q} - h_w\frac{(n_w-1)Q}{2}$$

Wholesaler lot-sizing stage At his lot-sizing decision nodes, the quantity discount characteristics and the Retailer order quantity have already been determined. As the objective function only differs by a constant $\alpha_w 1_{\{Q\in R_w\}}P_w D$ from $\Pi_w^u(Q,n_w)$ in Problem (5.2.3), Equation (5.2.4) remains optimal for

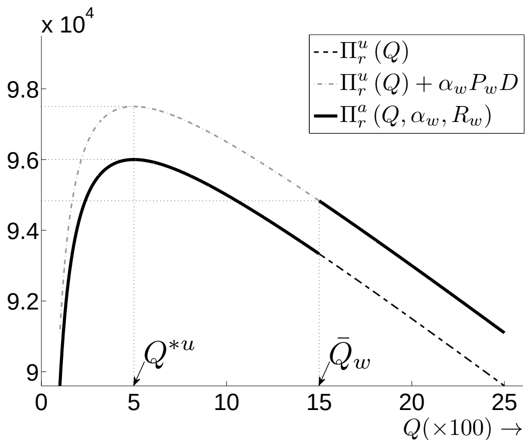
$$\begin{aligned}\max_{n_w}\Pi_w^a(Q,n_w,\alpha_w,R_w) &= \left(\left(1-\alpha_w 1_{\{Q\in R_w\}}\right)P_w - P_m\right)D - A_w\frac{D}{n_w Q} - h_w\frac{(n_w-1)Q}{2}\\ \text{subject to:}&\\ n_w &\in \mathbb{N}\end{aligned} \tag{5.4.1}$$

Retailer response The decision on the downstream inventory policy is no longer static, but depends on the vector (α_w, R_w). The reaction to a particular quantity discount layout follows from solving

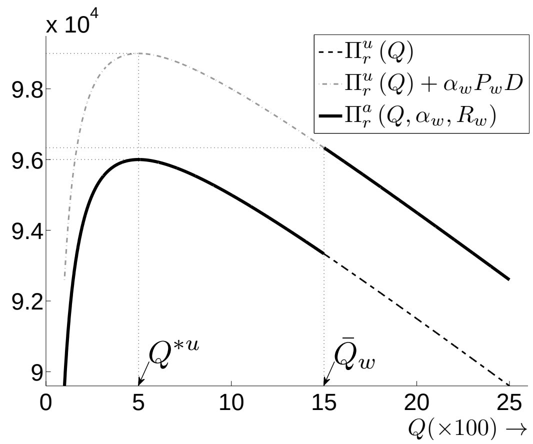
$$\begin{aligned}\max_{Q}\Pi_r^a(Q,\alpha_w,R_w) &= \left(P_r - \left(1-\alpha_w 1_{\{Q\in R_w\}}\right)P_w\right)D - A_r\frac{D}{Q} - h_r\frac{Q}{2}\\ \text{subject to:}&\\ Q &> 0\end{aligned}$$

The discontinuity of the profit function at $\bar{Q}_w$ makes finding the best order size somewhat more difficult than in case of Problem (5.2.5). Given an arbitrary set of price, demand and cost data, Figure 5.1 illustrates how altering the conditions of the AQD scheme influences the

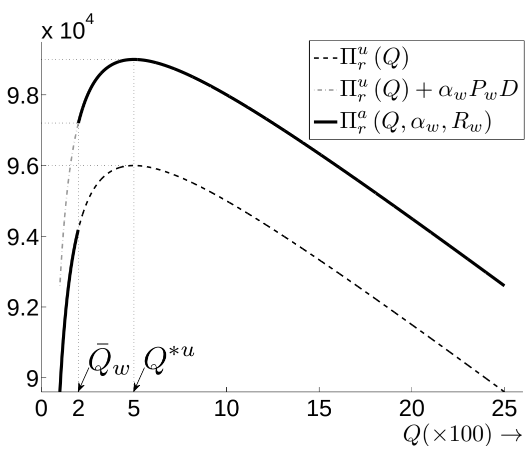
Retailer's options. In the no discount region ($Q \notin R_w$), profit just equals $\Pi_r^u(Q)$. For all order sizes in the discount region ($Q \in R_w$), $\Pi_r^a(Q, \alpha_w, R_w)$ coincides with the non-coordination profit shifted upwards by the average total discount per time unit. Because the two curves run parallel to one another, the unrestricted optimal order size for both is located at Q^{*u} in Equation (5.2.6).



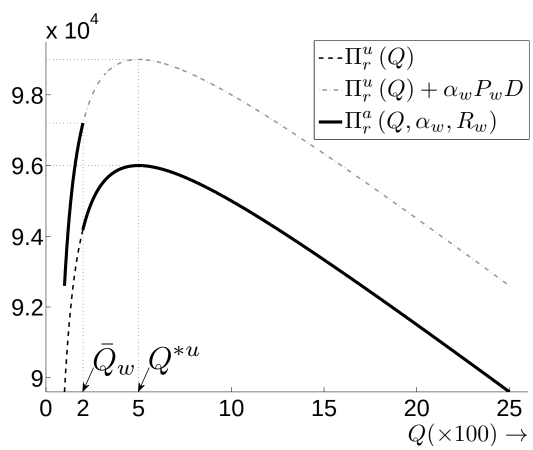
(a) $\alpha_w = 0.01$ and $R_w = [1500, \infty)$



(b) $\alpha_w = 0.02$ and $R_w = [1500, \infty)$



(c) $\alpha_w = 0.02$ and $R_w = [200, \infty)$



(d) $\alpha_w = 0.02$ and $(0, 200]$

Figure 5.1: *Optimization with $P_r = 25$, $P_w = 15$, $D = 10000$, $A_r = 100$ and $h_r = 8$ by the Retailer facing an AQD scheme*

Typical examples of the trade-off with $\bar{Q}_w \geq Q^{*u}$ are contained in the first two subfigures. The Retailer compares his profit at the breakpoint, which is the best among all discount order sizes, with the maximum non-coordination profit Π_r^{*u}. Figure 5.1(a) depicts a configuration wherein the Retailer will not deviate from his anarchy decision. After the discount has been increased to 2% in Figure 5.1(b), he chooses $\bar{Q}_w = 1500$. Figure 5.1(c) makes clear why a proper AQD schedule is incapable of provoking more frequent ordering downstream: the discount is received even when nothing changes at the Retailer. As shown in Figure 5.1(d), allowing quantity premiums removes this limitation.

The previous observations reveal the essence of the optimal response. The Retailer chooses $\bar{Q}_w$ if the default solution Q^{*u} is not part of the discount region minus the order breakpoint, and the profit at the breakpoint at least equals the maximum under anarchy:

$$Q^{*a}\left(\alpha_w, R_w\right) = \begin{cases} \bar{Q}_w & \text{if } Q^{*u} \notin \left(R_w \setminus \bar{Q}_w\right) \text{ and } \Pi_r^a\left(\bar{Q}_w, \alpha_w, R_w\right) \geq \Pi_r^{*u} \\ \text{and} & \\ Q^{*u} & \text{otherwise} \end{cases} \tag{5.4.2}$$

In view of reformulating the Wholesaler coordination problem, $Q^{*a}\left(\alpha_w, R_w\right)$ formally points at the breakpoint in the special case where both lot-sizing quantities are the same. We also assume that an indifferent Retailer accepts the schedule. Without the assumption, a Wholesaler not satisfied with Q^{*u}, could be forced to offer a scheme with a slightly higher discount to make the Retailer strictly prefer $\bar{Q}_w$. However, there is always a smaller discount that does the job, and, as a consequence, a Nash equilibrium might then not exist.

Coordination stage To take into account the indirect control over the Retailer order size, we rewrite the Wholesaler profit function as $\Pi_w^a\left(n_w, \alpha_w, R_w\right) = \Pi_w^a\left(Q^{*a}\left(\alpha_w, R_w\right), n_w, \alpha_w, R_w\right)$. The first stage problem is more easily solved by disregarding the rules in Equation (5.2.4), and instead include n_w in

$$\begin{aligned} \max \Pi_w^a\left(n_w, \alpha_w, R_w\right) = {} & \left(\left(1 - \alpha_w 1_{\{Q^{*a}(\alpha_w, R_w) \in R_w\}}\right) P_w - P_m\right) D \\ & - A_w \frac{D}{n_w Q^{*a}\left(\alpha_w, R_w\right)} - h_w \frac{\left(n_w - 1\right) Q^{*a}\left(\alpha_w, R_w\right)}{2} \\ \text{subject to:} & \\ & n_w \in \mathbb{N} \qquad \alpha_w \geq 0 \\ & R_w \in \left\{\left(0, \bar{Q}_w\right], \left[\bar{Q}_w, \infty\right)\right\} \qquad \bar{Q}_w > 0 \end{aligned} \tag{5.4.3}$$

Every downstream order size can be achieved by having the Retailer opt for the breakpoint of an AQD scheme. At the breakpoint it suffices that he is just as well of as under anarchy, i.e., the loss resulting from deviation of Q^{*u} is exactly compensated. A proper quantity discount should be chosen for $\bar{Q}_w > Q^{*u}$, while $\bar{Q}_w < Q^{*u}$ requires a quantity premium. In the special case where $\bar{Q}_w = Q^{*u}$ (with $\alpha_w = 0$), the remainder of the discount region may be located on either side of the breakpoint. In terms of n_w and $\bar{Q}_w$ we restate the problem as

$$\max \Pi_w^a \left(n_w, \bar{Q}_w\right) = \left(\left(1 - \frac{\Pi_r^{*u} - \Pi_r^u \left(\bar{Q}_w\right)}{P_w D}\right) P_w - P_m\right) D - A_w \frac{D}{n_w \bar{Q}_w} - h_w \frac{\left(n_w - 1\right) \bar{Q}_w}{2}$$

$$\text{subject to:} \qquad (5.4.4)$$

$$n_w \in \mathbb{N} \qquad \bar{Q}_w > 0$$

After substitution of the expression for $\Pi_r^u \left(\bar{Q}_w\right)$, and aggregating cost components per category, the reformulation becomes

$$\max \Pi_w^a \left(n_w, \bar{Q}_w\right) = \left(P_r - P_m\right) D - \left(A_r + A_w / n_w\right) \frac{D}{\bar{Q}_w} - \left(h_r + h_w \left(n_w - 1\right)\right) \frac{\bar{Q}_w}{2} - \Pi_r^{*u}$$

$$\text{subject to:}$$

$$n_w \in \mathbb{N} \qquad \bar{Q}_w > 0$$

In essence, apart from a constant and a change of variables, Problem (5.3.1) needs to be solved. The best integer n_w^{*a} equals n_w^{*s} in Equation (5.3.2), while the optimal breakpoint $\bar{Q}_w^*$ corresponds to the jointly optimal downstream order size Q^{*s}. Because the Retailer agrees to Q^{*s}, his pre-discount profit is Π_r^{*s}. From the solution we derive

$$\alpha_w^* = \frac{\Pi_r^{*u} - \Pi_r^{*s}}{P_w D} \qquad (5.4.5)$$

and the discount region

$$R_w^* = \begin{cases} \left(0, Q^{*s}\right] 1_{\{Q^{*s} < Q^{*u}\}} + \left[Q^{*s}, \infty\right) 1_{\{Q^{*s} \geq Q^{*u}\}} \\ \text{or} \\ \left(0, Q^{*s}\right] 1_{\{Q^{*s} \leq Q^{*u}\}} + \left[Q^{*s}, \infty\right) 1_{\{Q^{*} > Q^{*u}\}} \end{cases} \qquad (5.4.6)$$

By construction, two possible regions are defined if $Q^{*s} = Q^{*u}$. In fact, the location of R_w in a zero-discount scheme is not relevant at all. Moreover, for a Wholesaler satisfied with the non-coordination outcome, any schedule in which the Retailer is left with too little profit at the breakpoint, and therefore rejected by selecting Q^{*u}, is optimal.

Equilibrium strategies and efficiency increase Without further proof, we interpret the preceding analysis game theoretically in Proposition 5.4.1. Whether or not the supply chain needs the quantity discount scheme to operate more profitably, the presence of $n_w^{*u}(Q)$ makes that an infinite number of Nash strategy profiles is described.

Proposition 5.4.1. *The strategy profiles* $Q^{*a}(\alpha_w, R_w) \times [\alpha_w^*, R_w^*, n_w^{*u}(Q)]$ *satisfying Equations* (5.4.2), (5.4.5), (5.4.6) *and* (5.2.4), *are subgame perfect pure Nash equilibria in the quantity discount game. If* $Q^{*s} = Q^{*u}$, *every AQD set-up with* $\alpha_w = 0$, *or any schedule rejected by the Retailer, can be part of an equilibrium.*

Assuming the Retailer responds to α_w^* and R_w^* as anticipated (selecting the breakpoint $\bar{Q}_w^* = Q^{*s}$), the Wholesaler will be faced with a reduced Problem (5.4.4), which is solved by $n_w^{*u}(Q^{*s})$. The lot-sizing rule producing the same integer as in the coordination stage of the game shows that $n_w^{*a} = n_w^{*s}$ is situated on an equilibrium path. The number of outcomes (including those with rejected schedules) is infinite in case Q^{*u} is part of a jointly optimal inventory policy. Otherwise, because of the direct link with the solutions in Section 5.3, the maximum is 2.

While the Retailer does not improve upon his anarchy profit: $\Pi_r^{*a} = \Pi_r^{*u}$, the Wholesaler receives $\Pi_w^{*a} = \Pi^{*s} - \Pi_r^{*u}$ with $\Pi_w^{*u} \leq \Pi_w^{*a} \leq \Pi_w^{*s}$. The supply chain is fully aligned and any additional surplus accrues to the upstream actor. Complete synchronization of the supply chain has been observed before by Banerjee (1986) in a context where a lot-for-lot replenishment policy is imposed.

5.5 Holding cost compensation

Influence on profits Instead of setting a discount and an appropriate discount region, the Wholesaler targets the downstream holding costs in Equation (5.2.1) under the alternative scheme. The profit of a Retailer who gets a fraction β_w reimbursed is

$$\Pi_r^b(Q, \beta_w) = (P_r - P_w) D - A_r \frac{D}{Q} - (1 - \beta_w) h_r \frac{Q}{2}$$

After subtracting the total compensation per time unit in Equation (5.2.2), the Wholesaler profit becomes

$$\Pi_w^b(Q, n_w, \beta_w) = (P_w - P_m) D - A_w \frac{D}{n_w Q} - h_w \frac{(n_w - 1) Q}{2} - \beta_w h_r \frac{Q}{2}$$

Lot-sizing stages Like in Problem (5.4.1), the terms related to the coordination mechanism drop out in the last stage of the game, and, consequently, Equation (5.2.4) is maintained to maximize $\Pi_w^b(Q, n_w, \beta_w)$. Downstream, adjustment of Q^{*u} in Equation (5.2.6) to the new holding cost rate gives

$$Q^{*b}(\beta_w) = \sqrt{\frac{2 A_r D}{(1 - \beta_w) h_r}} \tag{5.5.1}$$

The Retailer cannot be triggered to decrease his order size: $Q^{*b}(\beta_w) \geq Q^{*u}$.

Coordination stage Before deciding on the best level of compensation, the Wholesaler updates his profit to $\Pi_w^b(n_w, \beta_w) = \Pi_w^b\left(Q^{*b}(\beta_w), n_w, \beta_w\right)$. Similar to Problem (5.4.3), the lot-sizing variable is part of

$$\begin{aligned} \max \Pi_w^b(n_w, \beta_w) &= (P_w - P_m) D - \left(\frac{A_w/n_w}{2A_r} + \frac{h_w(n_w - 1) + \beta_w h_r}{2(1-\beta_w) h_r}\right)\sqrt{2A_r(1-\beta_w) h_r D} \\ \text{subject to:} & \\ & n_w \in \mathbb{N} \qquad \beta_w \in [0, 1) \end{aligned} \tag{5.5.2}$$

For the moment we ignore the domain restriction on β_w, and, for a fixed n_w, take the derivative

$$\frac{\partial \Pi_w^b(n_w, \beta_w)}{\partial \beta_w} = Q^{*b}(\beta_w)\left(\frac{(A_w/n_w - A_r) h_r}{4A_r} - \frac{h_w(n_w - 1) + h_r}{4(1-\beta_w)}\right)$$

With $Q^{*b}(\beta_w) > 0$, the first order condition is solved by

$$\bar{\beta}_w(n_w) = 1 - \frac{A_r(h_w(n_w - 1) + h_r)}{(A_w/n_w - A_r) h_r} \tag{5.5.3}$$

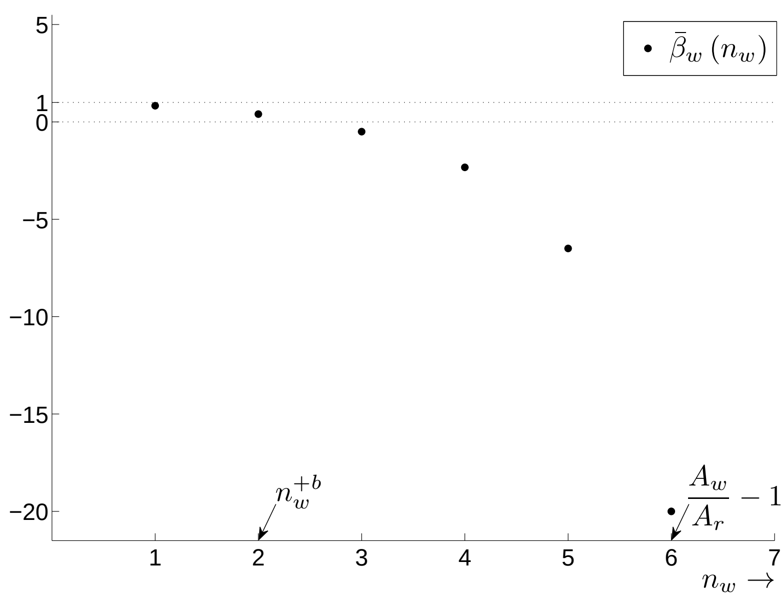


Figure 5.2: *Stationary points for relaxation of Wholesaler's HCC problem with $A_r = 100$, $h_r = 8$, $A_w = 700$ and $h_w = 4$*

Provided that $n_w \neq \frac{A_w}{A_r}$ (no division by zero), the solution denotes a stationary point in case $\bar{\beta}_w(n_w) < 1$, as $Q^{*b}(\beta_w)$, and thereby the derivative, are not defined for $\beta_w \geq 1$. The largest lot-sizing multiple with a stationary point (not necessarily positive) is either $\frac{A_w}{A_r} - 1$, if the ratio of fixed order costs is integer valued, or $\left\lfloor \frac{A_w}{A_r} \right\rfloor$, if it is not. Although we continue to have $A_r(h_w(n_w - 1) + h_r) > 0$, the denominator $(A_w/n_w - A_r)h_r$ becomes non-positive when going beyond these bounds. Typical behaviour is shown in Figure 5.2: $\bar{\beta}_w(n_w)$ starts below 1 at $n_w = 1$, and subsequently decreases.

The largest integer with a feasible stationary point ($\bar{\beta}_w \in [0, 1)$) equals 2. The behaviour of the function till $\frac{A_w}{A_r}$ allows n_w^{+b} to be described as

$$n_w^{+b} = \max\left\{ n_w : \frac{A_r(h_w(n_w - 1) + h_r)}{(A_w/n_w - A_r)h_r} \leq 1 | n_w \in \mathbb{N} \cup \{0\}, n_w < \frac{A_w}{A_r} \right\}$$

Zero is included in the domain, since the existence of a positive integer with a non-negative stationary point is not guaranteed. Turning the condition into an equality, taking the positive (non-integer) solution ν_w^{+b} and rounding down gives

$$n_w^{+b} = \left\lfloor \nu_w^{+b} \right\rfloor \tag{5.5.4}$$

where:

$$\nu_w^{+b} = \frac{1}{2} - \frac{h_r}{h_w} + \sqrt{\left(\frac{1}{2} - \frac{h_r}{h_w}\right)^2 + \frac{A_w h_r}{A_r h_w}}$$

While rewriting, the condition is multiplied on both sides by the possibly non-positive term $(A_w/n_w - A_r)h_r$. We show that this complication poses no problems.

Lemma 5.5.1. *The expression for the upper bound n_w^{+b} represents the maximum positive integer with a stationary point in the domain $[0, 1)$. In case there is no such integer, $n_w^{+b} = 0$.*

Proof. The equality $A_r\left(h_w\left(\nu_w^{+b} - 1\right) + h_r\right) = \left(A_w/\nu_w^{+b} - A_r\right)h_r$ is satisfied by the positive solution ν_w^{+b} to the quadratic equation $(n_w)^2 + \left(2\frac{h_r}{h_w} - 1\right)n_w - \frac{A_w h_r}{A_r h_w} = 0$. There are two scenarios to consider:
(i) When $0 < \nu_w^{+b} < \frac{A_w}{A_r}$, it is obvious that $n_w^{+b} = \left\lfloor \nu_w^{+b} \right\rfloor < \frac{A_w}{A_r}$. In this scenario, both sides of the equality are positive. With $A_r(h_w(n_w - 1) + h_r)$ an increasing and $(A_w/n_w - A_r)h_r$ a decreasing function of $n_w \in \mathbb{N}$, integers $\nu_w^{+b} < n_w < \frac{A_w}{A_r}$ have no feasible stationary point, while the reverse is true for each positive integer $n_w \leq n_w^{+b}$;
(ii) $\nu_w^{+b} \geq \frac{A_w}{A_r}$ means $\left(A_w/\nu_w^{+b} - A_r\right)h_r \leq 0$ and because of the equality characterizing ν_w^{+b}, we get as well $A_r\left(h_w\left(\nu_w^{+b} - 1\right) + h_r\right) \leq 0$. The last inequality requires $h_w > h_r$ and $\nu_w^{+b} < 1$.

With $\frac{A_w}{A_r} \leq \nu_w^{+b} < 1$, there are no positive integers with a stationary point in the relevant domain and so $n_w^{+b} = \lfloor \nu_w^{+b} \rfloor = 0$ is exactly the bound we need. $\square$

In Figure 5.2, n_w^{+b} looks quite small compared to $\frac{A_w}{A_r} - 1$. It turns out that $n_w^{+b} \leq \left\lfloor \frac{A_w}{2A_r} \right\rfloor$, once we note that $\frac{h_w(n_w-1)+h_r}{4(1-\beta_w)}$ in the expression for the derivative cannot be smaller than $\frac{h_r}{4}$ if $n_w \in \mathbb{N}$ and $\beta_w \in [0,1)$. At a feasible stationary point, we therefore have $\frac{(A_w/n_w - A_r)h_r}{4A_r} \geq \frac{h_r}{4}$, which, after some manipulation, is seen to be equivalent to $\frac{(A_w/n_w - 2A_r)h_r}{4A_r} \geq 0$. Another upper bound on n_w^{+b} is provided by the anarchistic lot-sizing multiple(s):

Lemma 5.5.2. *Equation (5.2.7) bounds the largest integer with a feasible stationary point from above: $n_w^{+b} \leq n_w^{*u}$.*

Proof. We start with a brief overview: ν_w^{-u} in Equation (5.2.8) is the positive solution of $(n_w)^2 + n_w - \frac{A_w h_r}{A_r h_w} = 0$, from $(n_w)^2 - n_w - \frac{A_w h_r}{A_r h_w} = 0$ we obtain ν_w^{+u} in Equation (5.2.9), and ν_w^{+b} solves $(n_w)^2 + \left(2\frac{h_r}{h_w} - 1\right) n_w - \frac{A_w h_r}{A_r h_w} = 0$:
(i) $h_r \geq h_w$ implies $2\frac{h_r}{h_w} - 1 \geq 1$ and $\nu_w^{+b} \leq \nu_w^{-u} < \nu_w^{+u}$. The inequality $\lfloor \nu_w^{+b} \rfloor \leq \lceil \nu_w^{-u} \rceil$ clearly holds. With $\nu_w^{+u} - \nu_w^{+b} \geq \nu_w^{+u} - \nu_w^{-u} = 1$, we get $\lfloor \nu_w^{+b} \rfloor < \lfloor \nu_w^{+u} \rfloor$;
(ii) $h_r < h_w$ results in $-1 < 2\frac{h_r}{h_w} - 1 < 1$. Since $\nu_w^{-u} < \nu_w^{+b} < \nu_w^{+u}$, $\lfloor \nu_w^{+b} \rfloor \leq \lfloor \nu_w^{+u} \rfloor$ immediately follows. Rounding up ν_w^{-u} returns the single integer in the set $[\nu_w^{-u}, \nu_w^{+u})$, rounding down ν_w^{+b} gives the same integer or the largest one smaller than ν_w^{-u}. $\square$

Next, we formally establish the optimal compensation level per integer $\beta_w^* (n_w)$. Its non-increasing nature is a direct consequence of the relationship with $\bar{\beta}_w (n_w)$: to curtail the detrimental effect of larger integer multiples on inventory holding costs, the Wholesaler is less inclined to stimulate the Retailer to lengthen his order cycle. Ceteris paribus, relatively expensive ordering and an attractive holding cost rate upstream tend to increase the level of compensation.

Lemma 5.5.3. *For each lot-sizing integer $n_w \in \mathbb{N}$, the Wholesaler's optimal holding cost compensation level is given by $\beta_w^* (n_w) = \bar{\beta}_w (n_w) 1_{\{n_w \leq n_w^{+b}\}}$.*

Proof. We break down $\mathbb{N}$ into three regions. In each we use that $Q^{*b} (\beta_w) > 0$:
(i) $n_w \leq n_w^{+b}$: The characteristics of these lot-sizing integers are $\frac{(A_w/n_w - A_r)h_r}{4A_r} > 0$ and non-negativity of the stationary point. The term $\frac{h_w(n_w-1)+h_r}{4(1-\beta_w)}$ increases in $\beta_w < 1$. The derivative $\frac{\partial \Pi_w^b (n_w, \beta_w)}{\partial \beta_w}$ positive for $\beta_w \in \left[0, \bar{\beta}_w (n_w)\right)$ and negative for $\beta_w \in \left(\bar{\beta}_w (n_w), 1\right)$ shows that the stationary point maximizes profit;
(ii) $n_w^{+b} < n_w < \frac{A_w}{A_r}$: As $\frac{(A_w/n_w - A_r)h_r}{4A_r} > 0$ continues to hold, a stationary point still exists. Its location below zero means $\frac{(A_w/n_w - A_r)h_r}{4A_r} < \frac{h_w(n_w-1)+h_r}{4(1-\beta_w)}$ and consequently $\frac{\partial \Pi_w^b (n_w, \beta_w)}{\partial \beta_w} < 0$

on the domain $[0, 1)$. Moving the compensation level away from 0 decreases profit;
(iii) $n_w \geq \frac{A_w}{A_r}$: Because $\frac{(A_w/n_w - A_r)h_r}{4A_r} \leq 0$ and $\frac{h_w(n_w-1)+h_r}{4(1-\beta_w)} > 0$, the derivative is negative for each feasible compensation level. Again, no compensation is optimal. $\square$

Based on the previous lemma, we reformulate Problem (5.5.2) entirely in terms of n_w. For $n_w \leq n_w^{+b}$, the feasible stationary points are optimal and $\bar{\beta}_w(n_w)$ is inserted. For larger integers, the Retailer chooses Q^{*u} in response to $\beta_w^*(n_w) = 0$, and the Wholesaler's profit boils down to $\Pi_w^u(Q^{*u}, n_w)$ (compare Equation (5.2.11)):

$$\begin{aligned}
\max \Pi_w^b(n_w) = & (P_w - P_m) D - 1_{\{n_w \leq n_w^{+b}\}} \sqrt{2 (A_w/n_w - A_r)(h_w(n_w - 1) + h_r) D} \\
& - 1_{\{n_w > n_w^{+b}\}} \left(\frac{A_w/n_w}{2A_r} + \frac{h_w(n_w - 1)}{2h_r} \right) \sqrt{2A_r h_r D} \\
\text{subject to:} & \\
& n_w \in \mathbb{N}
\end{aligned} \tag{5.5.5}$$

As long as $\beta_w^*(1) > 0$ is satisfied, imposing $n_w \geq 1$ instead of $n_w \in \mathbb{N}$ results in $\Pi_w^b(n_w)$ consisting of two continuous curves with the point of intersection at $\nu_w^{+b} > 1$. The derivative of the first curve with respect to n_w is

$$\frac{\partial \Pi_w^b(n_w)}{\partial n_w}_{|1<n_w<\nu_w^{+b}} = -\frac{\sqrt{D}\left(A_w(h_w - h_r)/(n_w)^2 - A_r h_w\right)}{\sqrt{2(A_w/n_w - A_r)(h_w(n_w - 1) + h_r)}}$$

The solution to the reduced first order condition $A_w(h_w - h_r)/(n_w)^2 - A_r h_w = 0$ equals

$$\bar{\nu}_w^b = \sqrt{\frac{A_w(h_w - h_r)}{A_r h_w}}$$

If $h_r < h_w$, the derivative is negative below and positive beyond $\bar{\nu}_w^b$. Profit always strictly increases between 1 and ν_w^{+b} when $h_r \geq h_w$. Finding the best integer n_w^{*b} and the associated optimal compensation β_w^* for an arbitrary problem instance with $n_w^{+b} \geq 1$ thus requires checking three integers: 1 and n_w^{+b}, being the extremes within the set of integers having a feasible stationary point, and the anarchy solution n_w^{*u}. The non-coordination, zero compensation, policy must be optimal in case $n_w^{+b} = 0$.

As shown below, the lack of any integer with a positive optimal compensation level can be tied to certain parameter characteristics. Additionally, there are circumstances for which we can establish beforehand that the status quo persists despite $\beta_w^*(n_w) > 0$ for some lot-sizing multiple.

Proposition 5.5.1. *The anarchy integer n_w^{*u} solves Problem (5.5.5), and the Wholesaler chooses $\beta_w^* = 0$, if $\frac{A_w}{A_r} \leq 2$ or $\frac{h_r}{h_w} \geq 1$.*

Proof. (i) $\frac{A_w}{A_r} \leq 2$ guarantees $\nu_w^{+b} \leq 1$, so that n_w^{+b} equals either 0 or 1 with $\beta_w^*(1) = 0$. As no compensation is optimal for each integer, the Wholesaler faces his non-coordination problem; (ii) For the alternative condition, we can limit ourselves to $\frac{h_r}{h_w} \geq 1$ with $\frac{A_w}{A_r} > 2$. Of course, n_w^{*u} yielding Π_w^{*u} is optimal among all integers with $\beta_w^*(n_w) = 0$. The derivative $\frac{\partial \Pi_w^b(n_w)}{\partial n_w}\Big|_{1<n_w<\nu_w^{+b}}$ is strictly positive up till ν_w^{+b}, and, as the second curve is concave with its maximum at ν_w^{*u} in Equation (5.2.13), profit further increases between ν_w^{+b} and ν_w^{-u} (the inequality $\nu_w^{+b} \leq \nu_w^{-u}$ follows directly from the proof of Lemma 5.5.2). Complementing with $\Pi_w^{*u} \geq \Pi_w^u(Q^{*u}, \nu_w^{-u})$ in Equation (5.2.12) shows that n_w^{*u} without compensation outperforms as well any integer having $\beta_w^*(n_w) > 0$. □

Before identifying the solution for two other classes of scenarios, we take the inventory related costs for $n_w = 1$, expressed as $\sqrt{\frac{A_w}{A_r} - 1}\sqrt{2A_r h_r D}$, and $\left(\sqrt{\frac{A_w h_w}{A_r h_r}} - \frac{h_w}{2h_r}\right)\sqrt{2A_r h_r D}$ in Equation (5.2.14). Assuming $\frac{A_w}{A_r} > 2$, the profit inequality $\Pi_w^b(1) \geq \Pi_w^u(Q^{*u}, \nu_w^{*u})$ is satisfied if $\sqrt{\frac{A_w}{A_r} - 1} + \frac{h_w}{2h_r} \leq \sqrt{\frac{A_w h_w}{A_r h_r}}$. The (concave) square root at least equals the increasing linear term for $\frac{h_w}{h_r}$ in the compact set enclosed by

$$\frac{h_w}{h_r}^{-b,+b} = 2\left(\frac{A_w}{A_r} - \sqrt{\frac{A_w}{A_r} - 1} \mp \sqrt{\frac{A_w}{A_r}\left(\frac{A_w}{A_r} - 2\sqrt{\frac{A_w}{A_r} - 1}\right)}\right)$$

Both bounds solve $\frac{1}{4}\left(\frac{h_w}{h_r}\right)^2 + \left(\sqrt{\frac{A_w}{A_r} - 1} - \frac{A_w}{A_r}\right)\frac{h_w}{h_r} + \left(\frac{A_w}{A_r} - 1\right) = 0$.

Proposition 5.5.2. *A lot-for-lot policy is optimal for Problem (5.5.5), and the Wholesaler chooses $\beta_w^* = 1 - \frac{A_r}{A_w - A_r}$, if $\frac{A_w}{A_r} > 2$, and either $\frac{A_w}{A_r} \leq 2\frac{h_w}{h_r}$ or $\frac{h_w}{h_r} \in \left[\frac{h_w}{h_r}^{-b}, \frac{h_w}{h_r}^{+b}\right]$.*

Proof. The condition $\frac{A_w}{A_r} > 2$ causes $\beta_w^*(1) = \bar{\beta}_w(1) = 1 - \frac{A_r}{A_w - A_r}$ and $n_w^{+b} \geq 1$. In both cases, $\frac{h_w}{h_r} > 1$ is implicit. For the combination with
(i) $\frac{A_w}{A_r} \leq 2\frac{h_w}{h_r}$, we have $\nu_w^{-u} \leq 1$, $\lceil \nu_w^{-u} \rceil \leq 1$, and by Lemma 5.5.2, $n_w^{+b} \leq 1$. The(an) anarchy integer coincides with $n_w^{+b} = 1$, for which, in fact, a positive compensation level is optimal;
(ii) $\frac{h_w}{h_r} \in \left[\frac{h_w}{h_r}^{-b}, \frac{h_w}{h_r}^{+b}\right]$, knowing that $\bar{\nu}_w^b > 1$ holds, the Wholesaler profit first decreases from $\Pi_w^b(1) \geq \Pi_w^u(Q^{*u}, \nu_w^{*u})$, and, may subsequently increase again, till $\Pi_w^b\left(\nu_w^{+b}\right) \leq \Pi_w^u(Q^{*u}, \nu_w^{*u})$ on the first continuous curve. The smallest element in the set $\mathbb{N}$ outperforms all other integers with $\beta_w^*(n_w) > 0$, and, given $\Pi_w^{*u} \leq \Pi_w^u(Q^{*u}, \nu_w^{*u})$ in Equation (5.2.12), is at least as profitable as the anarchy integer without any compensation. We can rule out $\bar{\nu}_w^b \leq 1$, as the profit strictly increasing along the same curve implies $\Pi_w^b(1) < \Pi_w^b\left(\nu_w^{+b}\right) = \Pi_w^u\left(Q^{*u}, \nu_w^{+b}\right) \leq \Pi_w^u(Q^{*u}, \nu_w^{*u})$, which is at odds with $\frac{h_w}{h_r}$ in the relevant set. □

Equilibrium Each actor's rational behaviour is combined in Proposition 5.5.3. Again, an infinite number of Nash strategy profiles is covered.

Proposition 5.5.3. *All subgame perfect pure Nash equilibria in the holding cost compensation game are given by the strategy profiles $Q^{*b}(\beta_w) \times [\beta_w^*, n_w^{*u}(Q)]$ satisfying Equations (5.5.1) and (5.2.4), and where β_w^* is the optimal compensation level associated with the integer n_w^{*b} solving Problem (5.5.5).*

Inserting β_w^* in $Q^{*b}(\beta_w)$ gives the Retailer's action Q^{*b}. We formally prove that n_w^{*b} is implemented in the last stage of the game. The Wholesaler profit for $n_w = 0$ is assumed to be $-\infty$.

Lemma 5.5.4. *The integer n_w^{*b} is part of an equilibrium path.*

Proof. The inequalities $\Pi_w^b\left(Q^{*b}, n_w^{*b}+1, \beta_w^*\right) \leq \Pi_w^b\left(Q^{*b}, n_w^{*b}, \beta_w^*\right)$ and $\Pi_w^b\left(Q^{*b}, n_w^{*b}-1, \beta_w^*\right) \leq \Pi_w^b\left(Q^{*b}, n_w^{*b}, \beta_w^*\right)$ must hold if n_w^{*b} is to follow from Equation (5.2.4);
(i) We get $\Pi_w^b\left(Q^{*b}, n_w^{*b}+1, \beta_w^*\right) \leq \Pi_w^b\left(Q^{*b}\left(\beta_w^*\left(n_w^{*b}+1\right)\right), n_w^{*b}+1, \beta_w^*\left(n_w^{*b}+1\right)\right)$ based on Lemma 5.5.3 and $\Pi_w^b\left(Q^{*b}\left(\beta_w^*\left(n_w^{*b}+1\right)\right), n_w^{*b}+1, \beta_w^*\left(n_w^{*b}+1\right)\right) \leq \Pi_w^b\left(Q^{*b}, n_w^{*b}, \beta_w^*\right)$ because of optimality. Combining both gives $\Pi_w^b\left(Q^{*b}, n_w^{*b}+1, \beta_w^*\right) \leq \Pi_w^b\left(Q^{*b}, n_w^{*b}, \beta_w^*\right)$;
(ii) The second condition is obtained similarly, using $n_w^{*b}-1$ instead of $n_w^{*b}+1$. ☐

With only 1, n_w^{+b} and n_w^{*u} potentially relevant, the number of equilibrium paths is bounded by 3. When we have two values for the anarchy integer, either both are optimal if $h_r \geq h_w$ (see Proposition 5.5.1), or, as can be inferred from the proof of Lemma 5.5.2 for the case $h_r < h_w$, the smallest coincides with n_w^{+b}.

The maximum profit Π_w^{*b} follows directly from solving Problem (5.5.5). For the Retailer $(1-\beta_w^*)\, h_r$ instead of h_r in Equation (5.2.10) gives

$$\Pi_r^{*b} = (P_r - P_w)\, D - \sqrt{2A_r\,(1-\beta_w^*)\, h_r D} \tag{5.5.6}$$

The Wholesaler does not set a positive compensation level, unless his anarchy profit is matched. Neither actor is worse off: $\Pi_r^{*b} \geq \Pi_r^{*u}$ and $\Pi_w^{*b} \geq \Pi_w^{*u}$.

Demand does not affect $\bar{\beta}_w(n_w)$, nor the bound n_w^{+b}. It scales the Wholesaler lot-sizing costs for each integer in $\Pi_w^b(n_w)$ to the same extent, thereby leaving n_w^{*b} and β_w^* unaltered. The same scaling effect can be observed with respect to Q^{*b} and the resulting lot-sizing costs for the Retailer.

The procedure below is used to determine an equilibrium path. By only updating the policy if a strictly better alternative is found, we ensure, with $\bar{\beta}_w(1) \geq \bar{\beta}_w\left(n_w^{+b}\right) \geq 0$, that an indifferent Wholesaler opts for the maximum possible compensation.

Calculation of a 2 actor holding cost compensation outcome

Initialize : $\Pi_w^{*b} = -\infty$

`Determine` n_w^{+b} `in Equation` (5.5.4)

if $n_w^{+b} \geq 1$

  for $n_w \in \{1, n_w^{+b}\}$

    `Calculate` $\Pi_w^b(n_w)$ `in Problem` (5.5.5)

    if $\Pi_w^b(n_w) > \Pi_w^{*b}$

      `Set` $\Pi_w^{*b} = \Pi_w^b(n_w)$

      `Calculate` $\bar{\beta}_w(n_w)$ `in Equation` (5.5.3)

      `Calculate` $Q^{*b}\left(\bar{\beta}_w(n_w)\right)$ `with Equation` (5.5.1)

      `Set` $\beta_w^* = \bar{\beta}_w(n_w)$, $Q^{*b} = Q^{*b}\left(\bar{\beta}_w(n_w)\right)$ `and` $n_w^{*b} = n_w$

if $\Pi_w^{*u} > \Pi_w^{*b}$

  `Set` $\Pi_w^{*b} = \Pi_w^{*u}$

  `Set` $\beta_w^* = 0$, $Q^{*b} = Q^{*u}$ `and` $n_w^{*b} = n_w^{*u}$

`Determine` Π_r^{*b} `in Equation` (5.5.6)

`Set` $\Pi^{*b} = \Pi_w^{*b} + \Pi_r^{*b}$

Chapter 6

Simulation study

6.1 Introduction

To better assess the effects of each mechanism, we calculate some statistics for a set of numerical examples much resembling those in Munson & Rosenblatt (2001). In a fixed demand context, the (constant) total gross margins are irrelevant in the optimization problems and when determining differences in profits. After slightly modifying α_w^*, we can do completely without a specification of prices. Because the effect of demand is limited to scaling all resulting lot-sizing costs and Retailer order sizes, the common component $\sqrt{D}$ drops out in the calculation of the alternative quantity discount $\tilde{\alpha}_w^*$ and the performance measures: using multiple demand levels does not add extra information. We, therefore, fix D at 10,000. For the remaining parameters, we select 10 A_r values in the range from 20 to 200, 10 multiples of 50 as A_w values, and 12 equally spaced h_r values between 5 and 60. The holding cost rate h_w starts at 2 and increases till 35 in steps of 3. Solving all 14,400 problem instances and aggregating statistics in Matlab takes less than 2 minutes on a 2.1 GHz, 4 GB RAM computer. Section 6.2 gives an idea about how the Wholesaler implements each mechanism. We discuss the efficiency improvements in Section 6.3. The distribution of savings among the supply chain partners is the subject of Section 6.4.

6.2 Discount level & amount of compensation

Instead of $P_w D$ in Equation (5.4.5), we express the total quantity discount per time unit in terms of $\sqrt{2A_r h_r D}$ in Equation (5.2.10). Table 6.1 shows that the mean value of this new $\tilde{\alpha}_w^*$ is nearly 14%. The maximum discount even amounts to more than 1.5 times the Retailer's initial lot-sizing costs. Although the table suggests otherwise, there is no scenario with the

Wholesaler choosing a zero-discount scheme. All scenarios have an efficiency gap and require a reduction in price to move inventory policies to the joint optimum.

Looking at the statistics for β_w^*, we see that, on average, the Wholesaler's compensation is a modest 6.95%. Compared to the first mechanism, there is relatively more dispersion. Whereas the standard deviation divided by the mean is considerably less than 2 for $\tilde{\alpha}_w^*$, the same ratio exceeds 3 for β_w^*. On the one hand, the difference can be explained by the large percentage of scenarios without any compensation (a little more than 88%), while, on the other, as exemplified by the maximum fraction 0.95833, environments exist in which the Wholesaler is prepared to carry a major part of the Retailer's holding costs.

Value	Mean	Stdev.	Min.	Max.	Positive
$\tilde{\alpha}_w^*$	0.13855	0.23194	0.00000	1.64757	100.00 %
β_w^*	0.06950	0.21055	0.00000	0.95833	11.75 %

Table 6.1: *Summary statistics for coordination values*

In general, the Wholesaler pays a quantity discount larger than the efficiency gap. Particularly noteworthy in Table 6.2 is the maximum where almost 23 times the gap is paid to close it. The measure Φ^b peaks at a much lower level, which is typical of the less intensive use of the HCC mechanism. The minimum value for Φ^a confirms that $\tilde{\alpha}_w^* = 0$ does not occur.

Measure	Mean	Stdev.	Min.	Max.
Φ^a	116.97 %	108.59 %	26.47 %	2267.2 %
Φ^b	8.97 %	27.85 %	0.00 %	249.83 %

Table 6.2: *Summary statistics payment size*

We construct Figure 6.1 by calculating the mean and standard deviation for subsets of scenarios obtained after fixing a parameter at one of its values. Higher Retailer costs tend to decrease the proportional discount $\tilde{\alpha}_w^*$. More expensive lot-sizing upstream has the opposite effect. The spread keeps up with the average.

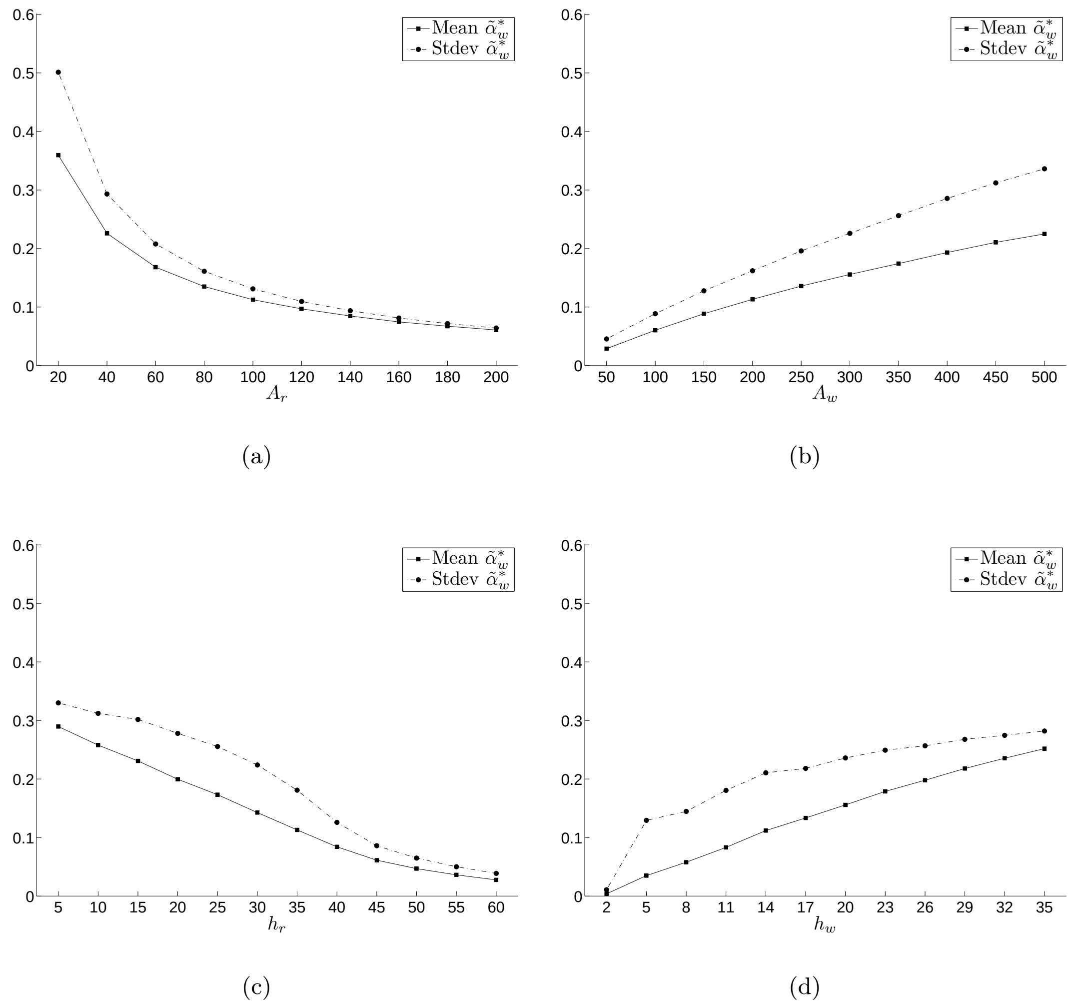


Figure 6.1: *Sensitivity of optimal total discount in terms of* $\sqrt{2A_r h_r D}$

A similar pattern is revealed by the sensitivity analysis for the optimal level of holding cost compensation in Figure 6.2. With A_r and h_r increasing (or A_w and h_w decreasing), it becomes harder for the Wholesaler to have the savings on his inventory related costs outweigh the payments to the Retailer. Specifically, a larger share of the scenarios in a subset gets covered by Proposition 5.5.1. For $h_r \geq 35$ or $h_w \leq 5$, no positive β_w^* is chosen, as each included parameter configuration then satisfies the condition $\frac{h_r}{h_w} \geq 1$.

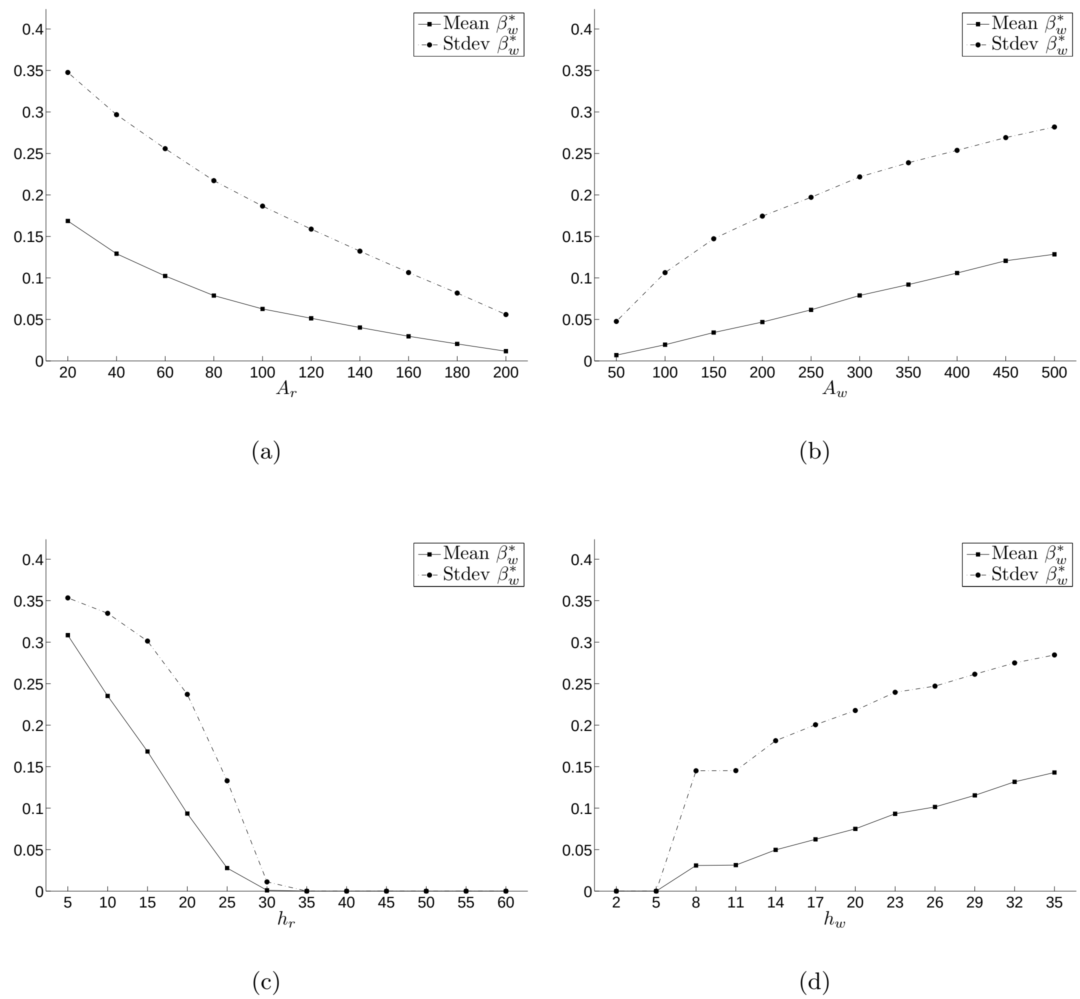


Figure 6.2: *Sensitivity of optimal HCC fraction*

6.3 Effect on the efficiency gap

The mean efficiency gap is 782.29 cost units, which corresponds to an average of 7.08% above the minimum joint lot-sizing expenses. The smaller impact of the HCC mechanism is obvious from Table 6.3. Except for the maximum of 99.93%, the Δ^b statistics stand out rather negatively. However, in the 11.75% (see Table 6.1) of all cases where some holding cost compensation is granted, the record with a, roughly estimated, average gap closure of $8.97\%/0.1175 \approx 76.3\%$, is actually not that bad.

Measure	Mean	Stdev.	Min.	Max.
Δ^a	100.00 %	0.00 %	100.00 %	100.00 %
Δ^b	8.97 %	26.03 %	0.00 %	99.93 %

Table 6.3: *Summary statistics efficiency gap closure*

The behaviour of the efficiency gain percentage in Figure 6.3 is closely related to the optimal compensation level. We see more clearly the implication of Proposition 5.5.1 with two dimensions: the Wholesaler never compensates when $h_r \geq h_w$ or $A_w \leq 2A_r$. There are pairs h_r, h_w in Figures 6.3(c) and 6.3(d) having more or less the same mean and standard deviation. If $A_w \leq 2A_r$ holds, no holding costs are reimbursed. The scenarios with $A_w > 2A_r$ satisfy at least one of the conditions in Proposition 5.5.2, and, therefore, have $\beta_w^* = 1 - \frac{A_r}{A_w - A_r}$. The same set of compensation levels results in almost the same set of efficiency data for each pair.

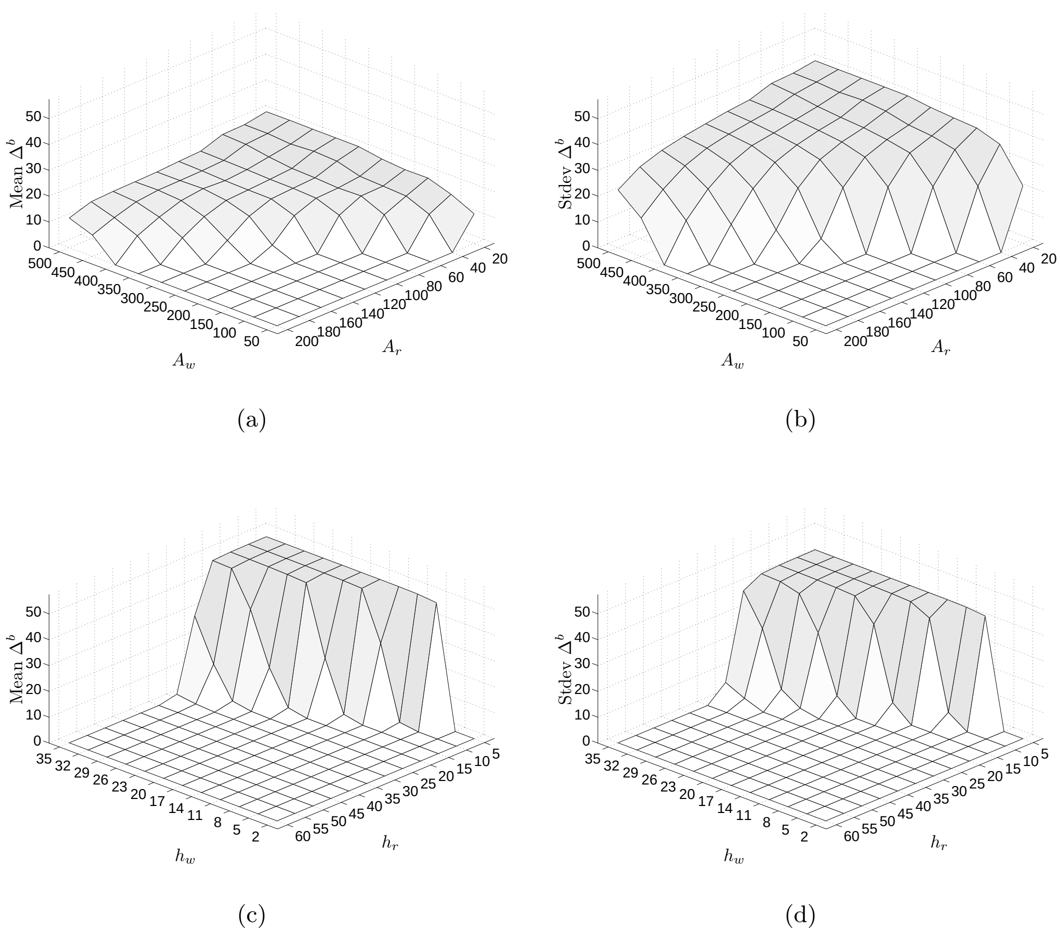


Figure 6.3: *Influence of varying parameters on HCC efficiency gain*

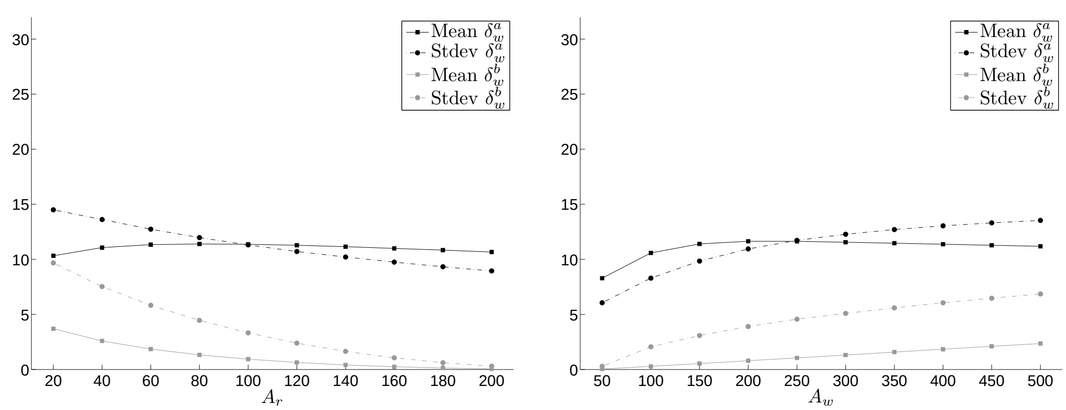
6.4 Savings throughout the supply chain

Table 6.4 gives an overview of how coordination benefits each actor. While a quantity discount scheme causes substantial savings upstream, the balance shifts markedly in favour of the downstream actor when coordination takes place by means of the second mechanism. Although the average δ_w^a value is a moderate 4.70%, the reduction in the Retailer's lot-sizing costs can be as high as 79.59%, which is well beyond the AQD maximum for the Wholesaler.

Measure	Mean	Stdev.	Min.	Max.
AQD				
δ_r^a	0.00 %	0.00 %	0.00 %	0.00 %
δ_w^a	11.04 %	11.45 %	0.00 %	58.62 %
HCC				
δ_r^b	4.70 %	14.92 %	0.00 %	79.59 %
δ_w^b	1.19 %	4.88 %	0.00 %	49.75 %

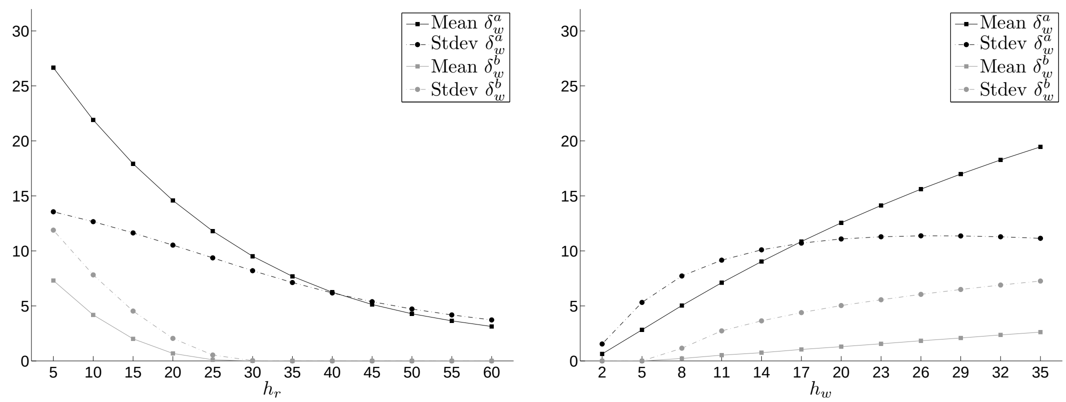
Table 6.4: *Summary statistics individual savings*

We get $\delta_r^b = 100\% \frac{\sqrt{2A_r h_r D} - \sqrt{2A_r(1-\beta_w^*)h_r D}}{\sqrt{2A_r h_r D}} = 100\% \left(1 - \sqrt{1-\beta_w^*}\right)$ by taking the lot-sizing components in Equations (5.2.10) and (5.5.6). The link between the optimal compensation level and savings at the upstream echelon becomes apparent from a comparison of Figure 6.4 with Figure 6.2. Just as for the Retailer, parameter settings encouraging a heavier use of the HCC scheme are associated with relatively larger decreases in costs. Figures 6.4(a) and 6.4(b) display a somewhat deviating pattern: the benefits for the Wholesaler of the AQD schedule seem rather insensitive to a change in A_r or A_w.



(a) (b)

Figure 6.4: *Effect of a changing environment on Wholesaler savings*



(c) (d)

Figure 6.4: *Effect of a changing environment on Wholesaler savings (continued)*

Chapter 7

Extensions

7.1 Introduction

The analysis in this thesis can be extended along several lines by relaxing the underlying assumptions. The models become more realistic, if, for instance, we introduce the element of imperfect knowledge, which could stem from stochastic demand (Li & Liu (2006)), or uncertainty about the characteristics of other actors (Corbett & de Groote (2000)). We consider two other natural extensions in more detail. Using our graphical approach, we provide preliminary insights in Section 7.2 for a larger serial supply chain. Section 7.3 roughly sketches some consequences of price-sensitive demand.

7.2 Serial supply chain with 3 echelons

Adding Manufacturer lot-sizing By going further upstream, we encounter the Manufacturer who replenishes the Wholesaler. Also faced with a lumpy demand pattern, this actor uses $n_m \in \mathbb{N}$ to determine an inventory policy. His profit function is a modified version of Equation (5.2.2). With a unit cost $C > 0$ paid to an outside source, a fixed order cost $A_m > 0$, a holding cost rate $h_m > 0$, and the size of incoming orders equal to $n_w Q$, we get

$$\Pi_m^u (Q, n_w, n_m) = (P_m - C)\, D - A_m \frac{D}{n_m n_w Q} - h_m \frac{(n_m - 1)\, n_w Q}{2}$$

Figure 7.1 contains the representation of the enlarged non-coordination game. It is the structure in Figure 3.1(a) complemented with the Manufacturer lot-sizing stage.

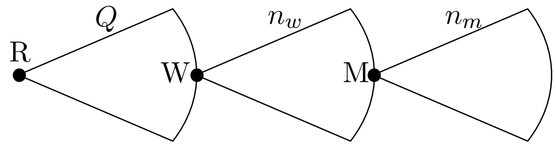


Figure 7.1: *Extensive form for 3 echelon non-coordination game*

In the non-coordination setting, the Retailer and Wholesaler hold on to respectively Equation (5.2.6) and (5.2.4). For the Manufacturer, straightforwardly adjusting the last one gives

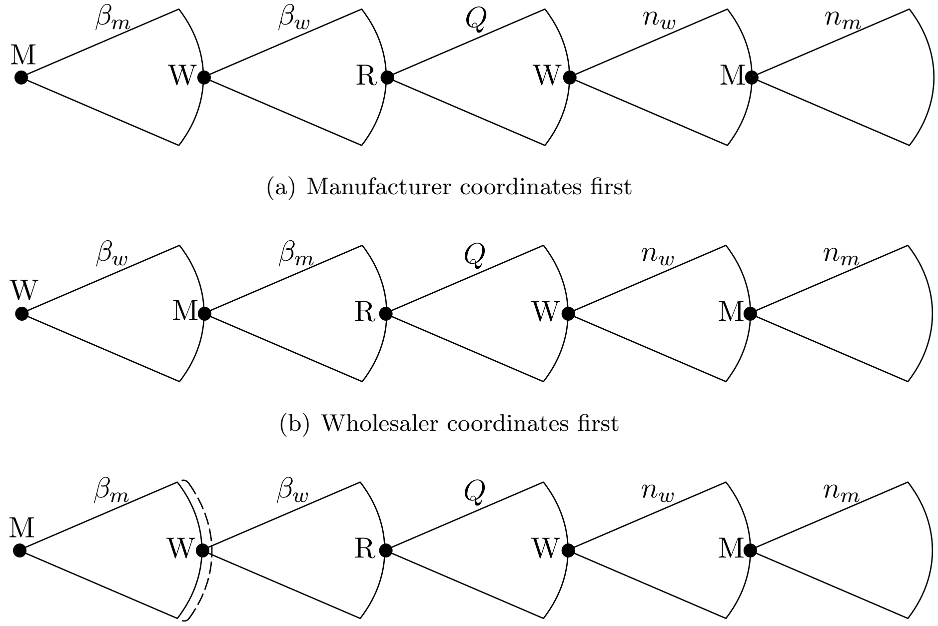
$$n_m^{*u}(Q, n_w) = \begin{cases} \left\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2A_m D}{h_m (n_w Q)^2}} \right\rceil \\ \text{or} \\ \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2A_m D}{h_m (n_w Q)^2}} \right\rfloor \end{cases}$$

The algorithm in Munson & Rosenblatt (2001) can be used to determine the values for Q, n_w and n_m which maximize total supply chain profit.

Coordination by multiple actors Analogous to the situation with 2 actors, coordination by the Manufacturer means targeting the downstream partner's costs. If he offers a combination of a discount fraction α_m and region R_m, his price is multiplied by $1 - \alpha_m 1_{\{n_w Q \in R_m\}}$, and the Wholesaler's gross margin becomes $\left(\left(1 - \alpha_w 1_{\{Q \in R_w\}}\right) P_w - \left(1 - \alpha_m 1_{\{n_w Q \in R_m\}}\right) P_m\right)$. The term $\beta_m h_w \frac{(n_w - 1)Q}{2}$ represents the costs of realizing a holding cost rate $(1 - \beta_m) h_w$ downstream.

As illustrated in Figure 7.2 for the second mechanism, interaction may take place in multiple ways. The more dominant upstream actor is granted the first mover advantage, giving him the most influence on how the rest of the game proceeds. Accordingly, lack of any difference in power between both is translated into simultaneous coordination in Figure 7.2(c). The new graphical element of a dashed line enclosing the β_m arc represents the fact that all decision nodes are in the same information set: the Wholesaler does not know the value of β_m when setting his level of compensation. We could have equally well reversed the first two stages.

Because the Retailer's lot-sizing problems do not change, Equation (5.4.2) and (5.5.1) still apply in a 3 actor context. Solving Problem (5.5.5) with $(1 - \beta_m) h_w$ instead of h_w gives the response to a particular β_m value. Irrespective of the scheme's details, the Wholesaler always at least earns his 2 echelon coordination profit. A pure strategy Nash equilibrium may not exist for the simultaneous moves scenario.



(a) Manufacturer coordinates first

(b) Wholesaler coordinates first

(c) Simultaneous coordination moves

Figure 7.2: *Extensive forms for 3 echelon coordination games*

The larger games can be used to investigate whether parties' coordinative actions are strategic substitutes or complements, and to what extent the sequence of decision making matters for the success of streamlining the supply chain. In contrast to the simpler setting, introduction of a mechanism could be bad for efficiency. Alignment of the two downstream echelons may create new lot-sizing externalities at the most upstream level, causing a decrease in the Manufacturer's profit which exceeds the total benefits for the Retailer and Wholesaler.

7.3 Price-sensitive demand

Influence on revenue In practice, there are many situations where demand, in fact, depends on the price charged to end-consumers. In our lot-sizing environment, a convenient way to model price-sensitivity is to let the Retailer price in Equation (5.2.1) decrease linearly in D:

$$\Pi_r^u(D, P_w, Q) = \begin{cases} ((P_0 - \rho D) - P_w)\, D - A_r \dfrac{D}{Q} - h_r \dfrac{Q}{2} & \text{if } 0 < D \leq \frac{P_0}{\rho} \\ \text{and} & \\ 0 & \text{if } D = 0 \end{cases}$$

The Wholesaler equivalent largely corresponds to Equation (5.2.2). The endogenous determination of prices and demand is reflected in

$$\Pi_w^u(D, P_w, Q, n_w) = \begin{cases} (P_w - P_m) D - A_w \dfrac{D}{n_w Q} - h_w \dfrac{(n_w - 1) Q}{2} & \text{if } 0 < D \leq \frac{P_0}{\rho} \\ \text{and} & \\ 0 & \text{if } D = 0 \end{cases}$$

The profit functions under the AQD and HCC schemes are obtained by incorporating the variables α_w, R_w and β_w as in the fixed demand context.

Referring to Figure 3.1, the Wholesaler moves first in each game by setting his price P_w (and the characteristics of the coordination mechanism). The decision on D is included in the next (Retailer lot-sizing) stage. A non-operational supply chain is the result of the downstream actor choosing $D = 0$.

Demand & lot-sizing Very little changes in the last stage of the game. The Wholesaler's lot-sizing rule $n_w^{*u}(D, Q)$ equals Equation (5.2.4) with D as an additional variable. The Retailer's anarchy reaction follows after inserting Equation (5.2.6) in $\Pi_r^u(D, P_w, Q)$ and subsequently solving

$$\max_D \Pi_r^u(D, P_w) = ((P_0 - \rho D) - P_w) D - \sqrt{2 A_r h_r D}$$
$$\text{subject to:}$$
$$0 \leq D \leq \frac{P_0}{\rho}$$

The response to a vector (P_w, β_w) is determined similarly using Equation (5.5.1). For the AQD scheme, we may apply the approach in Viswanathan & Wang (2003).

Performance Price-elasticity adds double marginalization as another source of inefficiencies (see Chapter 2). The effectiveness of a mechanism will not just depend on its capacity to fine-tune lot-sizing policies, but also on its potential to generate extra revenue. Setting higher quantity discounts or granting more compensation lowers the marginal costs for the Retailer and will stimulate him to sell more to end-consumers. Based on Weng (1995*a*), Weng (1995*b*) and Viswanathan & Wang (2003), we expect the AQD's efficiency performance to drop below 100%, and even more so if demand becomes more price-elastic.

Chapter 8

Conclusion

In order to investigate the potential of a quantity discount mechanism and a holding cost compensation scheme to align lot-sizing decisions in a 2 actor serial supply chain, we derived subgame perfect pure Nash strategies using backward induction. It was shown that introduction of an AQD schedule leads to full optimization of the supply chain, as the Wholesaler reaps all the benefits of coordination, and, therefore, has an incentive to maximize surplus. We provided a general solution algorithm to calculate an equilibrium path in case the HCC mechanism is used. Additionally, the optimal level of compensation was theoretically established for two classes of parameter configurations. The simulation study revealed that the Wholesaler sets a higher fraction, both actors realize larger savings percentages, and the efficiency gap is closed to a larger extent when lot-sizing becomes more expensive upstream/less expensive downstream. The Retailer receiving a considerable share of the savings makes the Wholesaler less inclined to coordinate.

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